Foundations of Analysis II
Week 13

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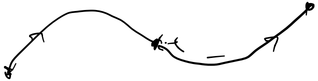
- $\sum_{\sigma\in Q_k(U)} a_\sigma \sigma = \sum_{\sigma\in Q_k(U)} b_\sigma \sigma$
  $\iff$
  $a_\sigma = b_\sigma$ for all $\sigma \in Q_k(U)$
- $\sum a_\sigma \sigma + \sum b_\sigma \sigma = \sum (a_\sigma + b_\sigma)\sigma$
- $a(\sum a_\sigma \sigma) = \sum (a a_\sigma)\sigma$.
- The elements of $S_k(U)$ are called *singular cubical chains* in U.

singular 1- cube:



I^2

$\varepsilon = 0$ or 1

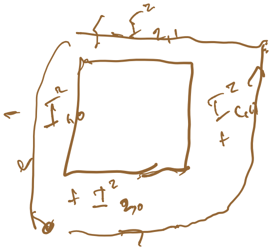
$I^2_{1,0}, I^2_{1,1}, I^2_{2,0}, I^2_{2,1}$

0 or 1

- $id : I^k \to I^k$ is an element of $Q_k(I^k)$.
- Faces of I^k come in pairs: as subsets $I^k_{i,\epsilon} = \{t_1, \dots t_{i-1}, \epsilon, t_{i+1}, \dots t_k) \in I^k\}$ for $\epsilon =$ ~~± 1~~
- As singular $k-1$– cubes in I^k the maps $\phi^k_{i,\epsilon}$

$$\phi^k_{i,\epsilon}(t_1, \dots, t_{k-1}) = (t_1, \dots, t_{i-1}, \epsilon, t_i, \dots, t_{k-1})$$

- Define $\partial(I^k)$, the boundary of I^k, to be



$$\boxed{\partial I^k = \sum (-1)^{i+\epsilon} \phi^k_{i,\epsilon} \in S_{k-1}(I^k)}$$

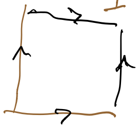
scratch wor

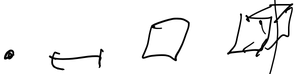
$= -I^2_{1,0} + I^2_{1,1} + I^2_{2,0} - I^2_{2,1}$

$= -\varphi^2 + \varphi^2 + \varphi^2$

$\phi^2_{2,1}$

$I^1 = \{t : 0 \le t \le 1\}$





- If $\sigma : I^k \to U$ is a singular k cube in U, define its boundary to be

$$\partial\sigma = \sum_{i,\epsilon} (-1)^{i+\epsilon}\sigma \circ \phi^k_{i,\epsilon}$$

- If $c = \sum_\sigma a_\sigma \sigma$ is a singular k-chain in U, define its boundary to be

$$\partial c = \partial(\sum a_\sigma \sigma) = \sum a_\sigma \partial\sigma$$

- Check $\partial^2 = 0$

given α

find β

$d\beta = \alpha$

Standard $I^k \longmapsto \mathrm{id}_{I^k} \in \boxed{Q_k(I^k)}$

$\in \boxed{S_k(I^k)}$

$$\partial(I^k) = \partial(\mathrm{id}_{I^k})$$

$$= \sum (-1)^{i+\varepsilon} \varphi^k_{i,\varepsilon} \in S_{k-1}(I^k)$$

$I^k = \{(t_1,\dots,t_k) \in \mathbb{R}^k \mid 0 \le t_i \le 1\}$ $(t_1,\dots,t_{k-1})$

$\{t_i\}_{k-1}$ $I^{k-1} = \{(t_1,\dots,t_{k-1}) \in \mathbb{R}^{k-1} : 0 \le t_i \le 1\}$

$\partial(k)$ maps $: I^{k-1} \to I^k$ $I^k_{i,\varepsilon}$ $(t_1,\dots,0,t_{k-1})$

