Math 4200, Review for Midterm 2

Be familiar with Chapters 2 and 3 of the text, except for 2.3 which we'll do later. The best way to prepare for the test is to know how to do all the assigned homework problems, including the review problems. Here is a reminder of the most important topics:

1. Definition of line integrals (contour integrals) $\int_\gamma f(z)dz$ (Def 2.1.1), how to compute them, and how to *estimate* them: (Prop 2.1.6) if $|f(z)| \leq M$ on γ, then

$$\int_\gamma |f(z)dz| \leq \int_\gamma |f(z)||dz| \leq Ml(\gamma) \quad (\text{where } l(\gamma) = \text{ length of } \gamma). \tag{1}$$

2. Path Independence of line integrals (Thm 2.1.9) and its relation to anti-derivatives (Thm 2.1.7)

3. *Cauchy's Theorem* (Preliminary version, variation of Thm 2.2.1): If γ is a simple closed curve which bounds a domain D on which f is analytic and f' is continuous, then

$$\int_\gamma f(z)\ dz = 0.$$

   Know how to prove this from Green's theorem and the Cauchy-Riemann equations, and how to apply this to compute integrals.

4. **Cauchy's Integral Formula** (Variation on Thm 2.4.4): Suppose f is analytic in a region $A \subset \mathbb{C}$, γ is a simple closed curve in A that is the boundary of a domain $D \subset A$. Then, for all z in the interior of D we have

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\ d\zeta}{\zeta - z}. \tag{2}$$

   This is the most important formula in the whole course! Make absolutely sure that you remember it, know what it says, and know its most important consequences. The best way to remember some of the consequences would be to have a general idea of how to derive them from Cauchy's formula and general principles. Here's a list of important consequences, and a brief explanation of why each one is true. Look through this and try to get a global picture of how things fit together. *It will always be assumed that $A \subset \mathbb{C}$ is an open, connected set, and $f : A \to \mathbb{C}$ an analytic function.*

   (a) Cauchy's formulas for the derivatives (Thm 2.4.6): under the same assumptions of f and γ,

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)\ d\zeta}{(\zeta - z)^{k+1}} \quad \text{for } k = 1, 2, 3, \ldots : \tag{3}$$

   in particular, f is infinitely differentiable. (Differentiate (2) under the integral sign.)

(b) Cauchy's inequalities (Thm 2.4.7): If $\{|z - z_0| \le R\} \subset A$, γ is the boundary circle $|z - z_o| = R$, and $|f(z)| \le M$ on γ, then for $k = 1, 2, 3, \ldots$,

$$|f^{(k)}(z_0)| \le \frac{k!}{R^k} M$$

(Specialize (3) to $z = z_0$ and γ the circle $|z - z_o| = R$, and apply the basic estimation principle (1).)

(c) Liouville's Theorem (Thm 2.4.8): If f is a bounded entire function (meaning f analytic on all of $\mathbb{C}$ and there exists a constant M so that $|f(z)| \le M$ for all $z \in \mathbb{C}$), then f is constant. (For any $z_0 \in \mathbb{C}$, apply the last inequality with $k = 1$ and let $R \to \infty$ to get $f'(z_0) = 0$. Since z_0 was arbitrary, $f'(z) \equiv 0$, hence f is constant.)

(d) Fundamental Theorem of Algebra (Thm 2.4.9): If $p(z)$ is a polynomial of degree at least one, then it has a complex root: there exists $z_0 \in \mathbb{C}$ so that $p(z_0) = 0$. (If not, $1/p(z)$ would be an entire function, and, since $|p(z)| \to \infty$ as $|z| \to \infty$, $1/p(z) \to 0$, hence $1/p(z)$ would be bounded and entire, hence constant by Liouville, contradicting that the degree of $p(z)$ is at least one.)

(e) Mean Value Property of analytic functions (Thm 2.5.2): $f : A \to \mathbb{C}$ analytic, and the closed disk $|z - z_o| \le r \subset A$. then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta; \tag{4}$$

in words, the value of an analytic function at the center of a disk is the average of its values in he boundary circle. (Specialize (2) to $z = z_0$ and γ the circle $|z - z_o| = r$.)

(f) Sub-Mean Value Inequality (not explicitly stated in the text): Under the same assumptions as in the last paragraph.

$$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta; \tag{5}$$

in words, the absolute value at the center of a disk is not larger than its average on the boundary circle. (Apply the first standard inequality (1) to equation (4).)

(g) Maximum Modulus Principle, Local Version (Thm 2.5.1): If $f : A \to \mathbb{C}$ is analytic and $|f|$ has a relative maximum at $z_0 \in A$, then f is identically constant in a neighborhood of z_0. Said more informally: $|f|$ can have no local maxima. (Since $|f|$ is continuous, at a local maximum the right hand side of (5) must be strictly smaller (for small r) than the left hand side.)

(h) Maximum Modulus Principle, Global Version (Thm 2.5.6): If $f : A \to \mathbb{C}$ analytic and not constant, $D \subset A$ is closed and bounded subset, then the maximum value of $|f|$ on D must occur on the boundary of D. (Follows easily from the local version).

(i) Mean Value property of Harmonic Functions (Thm 2.5.9): if $u : A \to \mathbb{R}$ is harmonic, then, for any disk $\{|z - z_0| \le r\} \subset A$, we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta; \tag{6}$$

(On disks can always find conjugate harmonic function v, apply (4) to $f = u + iv$).

(j) Local and Global Maximum and Minimum Principle for Harmonic Functions (Thms 2.5.10 and 2.5.11): harmonic functions can have no local maxima nor local minima (critical points must be saddle points), etc.

