Math 2210 - 4, Review for Final Exam

The final exam will be comprehensive. You need to review the the topics already covered in the three midterms, as well as the new material. There will be more emphasis on the new material. The final will be roughly half on the new material, half on the old. The last page is a list of formulas you will have available during the final.

Topics

1. Dot product and cross product, equations of planes and lines (11.3, 11.4 and 11.6).
2. Length of curves (11.1). Velocity, speed and acceleration for motion on a curve (11.5).
3. Functions of two or more variables (12.1), partial derivatives (12.2), differentiability (12.4), gradient, directional derivatives, direction of greatest increase (12.5).
4. Level curves for functions of two variables, level surfaces for functions of three variables (12.1). Gradient ∇f is perpendicular to the level curves or level surfaces of f (12.6). ∇f is a normal vector to the tangent plane of the surface $f = 0$. This gives the equation of the tangent plane to a surface (12.7) $f = 0$.
5. Chain rule (12.6).
6. Stationary points (where $\nabla f = 0$), classification into local maxima and minima and saddle points (12.8).
7. Global (or absolute) maxima and minima on a closed bounded set, checking interior and boundary (12.8, 12.9).
8. Double and triple integrals: over boxes (constant limits of integration: 13.2, 13.7).
9. Double and and triple integrals over more complicated regions (limits of integration no longer constant): describing regions by inequalities, using the inequalities to find limits of integration; how to change order of integration (13.3, 13.7).
10. Changing integrals from rectangular to polar, cylindrical or spherical coordinates (13.4, 13.8).
11. Area of a surface (13.6).
12. Vector fields in two and three dimensions (14.1 to 14.4):
    (a) Sketching vector fields, physical interpretation (14.1).
    (b) Divergence and curl of a vector field, $curl(\nabla f) = 0$ (14.1)
    (c) Line integrals: Computation and interpretation of $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ (14.2).
    (d) Conservative fields, potential functions (14.1, 14.3).
    (e) Path independence (14.3):

i. If $\mathbf{F}$ is a vector field on a region D, to say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent means: for any curve C in D, $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends just on the endpoints of C.

ii. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent in D if and only if $\mathbf{F}$ is a gradient. This means that there is a differentiable function f on D so that $\nabla f = \mathbf{F}$. The function f is called a *potential function*, and the field $\mathbf{F}$ is said to be *conservative*.

iii. If $\mathbf{F} = \nabla f$, and C is any curve in D starting at a point P in D and ending at Q in D, then

$$\int_C \mathbf{F} \cdot \, d\mathbf{r} \quad = \quad f(Q) - f(P). \tag{1}$$

iv. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative, to find a potential function f:

A. First check that $curl(\mathbf{F}) = 0$. Otherwise there is no solution (because $curl(\nabla f) = 0$). In two dimensions this simply means to check that $M_y = N_x$.

B. Solve the equation $\nabla f = \mathbf{F}$ for f by following the pattern of 14.3, Example 2 (in two dimensions) , and of Example 4 (in three dimensions) .

(f) *Green's Theorem*

i. If C is a closed curve that is the boundary of a region S in the plane, if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is a vector field defined and differentiable on S, and you travel along C so that S is always on your *left* (see Figure 2 of 14.4) then

$$\oint_C M \; dx + N \; dy = \int\int_S (N_x - M_y) dA. \tag{2}$$

ii. This formula is also true if S has holes, and C consists of several pieces $C_1, C_2, \ldots$, provided that we always travel along C so that S is on the left, see Figure 3 of 14.4.

iii. Divergence theorem in the plane: If $\mathbf{F}, S, C$ are as above, then

$$\oint_C \mathbf{F} \cdot \mathbf{n} \; ds = \int\int_S div(\mathbf{F}) dA, \quad \text{where } \mathbf{n} = \text{ outward unit normal.} \tag{3}$$

see 14.4, Figure 6.

iv. Interpretation of (2) and (3): If C_r is a circle of radius r centered at a point p, which is the boundary of the disk D_r, and $\mathbf{F}$ is a vector field, then