$(t_1,\dots,1,\dots)$

each map parametrize one the face

$t_1 \dots t_{i-1}\, \underset{0}{\boxed{1}}\, t_i \dots t_{k-1}$

Define $\partial(I^k)$, shorthand for $\partial(\mathrm{id}_{I^k})$

$\boxed{\in S_{k-1}(I^k)}$

I^k

Suppose $U \subset \mathbb{R}^n$ open

$I^k \xrightarrow{\sigma} U$ smooth

called a singular cube in U

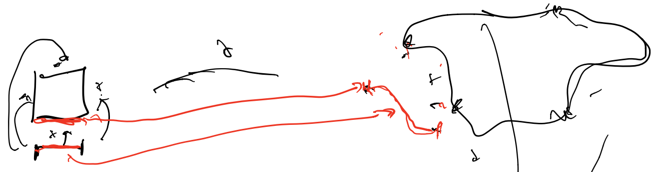
Want to define $\partial\sigma$

I^k

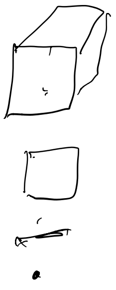
$\varphi^{k}_{i,\epsilon} : I^{k-1} \to I^k$

$$\begin{tikzcd} I^{k-1} \arrow[r, "\varphi^k_{i,\epsilon}"] \arrow[rr, bend right, "\text{sing }(k-1)\text{-cube in } U"'] & I^k \arrow[r, "\sigma"] & U \end{tikzcd}$$

$\sigma \circ \varphi^k_{i,\epsilon}$ "faces" of σ



$\partial\sigma = \sum (-1)^{i+\epsilon} \sigma\circ\varphi^k_{i,\epsilon}$



$\partial\left(\sum a_\sigma \sigma\right)$

$= \sum a_\sigma \partial\sigma$

$d(\alpha\wedge\beta)$

$= d\alpha\wedge\beta + (-1)^{\deg(\alpha)} \alpha\wedge d\beta$

$d(dx\wedge dy) = 0$

$\underset{0}{d^2x}\wedge dy + (-1)^1 dx\wedge \underset{0}{d^2y}$

$(dx\wedge dy)$

$df\wedge dg$

$f^* d = d f^*$

$f^*(\alpha\wedge\beta) = f^*\alpha\wedge f^*\beta$

$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge d\beta$

$c = \sum a_\sigma \sigma$ chain in U

$\partial c = \sum a_\sigma \partial \sigma$

$\sigma : I^k \to U$ sing k-cub

$\omega \in A^k(U)$

$$\boxed{\int_\sigma \omega = \int_{I^k} \sigma^* \omega}$$

$$\int_c \omega = \sum a_\sigma \int_\sigma \omega$$

- If $\alpha \in A^k(U)$ is a k-form and $c = \sum a_\sigma \sigma \in S_k(U)$ is a k-chain, define

$$\int_c \alpha = \sum_\sigma a_\sigma \int_\sigma \alpha = \sum_\sigma a_\sigma \int_{I^k} \sigma^* \alpha$$

- Theorem (Stokes's Theorem)

For all $\alpha \in A^{k-1}(U)$ *and for all* $c \in S_k(U)$

$$\boxed{\int_c d\alpha = \int_{\partial c} \alpha}$$

$c = \sum a_\sigma \sigma$

$\sigma : I^k \to U$

d c ∂ dual

for each σ

$\sum_\sigma a_\sigma \int_{I^k} \sigma^* d\alpha$

~~$\sum$ [illegible]~~ σ

$\cdot \, \partial$

Pf enough to prove for ~~I^k~~

$\alpha \in A^{k-1}(I^k)$

$$\int_\sigma d\alpha = \int_{I^k} \sigma^*(d\alpha) = \int_{I^k} d(\sigma^*\alpha)$$

$$\int_{\partial\sigma} \alpha = \int_{\partial I^k} (\sigma^*\alpha)$$

$$\forall\, \boxed{\eta} \in A^{k-1}(I^k) \quad \int_{\partial I^k} \eta = \int_{I^k} d\eta$$

Need

$$\eta = \sigma^{*}\alpha \in \underline{A}^{k-1}(\underline{I}^{2})$$

$$\oint \underbrace{\int_{\partial \underline{I}^{k}} \alpha}_{\shortparallel}$$

$$\sum (-1)^{i+\varepsilon} \int_{\underline{I}^{k-1}} (\varphi_{i,\varepsilon}^{k})^{*}\alpha$$

$$\eta \in A^{k-1}(\underline{I}^{k})$$

$$\underline{\sum f_i(t_1,\dots,t_k)}\ \underbrace{dt_1 \wedge \dots \wedge \widehat{dt_i} \wedge \dots dt_k}_{dt_2 \wedge \dots dt_k \ \& \ dt_1 \dots dt_{k-1}}$$