(k) Power Series Expansions of Analytic Functions (Taylor's Theorem, Thm 3.2.7) Suppose $f : A \to \mathbb{C}$ is analytic, $z_0 \in A$ and suppose $\{|z - z_0| < R\} \subset A$. Then on $\{|z - z_0| < R\}$ we have a convergent power series that converges to f:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{7}$$

(For any r, $0 < r < R$, start from Cauchy's integral formula (2) choosing $D = \{|z - z_0| \leq r\}$ and $\gamma = \{|z - z_0| = r\}$, and expand the factor $1/(\zeta - z)$ that appears inside the integral (2) by a convergent geometric series:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{(\zeta - z_0)} \frac{1}{(1 - \frac{z - z_0}{\zeta - z_0})} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

Since the continuous function $|f(\zeta)| \leq M$ on γ for some constant M, we see that, for each fixed z, $f(\zeta) \sum (z - z_0)^n/(\zeta - z_0)^{n+1}$ is majorized by the series of constants $(M/r) \sum (|z - z_0|/r)^n$, hence converges uniformly in ζ to $f(\zeta)/(\zeta - z)$. Substitute this sum for the integrand in Cauchy's formula (2) and then appeal to uniform convergence to interchange summation and integration:

$$f(z) = \frac{1}{2\pi i} \int f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \, d\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) \, d\zeta}{(\zeta - z_0)^{n+1}} \right) (z - z_0)^n$$

Finally, use Cauchy's formulas for derivatives (3) to identify the coefficient of $(z - z_0)^n$ to be the same as its coefficient in Taylor's formula (7).)

(l) Analytic Convergence Theorem (Thm 3.1.8): If $f_n : A \to \mathbb{C}$ is a sequence of analytic functions, $f : A \to \mathbb{C}$ is a function, and suppose that for every closed disk $D \subset A$, $f_n \to f$ uniformly on D. Then f is analytic and $f_n' \to f'$ uniformly on every closed disk $D \subset A$. (If $f_n \to f$ uniformly on D, then the Cauchy integrals (2) for f_n converge to that of f, so f is represented by a Cauchy integral, so it is analytic. (a quicker proof base on Morera's theorem, that we have not yet discussed, is given in the text). Since $f_n \to f$ uniformly on D, the integral formulas (3) for f_n' converge to that for f', and the convergence is uniform on every smaller sub-disk).

5. Be familiar with the basic tests for convergence of infinite series (Prop 3.1.3) and for uniform convergence of series of functions (Thm 3.1.7). Have the notion of uniform convergence (Def 3.1.4) very clear in your mind. Know the most important consequences: uniform limit of continuous functions is continuous (Prop 3.1.6), integral of a uniform limit is the limit of the integrals (Prop 3.1.9)).

6. *Power Series*: A series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has a *radius of convergence* $R \in [0, \infty]$ characterized by: the series converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. (Thm 3.2.1). It represents an analytic function inside its circle of convergence (Thm 3.2.3) and it can be differentiated term by term (Thm 3.2.4). These are just special cases of the Analytic Convergence Theorem.

7. Know the geometric and exponential series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ for } |z| < 1, \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and how to derive others from these, for example, $1/(1-z)^2$ either by squaring or by differentiation, $\log(1-z)$ by integration, $\cos(z) = (e^{iz} + e^{-iz})/2$, etc.

8. Order of zeros, isolation of zeros: If $f : A \to \mathbb{C}$ is analytic, $c \in A$ and $f(c) = 0$, then there are two possibilities (Prop 3.2.9):

   (a) It is a *zero of order* n if and only if $f(c) = f'(c) = \ldots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \neq 0$. If c is a zero of order n, then there is an $r > 0$ and an analytic function $\phi(z)$ so that $\phi(z) \neq 0$ for any z with $|z - c| < r$ and $f(z) = (z - c)^n \phi(z)$ for $|z - c| < r$.

   (b) It is a zero of infinite order: $f^{(n)}(c) = 0$ for all n. In this case there is an $r > 0$ so that $f(z) = 0$ for all z with $|z - c| < r$.

9. *Laurent Expansion*; If $0 \leq R_1 < R_2 \leq \infty$, and $z_0 \in \mathbb{C}$, let $A = R_1 < |z - z_0| < R_2\}$ be an annulus centered at z_0. If $f : A \to \mathbb{C}$ is analytic, then f has a *Laurent expansion* (Thm 3.3.1, in different notation):

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{8}$$

which converges absolutely in A and, for each r_1, r_2 such that $R_1 < r_1 < r_2 < R_2$, the series converges uniformly on $\{r_1 \leq |z - z_0| \leq r_2\}$. The coefficients are given by the formula

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}, \quad n \in \mathbb{Z} \tag{9}$$

where γ is any circle $|\zeta - z_0| = r$, $R_1 < r < R_2$.

10. If $R_1 = 0$ we say that f has an *isolated singularity* at z_0. If z_0 is an isolated singularity, then either $a_n \neq 0$ for infinitely many negative $n \in \mathbb{Z}$, called an *essential singularity*, or $a_n \neq 0$ for only finitely many n, which is called a *pole of order* k if $k \geq 1$, $a_{-k} \neq 0$ and $a_n = 0$ for all $n < -k$. If $a_n = 0$ for all $n < 0$ we say that f has a *removable singularity* at z_0.

11. Examples: $e^{1/z} = \sum (1/n!)(1/z^n)$ has an essential singularity at 0, while e^z/z^2 has a second order pole at 0 and $\sin(z)/z$ has a removable singularity at 0.