This is a continuation of Reviews 1 and 2. The final exam will be comprehensive and you should be familiar with the topics in the first two Reviews.

MH = Marsden-Hoffman, R1,R2 = Review1, 2.
As usual, when nothing else is said, $A \subset \mathbb{C}$ is a domain and $f : A \to \mathbb{C}$

1. *Isolated singularities*: If a function f is analytic in a region containing a punctured disk $\{0 < |z - z_0| \le r\}$ for some $r > 0$, we say that f has an *isolated singularity* at z_0. Recall from §9, 10 of R2, MH p226, that f has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{1}$$

convergent in the punctured disk, and for every r_1, r_2 such that $0 < r_1 < r_2 < r$ the convergence is *uniform* in the annulus $\{r_1 < |z - z_0| < r_2\}$.

2. The isolated singularities are classified as (MH Def 3.3.2, R2 §10):
   (a) *Essential Singularity* if $a_n \neq 0$ for infinitely many $n < 0$.
   (b) *Pole of order k* if $k \ge 1$, $a_{-k} \neq 0$, and $a_n = 0$ for $n < -k$.
   (c) *Removable Singularity* if $a_n = 0$ for all $n < 0$.

3. A function that is analytic in a region A except for poles is called *meromorphic* in A (MH Def 3.3.2, 5). Examples of meromorphic functions are quotients $f(z)/g(z)$ where $f, g : A \to \mathbb{C}$ are analytic and g is not identically zero. Then f/g has isolated singularities at the zeros of f, these are either removable singularities or poles. For example, $tan(z) = \sin(z)/\cos(z)$ has poles at $\{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$, the zeros of $\cos(z)$, and no other singularities, so it is meromorphic in $\mathbb{C}$. Examples of functions that are *not* meromorphic are $e^{1/z}$ or $\sin(1/z)$ which have essential singularities at 0.

4. A function f has a pole of order $k \ge 1$ at z_0 if and only if for some $\epsilon > 0$ there is an analytic function $\phi(z)$ defined on $|z - z_0| < 0$, with $\phi(z_0) \neq 0$ and $f(z) = (z - z_0)^{-k}\phi(z)$ on $\{|z - z_0| < \epsilon\}$. (See MH Prop 3.3.4 for this and other characterizations of poles that you should know). Compare this with the definition of a zero of order k (R2 §8). Looking at this, we get:

5. (MH Prop 3.3.5) If f, g have zeros of order k, l respectively at z_0, then f/g has
   (a) A zero of order $k - l$ at z_0 if $k > l$,
   (b) A removable singularity at z_0 with $f(z_0) \neq 0$ if $k = l$,
   (c) A pole of order $l - k$ at z_0 if $k < l$.

6. *Exercise*: We see from the last statement that we could think of a pole as a zero of negative order. Follow this idea by making the following definition: the *order* of a meromorphic function f at z_0, denoted $O(f, z_0)$, to be k if f has a zero of order $k \ge 0$ at z_0, and to be $-k$ if f has a pole of order k at z_0. Prove that $O(f/g, z_0) = O(f, z_0) - O(g, z_0)$.

7. **Residue**: If z_0 is an isolated singularity of f, the *Residue* of f at z_0, denoted $Res(f; z_0)$ is the coefficient a_{-1} in the Laurent expansion (1) of f at z_0. By the uniform convergence property of (1) stated above, we have

$$Res(f; z_0) = a_{-1} = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} f(z)\, dz \quad \text{for any } \epsilon \text{ such that } 0 < \epsilon < r. \qquad (2)$$

8. *Calculation of Residues*: (§4.1 of MH): There is a table of formulas in MH, Table 4.1.1 on p 250. I would not recommend memorizing much, instead know how to expand. One formula that comes up very often is 4: if $g(z_0) \neq 0$ and h has a simple zero at z_0, then g/h has a simple pole at z_0 (clear from above) and $Res(g/h; z_0) = g(z_0)/h'(z_0)$. If you don't remember this, just do the following : $g(z) = g(z_0) + \dots$ and $h(z) = h'(z_0)(z - z_0) + \dots$, where $\dots$ denotes higher order. Then

$$\frac{g(z)}{h(z)} = \frac{g(z_0) + \dots}{h'(z_0)(z - z_0) + \dots} = \frac{g(z_0)(1 + \dots)}{h'(z_0)(z - z_0)(1 + \dots)} = \frac{\frac{g(z_0}{h'(z_0)}}{z - z_0}(1 + \dots),$$

where the last equality uses the basic principles $1/(1 + \dots) = 1 + \dots$ (say, by geometric series) and $(1 + \dots)(1 + \dots) = 1 + \dots$. For example, $\tan z = \sin z/ \cos z$ has simple poles at each $z_n = (n + \frac{1}{2})\pi$, for each $n \in \mathbb{Z}$, with residue $\sin(z_n)/ \cos'(z_n) = -1$. A more complicated example, needed below, would be:

Example: Let $f(z) = \pi \cot(\pi z)$. Find $Res(f(z)/z^2; 0)$ and $Res(f(z)/z^4; 0)$.