$k=2$ $\quad f_1(t_1,t_2)\, dt_2 + f_2(t_1,t_2)\, dt_1$

$k=3$ $\quad f_1(t_1,t_2,t_3)\, dt_2 \wedge dt_3 + f_2(t_1,t_2,t_3)\, dt_1 \wedge dt_3 + f_3(t_1,t_2,t_3)\, dt_1 \wedge dt_2$

↳ shorthand for $dt_1 \wedge \dots \wedge \underbrace{dt_{i-1} \wedge dt_{i+1}}_{\widehat{dt_i}} \wedge \dots dt_k.$

$$d\eta = \sum_i \underline{df_i \wedge dt_1 \dots \widehat{dt_i} \dots dt_k}$$

$$\underline{df_1} \wedge dt_2 \wedge \dots dt_k$$

$$+ \overline{df_2 \wedge} dt_1 \wedge dt_2 \wedge \dots$$

$$d(f_1\, dt_2 \wedge \dots \wedge dt_k) \overset{\text{def}}{=} (df_1) \wedge dt_2 \wedge \dots dt_k.$$

$$d\eta = \sum_{i=1}^{k} df_i \wedge (dt_1 \wedge \dots \wedge \widehat{dt_i} \dots dt_k)$$

R. carefully check $k=1$ $\quad \eta = f(t)$

$k-1=0$

$\partial I = 1, 0$

$$\boxed{\boxed{\int_0^1 df = f(1) - f(0)} \atop \int_I \qquad \int_{\partial I}}$$

$$d\eta = \sum k \text{ terms}$$

$$\underline{\eta} = \sum k \text{ terms.}$$

$\underline{I^{k}_{i,\varepsilon}}$ $\quad \underline{dt_1 \dots \widehat{t_j} \dots dt_k}$ $\quad \varphi_{i,\varepsilon}^{*}\eta$

$$\eta\Big|_{I_{i,\varepsilon}} = \text{one term}$$

$$f_i\, dt_1 \dots \underline{\widehat{dt_j}} \dots dt_k$$

$j \neq i.$

$$\int_{I^k_{i,\varepsilon}} f_j \, dt_1 \dots \widehat{dt_j} \dots dt_k = 0 \text{ unless } i = j$$

$$\int_{\partial I^k} \eta = \sum (-1)^{i+\varepsilon} \int_{I^k_{i,\varepsilon}} f_i \, dt_1 \dots \widehat{dt_i} \dots dt_k$$

$$-(-1)^i \int f_i(t_1, \dots, 0, \dots, t_k) - \int f_i(\dots, 1, \dots) \, dt_k$$

$I(t_1, \dots \hat{t_i} \dots t_k)$

$$f_i(t_1, \dots, 1, \dots, t_k) - f_i(\dots, 0, \dots)$$

$$= \int_0^1 \frac{\partial f_i}{\partial t_i} \, dt_i$$

$(i, 1)$ $(i, 0)$

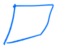
$$\int_{\varphi_{i,1}} \eta - \int_{\varphi_{i,0}} \eta = \int_{I^k} dt_1 \dots \frac{\partial f}{\partial t_i} dt_i \dots dt_k$$

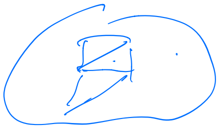
$$\int_{I^k} dt_1 \dots \frac{\partial f_i}{\partial t_i} dt_i \dots dt_k \quad - \quad df_i$$

Check

sign cai

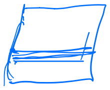
$$I^k d\eta = \pm \sum \int_{I^k_{i,\varepsilon}} \eta$$





