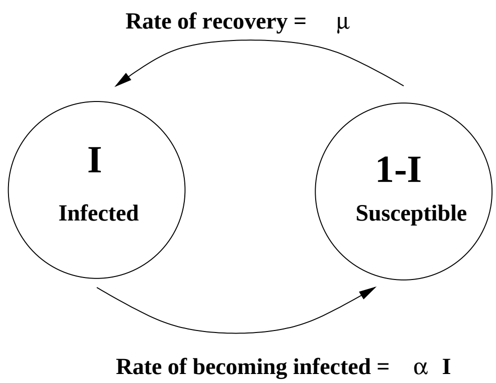
Math 1180 - Mathematics for Life Scientists

Computer Assignment 3

Due Tuesday, January 31, 2006

A Model of a Disease

Reference: FR Adler, Modeling the Dynamics of Life, 2nd ed., pp. 433-435



Suppose a disease is circulating in a population. Individuals will recover but are then immediately susceptible to another infection. Let I be the *fraction* of infected individuals in a population. Each susceptible individual has a chance of getting infected upon contact with an infected individual. Assume then that the rate of contact, i.e. the rate of becoming infected for *each* uninfected individual is proportional to the number of those infected,

$$\text{Per capita rate at which suscpetible individual becomes infected} = \alpha I$$

Each individual is assumed to be either infected or susceptible, so

$$\text{Fraction of susceptible individuals} = 1 - I$$

Thus, the *total* rate at which new individuals are infected is the product of the per capita rates and the number of susceptibles,

$$\text{Total rate at which infection occurs} = \alpha I(1 - I)$$

Now, suppose individuals recover from the disease at a rate proportional to the fraction of infected individuals,

$$\text{Rate at which infected individuals recover} = \mu I$$

Putting all of these together, we have

$$\begin{aligned} \frac{dI}{dt} &= \text{rate at which infection occurs} - \text{rate of recovery} \\ &= \alpha I(1 - I) - \mu I \end{aligned}$$

Starting with a few infected individulas, we would like to find out if the disease is going to persist. In particular, how do changes in parameter values of α and μ affect the outcome?

Problems

1. **Equilibrium and Stability**
First, enter your differential equation as you normally would. In addition, you will want to set a function that corresponds to the right-hand side of the differential equation. Then, solve for when this equals zero to find equilibrium points.

```
> de:= diff(i(t),t) = alpha*(1-i(t))*I(t) - mu*i(t);
> f:= i -> alpha*(1-i)*i - mu*i;
> eq:= {solve(f(i)=0, i)};
```

One thing to note, capital I is reserved for the complex/imaginary number i in Maple. Try to avoid using I in this lab.

   (a) Determine the equilibrium points using the Maple commands above. Try to explain what each equilibrium represents. Think about what the state variable I physically represents: what are the range of I values that would make sense? How many plausible equilibrium points are there when $\alpha > \mu$? How about when $\alpha < \mu$?

   (b) Draw the phase line diagram (plot $\frac{dI}{dt}$ against I). Do so for the two cases: $\alpha > \mu$ and $\alpha < \mu$, use $\mu = 1$ and $\alpha = 0.5, 1.5$. Draw directional arrows, indicating the direction of the flow of the system, by hand. Determine the stability of equilibrium points for the two cases above. Are the results that you obtain making sense biologically? Explain. When does the disease remain the population (become endemic)?

   (c) Draw the solution $I(t)$ using the initial condition $I(0) = 0.2$ for the two cases above. Recall how you should use the command **dsolve** to do this.

```
> mu:= 1; alpha:= 0.5;
> de1:= {de,i(0)=0.2};
> I1:= unapply(rhs(dsolve(de1,i(t))),t);
> plot(I1(t),t=0..5);
```

   (d) If you are to design a control measure to eradicate the disease, how would you do so? Would you need to completely reduce the transmission rate to zero?

2. **Bifurcation Analysis**
When parameter values change, the number and stability of equilibrium points change sometimes. Such changes are called *bifurcations*. It may have important biological implications. A bifurcation diagram is a plot indicating how equilibrium value and stability would change as a function of parameter value. Doing a bifurcation diagram analysis may help explain why the dynamics of the system can change drastically when a parameter value only changes very slightly.
Suppose $\mu = 1$. Even though it may not make biological sense, graph how the values and stabilities of equilibrium points change as α goes from 0 to 2. For each point I^*, plot the value $I^*(\alpha)$ and $f'(I)$ evaluated at $I^*(\alpha)$.

```
> restart;
> f:= i -> alpha*(1-i)*i - mu*i;
> eq:= {solve(f(i)=0,i)};
>
> eq1:= a ->  subs(alpha=a,eq[1]);
> # this is just a way to turn the 1st eq. value into a fcn. which you can plot
```

```
>
> fp1:= a -> subs({alpha=a, i=eq1(a)},diff(f(i),i));
> # computing f'(i) at the 1st eq. and turn it into a function
>
> # do the same thing for the 2nd point
> eq2:= a -> subs(alpha=a,eq[2]);
> fp2:= a -> subs({alpha=a, i=eq2(a)},diff(f(i),i));
>
> mu:= 1;
> plot([eq1(a),fp1(a)], a=0..2, y=-10..2, color=[blue,red]);
> plot([eq2(a),fp2(a)], a=0..2, y=-10..2,  color=[blue,red]);
```

On each of the plot above, indicate the stability of the equilibrium depending on the value of α. You will need to recall the stability theorem which we talked about in class. Explain the changes that occur at $\alpha = 1$. This type of change is referred to as *transcritical bifurcation*.