

## Math 1180 - Mathematics for Life Scientists

### Computer Assignment 4 - Phase Plane Analysis

Due **Friday**, February 10, 2006

#### Getting Started

Log in to your computer, and click the computer icon at the bottom of the screen to open an xterm window. As usual, create a new folder **lab4** by typing

```
> mkdir lab4
```

Now, open a web-browser by typing **netscape &** or **firefox &** on your terminal. A useful software for doing phase-plane analysis can be downloaded for free from

<http://math.rice.edu/~dfield/index.html>

We will utilize **pplane** for this lab. On your page, click on the link for **For use with version 7.0 of MATLAB**. Then, *right click* on **pplane7.m**. Choose **Save Link As** and save the file in the directory for **lab4**. Make sure you are saving the file in the correct directory.

We will use MATLAB to do this lab instead of Maple. Change into the **lab4** directory and call MATLAB 7 by typing this on your xterm window

```
> cd lab4
> matlab &
```

Now call **pplane** from MATLAB

```
>> pplane7
```

A window for **pplane** should appear on your screen.

#### Lotka-Volterra Predator and Prey Model

A predator-prey interaction occurs when one species, the prey, serves as a food source for another species, the predator. The simplest and earliest mathematical model that describes such interaction is the Lotka-Volterra model:

$$\frac{dx}{dt} = ax - bxy. \quad (1)$$

$$\frac{dy}{dt} = -cy + dxy \quad (2)$$

where  $x(t)$  represents the prey population and  $y(t)$  represents the predator population. The parameters  $a$ ,  $b$ ,  $c$ , and  $d$  are all positive rate constants representing the following:

- $a$  is the per-capita birth rate of preys.
- $b$  is the per-capita prey mortality rate due to predation by a single predator. It can be thought of as a measure of the predator's ability to capture prey
- $c$  is the per-capita mortality rate of predators
- $d$  is the per-capita predator reproduction rate for each prey captured. It can be thought of as hunting efficiency.

This model makes several critical assumptions which may be unrealistic in most real situations in nature. Specifically, the model assumed:

- Prey has unlimited resource; the prey population grows exponentially in absence of predators.
- There is no other threat to prey other than one specific predator.
- Predator has only one food source. It will die in the absence of the specific prey instead of resorting to another food source.

While very few such simplistic interaction exists in nature, data on the interaction between the Canadian lynx and the snowshoe hare indicates that the Lotka-Volterra system can be quite a reasonable model for this case. The data was collected on these populations for almost a century through the pelt-trading records of the Hudson Bay Company.

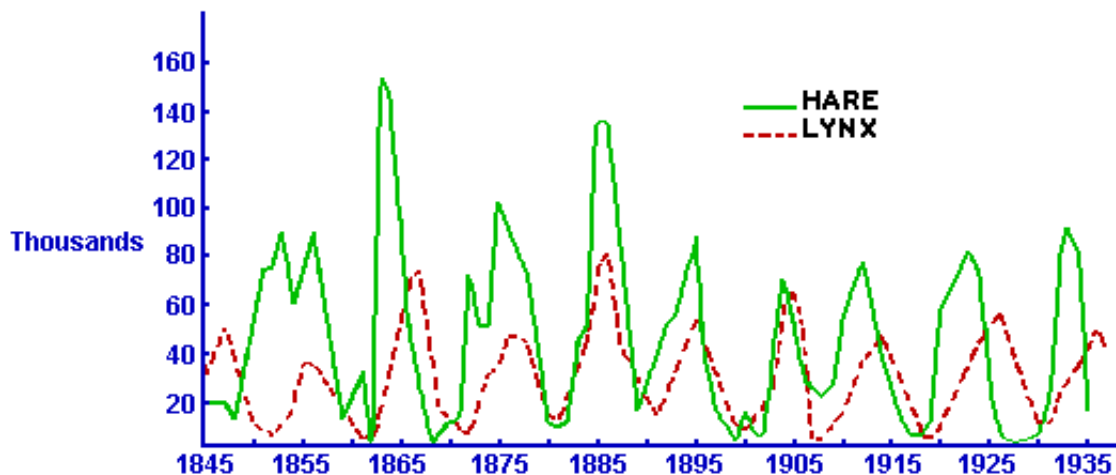


Figure 1: Source - <http://www.math.duke.edu/education/ccp/materials/diffeq/predprey/pred1.html>

The dominant feature of this picture is the oscillating behavior of both populations. In particular, on average, the peaks of the prey population slightly precede those of the predator population. The Lotka-Volterra model exhibits such behaviors as we talked about in class. Unfortunately, things are never quite as simple as they first appeared. In areas of Canada where lynx died out completely, there is evidence that the snowshoe hare population continued to oscillate; this suggests that lynx were not the only effective predator for hares. We will not pursue this in this lab however.