$k = 1$ $$\int_0^1 df = f(1) - f(0)$$

$k = 2$ $f\,dx + g\,dy$

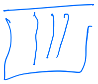


$$d(f\,dx + g\,dy) =$$

$$\int_{x=1} \qquad \int_{x=0} g\,dy$$

$$\int \left[g(1,y)\,dy - g(0,y)\,dy \right] = \iint \frac{\partial g}{\partial x}(x,y)\,dx\,dy$$

$$\int \qquad \iint dg \wedge dy$$



$f(x)\,dx$

$$\int (f(x,1) - f(x,0))\,dx$$

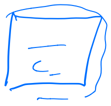
$$= \int \frac{\partial f}{\partial y}\,dy \wedge dx$$

$$= -\int \frac{\partial f}{\partial y}\,dx \wedge dy$$

$$\iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

$$d(f\,dx + g\,dy) = \frac{\partial f}{\partial y}\,dy \wedge dx + \frac{\partial g}{\partial x}\,dx \wedge dy$$

$$= \left(\partial g/\partial x - \partial f/\partial y \right) dx \wedge dy$$



$$\int_{\partial R} f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy$$

Problem: given $\alpha \in A^{k+1}(U)$

find $\beta \in A^k(U)$

s.t. $\boxed{d\beta = \alpha}$

First observation

suppose $\exists\, \beta$ s.t. $d\beta = \alpha$

$$d(d\beta) = d\alpha$$

$$\|$$

$$d^2\beta = 0$$

Necessary condition: $d\alpha = 0$

Terminology: α is <u>closed</u>

$\Longleftrightarrow d\alpha = 0$

α is <u>exact</u>

$\Longleftrightarrow \alpha = d\beta$ for some β.

exact $\Rightarrow$ closed

$\overset{??}{\Longleftarrow}$

Example in $A^1(\mathbb{R}^2 - \{0\})$

"$d\theta$"

$$\eta = \frac{x\,dy = y\,dx}{x^2+y^2} \quad \text{in } \mathbb{R}^2 - \{0\}$$

$d\eta = 0$

$$d\eta = d\left((x^2+y^2)^{-1}(x\,dy - y\,dx)\right)$$

$$= \left[\underline{-(x^2+y)^{-2}}(2x\,dx + 2y\,dy)\right]\wedge(x\,dy - y\,dx)$$

$$+ (x^2+y^2)^{-1}\wedge d(\underline{x\,dy - y\,dx})$$

$dx\wedge dy - dy\wedge dx = 2\,dx\wedge dy$

$=$

$$\underline{-(x^2+y^2)^{-2}}\underbrace{(2x\,dx + 2y\,dy)\wedge(x\,dy - y\,dx)}_{2x^2\,dx\wedge dy - 2y^2\,dy\wedge dx} + \underline{(x^2+y^2)^{-1}\,2\,dx\wedge dy}$$

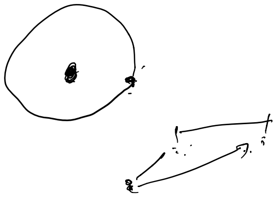
$$= \frac{-2(x^2+y^2)\,dx\wedge dy + (x^2+y^2)(2\,dx\wedge dy)}{(x^2+y^2)^2} = 0$$

η is a closed form, <u>not</u> exact

$[0,1) \to \cos(2\pi s$

$[0,1] \xrightarrow{\sigma} \mathbb{R}^2 - \{0\}$

$s \to (\cos 2\pi s, \sin 2\pi s)$



σ sing chain

$\partial\sigma = 0$

if $\eta = df$

$$\int_\sigma \eta = \int_\sigma df = \int_{\partial\sigma} f = f(1) - f(0) = 0$$

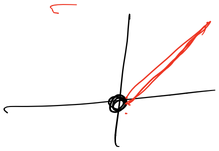
$$\int_\sigma \eta = \int_0^1 (\cos 2\pi s)\, 2\pi \cos(2\pi s) - \sin(2\pi s)(-2\pi \sin 2\pi s)$$

$$= 2\pi \int_0^1 \underbrace{(\cos^2(2\pi s) + \sin^2 2\pi s)}_{1}\, ds$$

$$= 2\pi$$

η closed, not exact in $\mathbb{R}^2 - \{0\}$

η is exact in $\mathbb{R}^2$ - ray thru 0



closed "local property"

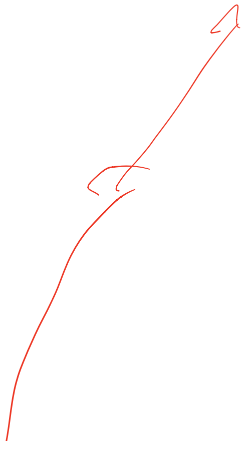
exact "global property"

$\mathbb{R}^n - \{0\}$

$$\frac{x_1\, dx_2 \cdots dx_n - x_2\, dx_1\, d\hat{x}_2 \cdots dx_n \cdots}{r^n}$$

$r = (\Sigma x_i^2)^{1/2}$

closed in $\mathbb{R}^n - \{0\}$, not exact.