A. The *circulation* of $\mathbf{F}$ around C_r is measured by $curl(\mathbf{F})$:

$$\oint_{C_r} \mathbf{F} \cdot \mathbf{T} ds = \int\int_{D_r} (curl(\mathbf{F}) \cdot \mathbf{k}) \; dA \approx (curl(\mathbf{F})(p) \cdot \mathbf{k}) \pi r^2.$$

B. The *flux* of $\mathbf{F}$ accross C_r is measured by $div(\mathbf{F})(p)$:

$$\oint_{C_r} \mathbf{F} \cdot \mathbf{n} ds = \int\int_{D_r} div(\mathbf{F}) dA \approx (div(\mathbf{F})(p)) \pi r^2.$$

(g) Make sure you understand all the examples in chapter 14, and the webwork problems on this material.

As usual, the best way to prepare for the test is to have done all the homework problems.

1. Length of curve $(x(t), y(t), z(t)), a \leq t \leq b$ is $\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt = \int_a^b ds$.
2. $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$.
3. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$.
4. $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin\theta =$ area of parallelogram, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} =$ volume of parallelipiped.
5. Plane $A(x - x_0) + B(y - y_0) + C(z - z_0) = Ax + By + Cz - D = 0 \perp A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.
6. Line $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$ through (x_0, y_0, z_0) in direction $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
7. Position $\mathbf{r}(t)$, velocity $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$, speed $||\mathbf{v}(t)|| = \frac{ds}{dt}$, acceleration $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}$
8. $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.
9. Directional derivative at $\mathbf{p}$ in direction of the unit vector $\mathbf{u}$: $D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u} \cdot \nabla f(\mathbf{p})$.
10. Chain rule: $\frac{d}{dt}(f(\mathbf{r}(t))) = \nabla f(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}$.
11. Tangent plane to $z = f(x, y)$ at (x_0, y_0, z_0): $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
12. Second derivative test at (x_0, y_0) where $\nabla f((x_0, y_0)) = 0$: If $D = (f_{xx}f_{yy} - f_{xy}^2)((x_0, y_0))$, then: $(1) : D > 0, f_{xx} > 0)$ local min;$(2) D > 0, f_{xx} < 0$ local max; $(3) : D < 0$ saddle pt.
13. Area of graph $z = f(x, y)$ over region S in xy-plane: $\int\int_S \sqrt{f_x^2 + f_y^2 + 1} dA$.
14. Polar coordinates: $x = r\cos\theta, y = r\sin\theta, dA = rdrd\theta$.
15. Spherical coordinates: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi, dV = \rho^2\sin\phi d\rho d\theta d\phi$.
16. $div(\mathbf{F}) = \nabla \cdot \mathbf{F} = M_x + N_y + P_z$.
17. $curl(\mathbf{F}) = \nabla \times \mathbf{F} = (P_y - N_z)\mathbf{i} + (M_z - P_x)\mathbf{j} + (N_x - M_y)\mathbf{k}$.
18. $curl(\nabla f) = \mathbf{0}$.
19. If C is a curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Mdx + Ndy + Pdz = \int_a^b (M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt})dt$ and $\int_C f\ ds = \int_a^b f(\mathbf{r}(t))||\frac{d\mathbf{r}}{dt}(t)||dt$.
20. If $\mathbf{F}$ is conservative means $\mathbf{F} = \nabla f$. If $\mathbf{F}$ conservative, then $curl(\mathbf{F}) = \mathbf{0}$.
21. If C is a curve from P to Q, $\int_C \nabla f \cdot d\mathbf{r} = f(Q) - f(P)$.
22. Green's Thm:$\oint_{\partial S} Mdx + Ndy = \int\int_S (N_x - M_y)dA$, travel along the boundary curve ∂S iso that S is always on the left.
23. Divergence Thm in plane: $\oint_{\partial S} \mathbf{F} \cdot \mathbf{n} ds = \int\int_S div(\mathbf{F})dA$, $\mathbf{n} =$ outward unit normal, ∂S as above.