

# Math 1180 - Spring 2006

## Practice Exam 1

February 2, 2006

1. Consider the following differential equation describing the concentration of salt (sodium ion) in a cell following the consumption of a bag of Doritos.

$$\frac{dN}{dt} = 2 - 10t \quad (1)$$

$$N(0) = 50\text{mmol/cm}^3 \quad (2)$$

- (a) What kind of an equation is this?
- (b) Find the solution to this equation. Then, sketch the graph of the concentration as a function of time.
- (c) What is the concentration of salt after  $t = 2$  seconds?
- (d) Find the change in salt concentration from  $t = 0.5$  s to  $t = 1$  s using definite integral. What is the change in the salt concentration in the same period of time if you ate two bags of Doritos so that the initial condition is  $N(0) = 100\text{mmol/cm}^3$  instead.
2. Verifying solutions of autonomous differential equations. **Do not solve these equations.**
- (a) Check that  $b(t) = 10e^{3t}$  solves  $\frac{db}{dt} = 3b$  with initial condition  $b(0) = 10$
- (b) Check that  $G(t) = 1 + e^t$  solves  $\frac{dG}{dt} = G - 1$  with initial condition  $G(0) = 2$
3. Find the equilibria to the following autonomous differential equations. Sketch the phase line diagram and indicate the stability of the equilibria.
- (a)  $\frac{dx}{dt} = 1 - x^2$   
Consider both positive and negative values of  $x$ .
- (b)  $\frac{dz}{dt} = \frac{1}{z} - 3$
- (c)  $\frac{dx}{dt} = 1 - ax$   
Take  $a$  to be a parameter. Identify all possible behavior depending on the value of  $a$ .
4. Consider the selection problem between two types of bacterial strains  $a$  and  $b$ . Suppose that type  $a$  grows at the per-capita rate of  $\mu$  and type  $b$  grows at the per-capita rate  $\lambda$ .
- (a) Write down the autonomous differential equations describing growth of  $a$  and  $b$ . That is write down what  $\frac{da}{dt}$  and  $\frac{db}{dt}$  are.

- (b) Let  $p(t)$  be the fraction of type  $a$  so  $p(t) = \frac{a}{a+b}$ . Then, the corresponding differential equation for  $p(t)$  is

$$\frac{dp}{dt} = (\mu - \lambda)p(1 - p) \quad (3)$$

Take  $\mu = 2.0$  and  $\lambda = 1.0$ . Find all equilibria points. Do a phase line-diagram analysis.

- (c) State the stability theorem for autonomous differential equation. Verify the stability of equilibria obtained from your phase-line diagram **using the stability theorem**.
- (d) Sketch the solution  $p(t)$  assuming that the initial condition is  $p(0) = 0.3$ .
- (e) Will one of the two strains take over? Which one? Can you get stable coexistence in this model? If yes, explain which stable equilibria correspond to stable coexistence of the two strains. If not, what needs to be accounted to improve the model?

5. Suppose a disease is circulating in a population. The spread of the disease can be described by the equation

$$\frac{dI}{dt} = \alpha(1 - I)I - \mu I \quad (4)$$

where  $I$  is the fraction of infected individuals in the population.

- (a) Explain each term in the equation. What is the fraction of the population that's susceptible (not infected)? What is the per-capita rate at which susceptible individuals becomes infected? What is the total rate at which infection occurs? What is the total rate of recovery?
- (b) Take  $\mu = 1.0$  and  $\alpha = 0.5$ . Find all equilibria that's biologically plausible. Draw a phase line diagram. Sketch a solution with initial condition  $I(0) = 0.1$ .
- (c) Take  $\mu = 1.0$  and  $\alpha = 2.0$ . Find all equilibria that's biologically plausible. Draw a phase line diagram. Sketch a solution with initial condition  $I(0) = 0.1$ .
- (d) In which of the two cases above is the disease going to persist (endemic situation)?

## 6. Bifurcation Diagram

Review the following problems from your homework.

- (a) Section 5.3 - 17
- (b) Section 5.3 - 18

Use the stability theorem to do your analysis.

7. Solve the following autonomous differential equation using separation of variables.

(a)  $\frac{db}{dt} = 3b$ ,  $b(0) = 10$

(b)  $\frac{dz}{dt} = z^2$ ,  $z(0) = 100$

At what time does your solution approach infinity? Sketch a graph of the solution.

(c)  $\frac{dy}{dt} = 1000$ ,  $b(0) = 1000$

Describe the behavior of your solution and explain why this makes sense.

8. Consider the following Lotka-Volterra equation describing the competition between two species.

$$\frac{da}{dt} = \left(1 - \frac{a + 2b}{3000}\right) a \quad (5)$$

$$\frac{db}{dt} = \left(1 - \frac{a + b}{2000}\right) b \quad (6)$$

- (a) In the absence of  $b$ , what is the carrying capacity for population  $a$ ?
- (b) Do a phase-plane analysis by completing the following steps
  - i. Find the nullclines and plot the nullclines on the phase-plane
  - ii. Find equilibrium points by solving the equation algebraically or if it is obvious, by looking at plots
  - iii. Draw directional arrows on the nullclines. Also indicate the flow direction in each of the region separated by the nullclines.
  - iv. Sketch trajectories starting from each region and predict where they go.
- (c) Discuss your result. First, explain what each of the equilibria means biologically. Do you get a stable coexistence between the two species?