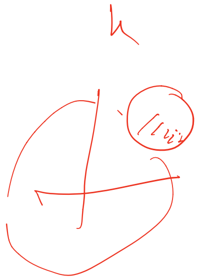


$\underline{d\alpha = 0}$

need to know α ~~at~~ at a pt.
can

$\alpha = \sum a_I \, dx_I$

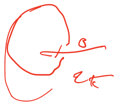
$\underline{(d\alpha)_{(x)}}$ depends on $\left.\begin{array}{l} a_I(p) \\ \frac{\partial a_I}{\partial x_i}(p) \end{array}\right\}$



$\eta = \underline{\underline{d\theta}}$

$\theta = \measuredangle$

θ ~~Not~~ not a fun on $\mathbb{R}^2 - \{0\}$



$\frac{y\,dx - x\,dy}{x^2+y^2}$ in $\mathbb{R}^2 - \{0\}$

$\mathbb{R}^n - \{0\}$

$\mathbb{R}^3 - \{0\}$

$\varphi = \boxed{\frac{x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}}$

$\in A^2(\mathbb{R}^3 - \{0\})$

$d\varphi = 0$

but $\int_{S^2} \varphi = 4\pi$

$\Rightarrow$ not exact.

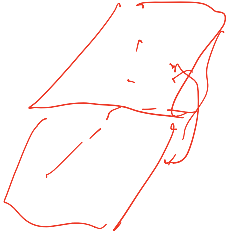
Proof of Stokes's Theorem

- Recall notation for faces of I^k.
- Faces of I^k come in pairs: as subsets $I^k_{i,\epsilon} = \{t_1, \ldots . t_{i-1}, \epsilon, t_{i+1}, \ldots . t_k) \in I^k\}$ for $\epsilon = 0$ or 1
- As singular $k-1$– cubes in I^k the maps $\phi^k_{i,\epsilon}$

$$\phi^k_{i,\epsilon}(t_1, \ldots, t_{k-1}) = (t_1, \ldots, t_{i-1}.\epsilon, t_i, \ldots, t_{k-1})$$

- Define $\partial(I^k)$, the boundary of I^k (Strictly speaking, boundary of $id : I^k \to I^k \in Q_k(I^k)$) to be

$$\partial I^k = \sum (-1)^{i+\epsilon} \phi^k_{i,\epsilon} \in S_{k-1}(I^k)$$



$0 \leq i \leq k \qquad I^k$

$I^k_{i,0}$

$x_1, \ldots, 0, \ldots, x_k$

$x_1, \ldots, 1, \ldots, x_k$

$$\partial I^k = \sum (-1)^{i+\varepsilon} I^k_{i,\varepsilon}$$

- Suffices to prove that for all $\eta \in A^{k-1}(I^k)$

$$\int_{\partial I^k} \eta = \int_{I^k} d\eta$$

- Every $\eta \in A^{k-1}(I^k)$ can be written as

$$\eta = \sum_{i=1}^{k} f(t_1, \dots, t_k) dt_1 \wedge \dots, \widehat{d\overset{t}{\cancel{x}}_i} \dots d\overset{t}{\cancel{x}}_k$$

$dt_1 \wedge -- \widehat{dt_i} \wedge -- dt_k$

for $f_i \in A^0(I^k)$, $i = 1, \dots, k$.

- Then

$$d\eta = \sum_{i=1}^{k} (-1)^{i-1} \frac{\partial f_i}{\partial t_i} dt_1 \wedge \cdots \wedge dt_k$$

- Thus
$$\int_{I^k} d\eta = \sum_{i=1}^{k} (-1)^{i-1} \int_{I^k} \frac{\partial f_i}{\partial t_i} dt_1 \ldots\ dt_k$$
- Now
$$\int_{I^k} \frac{\partial f_i}{\partial t_i} dt_1 \ldots\ dt_k = \int_{Q_i} \left(\int_0^1 \frac{\partial f_i}{\partial t_i} dt_i \right) dt_1 \ldots \widehat{dt_i} \ldots dt_k$$
with Q_i the $(k-1)$-cube $\{(t_1, \ldots \widehat{t_i}, \ldots t_k) : 0 \le t_j \le 1\}$
- $$\int_0^1 \frac{\partial f_i}{\partial t_i} dt_i = f(t_1, \ldots, 1, \ldots t_k) - f(t_1, \ldots, 0, \ldots, t_k)$$

