MATH 3220-3 HOMEWORK 5

DUE APRIL 22 BEFORE CLASS

(1) Recall the parametrized torus T of Homework 4 obtained by fixing $0 < b < a$ and defining $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ with image T by $\Phi(\phi, \theta) = (x(\phi, \theta), y(\phi, \theta), z(\phi, \theta))$ where

$$\begin{aligned} x &= (a + b\ \cos\phi)\cos\theta \\ y &= (a + b\ \cos\phi)\sin\theta \\ z &= b\ \sin\phi \end{aligned}$$

By periodicity, $T = \Phi(D)$ where $D = I_1 \times I_2$ for any two intervals I_1, I_2 of length 2π.

For this problem and the next you'll need to use the definition of singular cubes in U(as maps of the standard cube to U), singular cubical chains, their boundaries. See the posted lecture notes for week 12, starting page 46.

(a) Define a singular square $\sigma : [0,1]^2 \to \mathbb{R}^3$ with image in T (briefly, a singular square in T) by

$$\sigma(s,t) = \Phi(2\pi s, 2\pi t)$$

Check that $\partial\sigma = 0$.

(b) For any 2-form $\alpha \in A^2(\mathbb{R}^3)$ define $\int_T \alpha$ to be $\int_D \Phi^*\alpha$ for any $D = I_1 \times I_2$ as above, for example. $D = [0, 2\pi] \times [0, 2\pi]$. Prove that

$$\int_\sigma \alpha = \int_T \alpha$$

(c) Show, without any calculation, that $\int_T dy \wedge dz = 0$.

(d) More generally, apply Theorem 10.39 in Rudin to show if $\alpha \in A^2(\mathbb{R}^3)$ is any *closed* form, that is, $d\alpha = 0$, then $\int_T \alpha = 0$.

(2) (See Rudin, Chapter 10,Exercise 22) Let $\zeta \in A^2(\mathbb{R}^3 \setminus 0)$ be defined by

$$\zeta = \frac{1}{r^3}\Big(x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy\Big)$$

where $r = r(x,y,z) = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ is distance from the origin. Let $D = [0,\pi] \times [0, 2\pi]$ and $\Sigma : D \to \mathbb{R}^3$ the parametrized surface

$$\Sigma(u,v) = (x,y,z) = (\sin u \cos v, \sin u \sin v, \cos u)$$

(a) Prove that $d\zeta = 0$, that is, ζ is *closed*.

(b) Show that $\Sigma(D) = S^2 = \{x^2+y^2+z^2=1\}$ the unit sphere in $\mathbb{R}^3$ centered at 0. Show that Σ maps $(0,\pi)\times[0,2\pi)$ bijectively onto $S^2\setminus\{N,S\}$. where N,S are the north and south poles $(0,0,\pm 1)$ and collapses $0\times[0,2\pi]$ to N and $\pi\times[0.2\pi]$ to S. Sketch.

(c) Check that $\Sigma^*\zeta = \sin u du\wedge dv$, the area form of S^2 in spherical coordinates, so $\int_\Sigma \zeta = 4\pi \neq 0$.

(d) Prove that ζ is not exact. Follow the steps of Problem 1(a),(b). First let $\sigma : [0,1]\times[0,1] \to S^2 \subset \mathbb{R}^3\setminus\{0\}$ be the singular square defined by the map $\sigma(s,t) = \Sigma(\pi s, 2\pi t)$. Show that $\partial\sigma = \phi_N - \phi_S$, where ϕ_N, ϕ_S are the maps $[0,1]\to S^2 \subset \mathbb{R}^3\setminus\{0\}$ that send $[0,1]$ to N, S.

(e) Then show that for any $\alpha \in A^2(\mathbb{R}^3\setminus\{0\})$, $\int_\sigma \alpha = \int_\Sigma \alpha$. Then $\forall \eta \in A^1(\mathbb{R}^3\setminus\{0\})$,

$$\int_\Sigma d\eta = \int_\sigma d\eta = \int_{\partial\sigma} \eta = \int_0^1 \phi_N^*\eta - \int_0^1 \phi_S^*\eta = 0$$

since $\phi_N^*\eta = 0 = \phi_S^*\eta$ because ϕ_N, ϕ_S are constant maps. Thus ζ is not exact.

(f) But ζ is exact in $\mathbb{R}^3\setminus Z$, where $Z =$ the z-axis. In fact, let

$$\lambda = (-\frac{z}{r})\Big(\frac{x\,dy - y\,dx}{x^2+y^2}\Big) \in A^1(\mathbb{R}^3\setminus Z)$$

Show, by direct calculation, that $d\lambda = \zeta$.
Motivation: On $(0,\pi)\times[0,2\pi]$, $\Sigma^*\zeta = \sin u du\wedge dv = d(-\cos u dv)$ and

$$\cos u = \frac{z}{r}, \quad dv = \frac{x\,dy - y\,dx}{x^2+y^2}.$$

(g) Show that, if T is the torus of Problem 1, $\int_T \zeta = 0$

(3) Two problems relevant to Lebesgue measure and integration:

(a) Let $E = \mathbb{Q}\cap[0,1]$ and let $I_1,\dots,I_k$ ve a *finite* collection of open intervals in $\mathbb{R}$ covering E, that is, $E \subset \cup_{i=1}^k I_i$. Prove that $\sum_{i=1}^k \ell(I_i) \geq 1$, where $\ell(I_i)$ is the length of I_i.

(b) (Rudin, Chapter 11, problem 5) Let

$$\begin{aligned} g(x) &= \begin{cases} 0 & 0\le x\le \frac{1}{2}, \\ 1 & \frac{1}{2} < x \le 1, \end{cases} \\ f_{2i}(x) &= g(x) \quad 0\le x\le 1 \\ f_{2i-1}(x) &= g(1-x) \quad 0\le x\le 1 \end{aligned}$$

Show that

$$\liminf_{n\to\infty} f_n(x) = 0 \text{ for } 0\le x\le 1 \text{ but } \int_0^1 f_n(x)dx = \frac{1}{2}$$