To find these residues, we first find the first few coefficients of the Laurent expansion of $\cot z = \cos z/ \sin z$ by dividing the power series at 0 of $\cos z$ and $\sin z$:

$$\frac{\cos z}{\sin z} = \frac{1}{z} \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} + \dots}{(1 - \frac{z^2}{6} + \frac{z^4}{120} + \dots)} = \frac{1}{z}(1 - \frac{z^2}{2} + \frac{z^4}{24} + \dots)(1 + (\frac{z^2}{6} - \frac{z^4}{120} + \dots) + (\frac{z^2}{6} + \dots)^2 + \dots),$$

where the second factor is obtained by applying the geometric series. The terms of degree ≤ 4 in the second factor come from the first term in parenthesis and the square of the first term in the second parenthesis:

$$1 + \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \dots = 1 + \frac{z^2}{6} + \frac{7z^4}{360} + \dots,$$

so multiplying and keeping only the terms of degree ≤ 4 we get

$$\frac{1}{z}(1 - \frac{z^2}{2} + \frac{z^4}{24})(1 + \frac{z^2}{6} + \frac{7z^4}{360}) = \frac{1}{z}(1 + (-\frac{1}{2} + \frac{1}{6})z^2 + (\frac{7}{360} - \frac{1}{12} + \frac{1}{24})z^4 = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45}$$

. so the Laurent expansion of $\pi \cot(\pi z)$ looks like

$$\frac{1}{z} - \frac{\pi^2}{3} z - \frac{\pi^4}{45} z^3 + \dots.$$

Dividing by z^2, z^4 we get $Res(\pi \cot(\pi z)/z^2; 0) = -\frac{\pi^2}{3}$ and $Res(\pi \cot(\pi z)/z^4; 0) = -\frac{\pi^4}{45}$.

9. **Residue Theorem** (MH §4.2) Let $A \subset \mathbb{C}$ a region, let $z_1, \dots, z_n \in A$, and let $f : A \setminus \{z_1, \dots z_n\} :\to \mathbb{C}$ be analytic. Then, for any domain $D \subset A$ with boundary a simple closed curve γ,

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{z_j \in D} Res(f; z_j). \tag{3}$$

This is a variation of Theorem 4.2.1 of MH, using a somewhat different and less precise language, just as we did in proving Cauchy's integral formula from Green's theorem. The language of homotopy used in MH Thm 4.2.1 will be explained later.

Example

$$\int_{|z|=2} \tan(z)\, dz = -4\pi i$$

since there are two poles in $|z| < 2$, namely $\pm\pi/2$, each with residue -1.

10. *Applications of the residue theorem*

(a) *Counting Zeros*, also called the *argument principle* (Explained in class, also MH §6.2): Suppose $f : A \to \mathbb{C}$ is analytic, $D \subset A$ is a domain with boundary curve γ, and $f(z) \neq 0$ for all $z \in \gamma$. Let $a_1, \dots, a_n$ be the zeros of f in D, and let k_j denote the order of a_j. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)\, dz}{f(z)} = k_1 + \dots k_n, \tag{4}$$

in other words, the number of zeros in D counting multiplicities.

Proof: Since a_j is a zero of multiplicity k_j, by definition we have disk Δ_j centered at a_j and an analytic function $\phi_j : \Delta_j \to \mathbb{C}$ with $\phi_j(a_j) \neq 0$ and $f(z) = (z - a_j)^{k_j}\phi_j(z)$. By shrinking Δ_j we may assume that $\phi_j(z) \neq 0$ for $z \in \Delta_j$. Then on Δ_j we have $f'(z) = k_j(z - a_j)^{k_j - 1}\phi_j(z) + (z - a_j)^{k_j}\phi_j'(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{k_j}{z - a_j} + \frac{\phi_j'(z)}{\phi_j(z)} = \frac{k_j}{z - a_j} + \text{analytic function},$$

so a_j is a simple pole of f'/f with residue k_j, hence the residue theorem gives (4).