- Combine the last two equations
- We get $\int_{I^k} \frac{\partial f_i}{\partial t_i}\, dt_1 \dots dt_k$ is

$$\int_{I^k_{i,1}} f(t_1,\dots,1,\dots t_k)\, dt(\widehat{i}) - \int_{I^k_{i,0}} f(t_1,\dots,0,\dots,t_k)\, dt(\widehat{i})$$

  where $dt(\widehat{i}) \;=\; dt_1 \dots \widehat{dt_i} \dots dt_k$

- In terms of the maps $\phi^k_{i,\epsilon} : I^{k-1} \to I^k$ with image $I^k_{i,\epsilon}$

$$\int_{I^k}\int_{I^k} \frac{\partial f_i}{\partial t_i}\, dt_1 \dots dt_k = \int_{I^{k-1}} (\phi^k_{i,1})^*\eta - \int_{I^{k-1}} (\phi^k_{i,0})^*\eta$$

- Combi~~n~~ning with the previous formula

$$\int_{I^k} d\eta = \sum_{i=1}^{k} (-1)^{i-1} \int_{I^k} \frac{\partial f_i}{\partial t_i} dt_1 \dots\ dt_k$$

get

$$\int_{I^k} d\eta = \sum (-1)^{i-1} \Big(\int_{I^{k-1}} (\phi^k_{i,1})^* \eta - \int_{I^{k-1}} (\phi^k_{i,0})^* \eta \Big)$$

- Recalling the definition and comparing

$$\partial I^k = \sum (-1)^{i+\epsilon} \phi^k_{i,\epsilon} = \sum (-1)^{i+1} \phi^k_{i,1} + \sum (-1)^i \phi^k_{i,0}$$

- Done!

$t_1 \geq 0$

[illegible]

$t_1 + t_2 \leq 1$

t_1, t_2 [illegible] $t_1 + t_2 + t_3 \leq 1$

$0 \leq t_1, \ldots, t_n$

$t_1 + \cdots + t_n \leq 1$

$t_1 + t_2 + t_3 \leq 1$

6

$n!$

Cubes vs Simplices

- There are two standard ways to build chains on $U \subset \mathbb{R}^n$, or in more general spaces X
  - Cubical chains, as we did
  - Simplicial chains as in Rudin's book.
- One way to define simplicial chains (not exactly as in Rudin) :
  - Define the *standard k-simplex* Δ^k to be

$$\{(t_1, \ldots, t_k) \in \mathbb{R}^k \;:\; t_i \geq 0 \text{ and } t_1 + \cdots + t_k \leq 1\}$$

  - Follow the same procedure replacing I^k by Δ^k

- Singular k-simplices in U are smooth maps $\sigma : \Delta^k \to U$,
- Simplicial k-chains are $\sum a_\sigma \sigma$ in the $\mathbb{R}$-vector space $S_k(U)$ with basis the collection of all σ.
- To define the boundary of Δ^k, observe that Δ^k is the convex hull of the $k+1$ points $0, e_1, e_2, \ldots e_k$ in $\mathbb{R}^k$.
- Let $e_0 = 0$ and let $< e_0. \ldots, \widehat{e_i}, \ldots, e_k >$ denote the convex hull of the k points $e_0, \ldots \widehat{e_i}, \ldots, e_k$.
- $< e_o, \ldots, \widehat{e_i}, \ldots, e_k >$ is the face of Δ^k opposite the vertex e_i.
- Choose a parametrization $\phi_i^k : \Delta^{k-1} \to < e_0, \ldots, \widehat{e_i}, \ldots, e_k >$.
- Define $\partial \Delta^k = \sum_{i=0}^k (-1)^i \phi_i$

- Loosely, but suggestively,

$$\partial\Delta^k = \sum_{i=0}^{k}(-1)^i < e_0, \dots, \widehat{e_i}, \dots, e_k >$$