## Paper and Pencil Work

1. Compute the equilibria of the system. Express your answer in terms of the parameters  $a$ ,  $b$ ,  $c$ , and  $d$ .
2. Explain why we can write  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ .
3. Use the previous result to write a single differential equation for  $y$  as a function of  $x$ , with the time variable  $t$  eliminated from the problem. That is find the equation for  $\frac{dy}{dx}$ . Now, find the solution to this equation using separation of variables. You will not be able to write  $y$  in terms of  $x$  explicitly.

Do the solutions you just obtained give any useful information? If yes, explain any insights you obtain from staring at the solutions. If not, how should we study this system?

The solutions you just obtained exactly describe the trajectories on the phase plane. Different values of the integration constant give different trajectories, and different starting point on the phase plane lead to different integration constant hence a different trajectory.

## Phase Plane Analysis Using pplane

1. Type the Lotka-Volterra system into the pplane window. Use the parameter values  $a = 0.4$ ,  $b = 0.01$ ,  $c = 0.3$ , and  $d = 0.005$ . Plot the phase plane for both  $x$  and  $y$  ranging from 0 to 200.
2. Plot the nullclines. Click on **Solutions** then choose **Show nullclines**.
3. Identify and mark all equilibria. Click on **Solutions** then choose **Find an equilibrium point**. A toggle would appear; use your mouse to move it around the phase-plane then click on where you think an equilibrium point is. If you are correct, you should get a confirmation; pplane will also tell you the type of equilibrium point. Write down the values of the equilibria. What type of equilibrium points can be found for this system?
4. Mark three trajectories on the plane by clicking on three different places on the plane. You may need to hit the **Stop** button if pplane doesn't seem to stop computing after a while. Now plot the solution  $x(t)$  and  $y(t)$ . To do this, click on **Graph** then choose **Both**. Use the toggle that appear to choose which trajectories you want to plot. Plot for several different cases. Explain what you see. That is, how does your initial condition affect the solution - what determines the amplitude of the oscillation? Explain what this means.

Turn in the phase plane plot with nullclines, equilibria and three trajectories shown. To print, do to **File** then **Print**. Choose **lcb115** as the printer name. Also print plots of solutions  $x(t)$  and  $y(t)$  corresponding to the three trajectories above.

## Effect of Varying Parameter Values

1. Increase the value of  $a$  by 50%. How does this change the equilibrium points? Explain what this means biologically.

- Return  $a$  to its original value. Increase the value of  $b$  by 50%. How does this change the solutions? Does it make sense biologically?
- Return  $a$  to its original value. Increase the value of  $c$  by 50%. How does this change the solutions? Does it make sense biologically?

You do not have to turn in the plots for the phase plane. Just write down the values of the new equilibria and explain any other changes you see.

## Logistic Growth

- With the Lotka-Volterra model above, what will happen if you start with no predator in the population? Is this realistic?
- A different way of modeling population growth which we have talked about is to assume logistic growth. We can incorporate this and come up with a new model

$$\frac{dx}{dt} = a\left(1 - \frac{x}{K}\right)x - bxy. \quad (3)$$

$$\frac{dy}{dt} = -cy + dxy \quad (4)$$

Type this new system into your pplane window.

- First, compute the equilibria in terms of the parameters  $a$ ,  $b$ ,  $c$ , and  $d$ . How many equilibria are there? Are they all biologically plausible? Explain what this means biologically.
- Choose  $a = 0.4$ ,  $b = 0.004$ ,  $c = 0.3$ ,  $d = 0.006$ , and  $K = 1000$ . Plot the phase plane. Show the nullclines and identify all equilibria.

Before you plot any trajectory, go to **Options** then **Solution direction**, choose **Forward**.

Now, plot a trajectory on your phase-plane and also plot the solution  $x(t)$  and  $y(t)$ . What is the long-term behavior of your solution? How does it differ from the solution of the Lotka-Volterra model? Explain any biological significance to the changes.

Turn in the phase plane plot with nullclines, equilibria and a trajectory shown. In addition, also print the solution  $x(t)$  and  $y(t)$