(b) *Evaluation of Definite Integrals* (MH §4.3). Many real definite integrals can be computed by residues. To do this one has to find a sequence of contours (closed curves) to which the residue theorem applies, and show that the the integrals over the unwanted parts go to zero. Good examples are:

i. (MH Example 4.3.5): $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx$

ii. (MH Example 4.3.13) Principal value integrals: $\int_{-\infty}^{\infty} \frac{x}{x^3+1}\, dx$.

iii. (MH Example 4.3.15) Banch cuts: $\int_0^{\infty} \frac{\sqrt[3]{x}}{1+x^2}\, dx$

iv. (MH Example 4.3.20) $\int_{-\infty}^{\infty} \frac{1}{1+x^{2n}}\, dx$.

(c) *Evaluation of Infinite Series* (MH §4.4): The fact that $\pi\cot(\pi z)$ is meromorphic with poles exactly the integers, each simple of residue one, and the fact that, if C_N

denotes the square with vertices $(N+\frac{1}{2})(\pm 1 \pm i)$, $|\pi\cot(\pi z)| \leq 2\pi$ for N large (MH p 308), allows us to sum certain series, for instance,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{a_{2k-1}}{2},$$

where a_{2k-1} is the coefficient of z^{2k-1} in the Laurent expansion of $\pi\cot(\pi z)$ at 0. We computed above that $a_1 = -\frac{\pi^2}{3}$ and $a_3 = -\frac{\pi^4}{45}$, therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \text{etc.}$$

Exercises 5 and 8 of MH §4.4 indicate how to use $\pi/\sin(\pi z)$ to evaluate alternating sums $\sum(-1)^n/n^{2k}$.

11. *Elementary proof of Cauchy's formula without assuming continuity of f'* (MH §2.3) The end result is, that if $f : A \to \mathbb{C}$ is continuous, then f is analytic if and only if $\int_{\partial R} f(z)\,dz = 0$ for all rectangles $R \subset A$ with sides parallel to the axes. Proofs as follows:

    (a) f analytic $\Rightarrow$ $\int_{\partial R} f(z)\,dz = 0$ is the argument of subdividing R into 4 equal pieces and iterating, see proof of Thm 2.3.1 in MH.

    (b) If f is continuos and $\int_{\partial R} f(z)\,dz = 0$ for all rectangles $R \subset A$ (with sides parallel to axes), then given any disk $D \subset A$ there is function $F : D \to \mathbb{C}$ with $F'(z) = f(z)$. The proof of Theorem 2.3.2 of MH gives this.

    (c) The hypothesis of (a) can be weakened to assuming that f is continuous on A and analytic except at one point. This strengthening gives a proof of Cauchy's integral formula (without assuming continuity of f'), Thm 2.4.4 of MH, and therefore, by using Cauchy's formulas for derivatives, get f analytic $\to$ f' analytic (MH Thm 2.4.6, R2 §4(a)).

    (d) Converse to (a) (Morera's Theorem) now follows: f continuous and $\int_{\partial R} f(z)\,dz = 0$ for all $R \subset A$ $\Rightarrow$ for all disks $D \subset A$, there exists $F : D \to \mathbb{C}$ with $F'(z) = f(z)$ on D $\Rightarrow$ F is analytic $\to$ $F' = f$ is analytic in D $\Rightarrow$ f is analytic in A.

12. *Homotopic curves and simply connected regions*(MH §2.3) Two continuous (piecewise C^1) closed curves $\gamma_0, \gamma_1 : [0,1] \to A$ are *homotopic (as closed curves)* if there is a continuous (piecewise C^1) map $H : [0,1] \times [0,1] \to A$ so that $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$ for all $t \in [0,1]$ and $H(s,0) = H(s,1)$ for all $s \in [0,1]$ (MH Def 2.3.7, Figure 2.3.9).

    A region $A \subset \mathbb{C}$ is *simply connected* if every closed curve in A is homotopic (as closed curve) to a point. (MH Def 2.3.8)

    Example: Convex regions (MH Prop 2.3.9), star shaped regions (MH, Exercise 3 to 2.3) are simply connected.

    If γ_0, γ_1 are closed curves in A which are homotopic (as closed curves) and f is analytic in A, then $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$. (MH, Thm 2.3.12)