- Chose cubes rather than simplices for only one reason
- The proof

$$\int_{\partial I^k} \eta = \int_{I^k} d\eta$$

is much simpler, by the way the faces come in pairs, than the proof of

$$\int_{\partial\Delta^k} \eta = \int_{\Delta^k} d\eta$$

- See Rudin for the proof of the second equation.
- For most purposes simplices are much nicer than cubes.
- For example, triangulations of spaces.

$$\int_{\partial \sigma} \eta = \int_{\sigma} d\eta$$

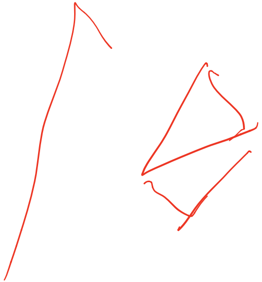
C = chn

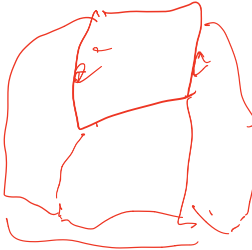
$$\boxed{\int_{\partial c} \eta = \int_{c} d\eta}$$

ch

→ coch

→ homol.





$$\int f\,dx + g\,dy = \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy$$

Suppose $d\alpha = 0$

when can you find β

s.t. $d\beta = \alpha$?

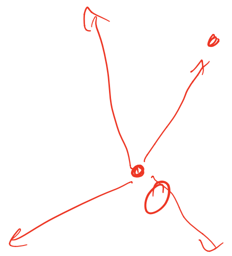
always true in $\mathbb{R}^n$

$A^{k-1}(\mathbb{R}^n) \to A^k(\mathbb{R}^n) \xrightarrow{d} A^{k+1}(\mathbb{R}^n)$

$\beta \xrightarrow{d} \alpha \longrightarrow 0$

Nice proof on any convex set

"starred" about a pt.



$0 \in U$

$\forall x \in U,$

$\overrightarrow{0x} \subset U$

$\alpha \to P(\alpha)$

"

$\sum a_I \, dx_I \to P(\alpha)$

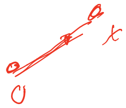
$\cancel{dx_{i_1} \cdots dx_{i_k}}$

$\int_0^1 \cancel{\omega(tx)}$

$(t, x) \to tx$

$I \times U \to U$

$t, x \to tx$



$I \times U \leftarrow a_I \, dx_I$

$F(t,x) = tx$

$F^*\alpha = \sum c_I^{dx} \wedge c_J^{dx} \, dt$

$F^*(dx_{i_1} \wedge \dots \wedge dx_{i_k})$

$d(tx_{i_1}) \wedge \dots$

$\underline{x_{i_1}\, dt \wedge \dots}$

$\alpha \in A^k(U) \to F^*\alpha \in A^k(U \times I)$

$\sum a_i(t,x)\, dx_I\,,\; dt \wedge dx_J$

$|J| = |I| - 1$

$(dx \wedge dy)$

$d(tx) \wedge d(ty)$

$= (x\,dt + t\,dx) \wedge (y\,dt + t\,dy)$

$= \boxed{tx\,dt \wedge dy + t\,dx \wedge dt} + t^2\,\underline{dx \wedge dy}$

$\int_0^1 (F^*\alpha)^{dt}\, dt$ = k-1 form

$P\alpha =$

$$\boxed{\boxed{dP\alpha + Pd\alpha = \alpha} \qquad d\alpha = 0,\; \alpha = dP\alpha}$$

"Poincaré Lemma"

Cartan

"Topology" of U

$d^2 = 0$

$$\left(\frac{\text{Closed forms}}{\text{exact forms}}\right)(U) = \text{de Rham cohomology of } U.$$

grad, curl,div

~~d~~

$$A^{k-1}(U) \xrightarrow{d} A^k(U) \xrightarrow{d} A^{k+1}(U)$$

$\{\alpha \longrightarrow d\alpha = 0\}$ subspace of $A^k(U)$ C

$\{d\beta : \beta \in A^{k-1}\}$ subspace.. E

C/E book ..

Classical Theorems

$$\boxed{\cancel{f(x,y)\,dx\wedge dy}} + g(x,z)\,dx\wedge dz \cdots$$

$$f(x$$

$$(x,y,t) \to (tx, ty)$$

$$f(tx, ty)\, d(tx)\wedge d(ty)$$

$$= f(tx, ty)\big((dt)x + t\,dx\big)\wedge\big((dt)y + t\,dy\big)$$

Spivak's

$$f(tx, ty)\,(x\,t\,dt \wedge dy + t y\,dx \wedge dt + t^2\,dx \wedge dy)$$

$$t\int_0^1 f(tx, ty)\,(t\,dt \wedge dy + t y\,dx \wedge dt)$$

$$\int_0^1 f(tx, ty)\,t\,dt\,(x\,dy$$

$$\int_0^1 f(tx, ty)\,t\,dt\,(x\,dy - y\,dx)$$

$$\int_0^1 f(tx, ty)\,t\,dt\;dx$$

$$\int_0^1 f(tx, ty)\,t\,dt\;dy$$

Closed Forms, Exact Forms

$d\theta$

grad, div, curl $\qquad \mathbb{R}^3$

$\mathbb{R}^3$ Forms | Vector fields

$$\omega = \alpha\, dx + \beta\, dy + \gamma\, dz \longleftrightarrow \alpha \vec{i} + \beta \vec{j} + \gamma \vec{k}$$

$$\omega = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \longleftrightarrow \nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

$$\underline{\underline{d\omega}} = d\alpha \wedge dx + d\beta \wedge dy + d\gamma \wedge dz$$

$$= \frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy +$$

$$\frac{\partial \alpha}{\partial y} dx \wedge dy - \frac{\partial \beta}{\partial x} dx \qquad \underline{\left(\frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial x}\right) dx + (\ldots) \ldots}$$

$$\Uparrow$$

curl F

$$\alpha \longleftrightarrow F$$

$$d\alpha \longleftrightarrow \text{curl } F$$

$$(\ )\, dx \wedge dy + (\ )\, dx \wedge dz + (\ )\, dy \wedge dz$$

$\vec{i}$ curl F ...

Stokes $\Longrightarrow$ all the classical thm

~~chm~~

$$\underline{\int_S (\nabla \times F)\cdot \hat{n} = \int_{\partial S} \vec{F} \cdot \hat{t}}$$

$\vec{F}$ $$\underline{\iiint \operatorname{div} F \; dV} = \iint_{\partial C} \underline{\vec{F}\cdot\hat{n}} \; dA$$

divergence thm

$F = \alpha \hat{i} + \beta \hat{j} + \gamma \hat{k}$

$$\int_{\partial D} \omega = \int_{D} d\omega$$

$$\underline{\operatorname{div} F} = \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}$$

$$\underline{\vec{F}\cdot\hat{n}} \hookrightarrow \underline{(\alpha \, dy\wedge dz - \beta \, dx\wedge dz + \gamma \, dx\wedge dy)} \quad \omega$$

$$d \left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right) dx\wedge dy\wedge dz \quad d\omega$$

$$\boxed{\int_{D} d\omega} = \boxed{\int_{\partial} \omega} \Rightarrow \text{div thm}$$

$$\int_{\partial I_c} \omega \; \longrightarrow \; d\omega$$

vol elem.

Next Lebesgue Integration

Ref: Rudin chap 11

Royden Real Analysis

Goals:

1) define a collection of subsets of $\mathbb{R}$ (and later of $\mathbb{R}^n$) called **measurable** sets

$\mathfrak{M}$ should be a "σ-algebra" (a "σ-ring" Rudin)

Closed under complements, countable unions

($\Rightarrow$ closed under countable intersections)

2) a **measure** $\mu: \mathfrak{M} \to \mathbb{R}^+$ $\quad \mathbb{R}^+ = [0, \infty]$

↑ allow inf

$A \in \mathfrak{M} \Rightarrow \mu(A) \geq 0$

$\{A_i\} \in \mathfrak{M}$ countable, $A_i \cap A_j = \emptyset \ \forall\, i \neq j$

$$\Rightarrow \mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$$

(countable additivity)

also want translation invariance:

$$\mu(A+x) = \mu(A) \quad \forall x \in \mathbb{R}.$$

3) define measurable funcs

and integrals $\int f\,d\mu$

Start with outer measure:

$A \subset \mathbb{R}$

$m^*(A) = \inf\{\sum \ell(I_i) : \{I_i\}$ countable collection of open intervals with $A \subset \bigcup_i I_i\}$

$\ell(I) = \text{length}(I)$.