Math 1180 - Spring 2006

Homework 3

Due Friday, February 3

1 Competition Model

In the Lotka-Volterra Competition model we worked on in-class, we assume that the per-capita growth rate is just a function of the total population. Take for example, for species a,

$$\text{Per capita growth rate ofr a} = \mu(1 + \frac{a+b}{K_a}) = \mu(1 + \frac{\text{total population}}{K_a})$$

In reality, interaction may affect each species differently. The resource space occupied by species a (rabbit) is not equal to that for species b (sheep). When the two interact, the conflict that results will reduce the growth rate for both. However, we can expect the effect to be more severe for the rabbits for example. To account for this, we write

$$\frac{da}{dt} = \mu(1 - \frac{a+\beta b}{K_a})a \quad (1)$$

$$\frac{db}{dt} = \lambda(1 - \frac{\alpha a + b}{K_b})b \quad (2)$$

In the previous example, we see that coexistence between the two species is impossible (competitive exclusion). There are only three equilibria representing the extinction of type a alone, the extinction of type b alone, or the extinction of both. We are interested if stable coexistence is possible in this improved model.

1. Take $\beta = 1/2$ and $\alpha = 2$. Set $\mu = \lambda = 1$ and $K_a = K_b = 1000$. Which of the species fo you think is at a disadvantage? Explain. Do a phase-plane analysis
2. Take $\beta = 1/2$ and $\alpha = 1/2$. Set $\mu = \lambda = 1$ and $K_a = K_b = 1000$. Do a phase-plane analysis
3. In the two cases above, do you get stable coexistence? Explain the biological significance of your calculation above.

What you need to do to complete a phase-plane analysis:

1. Find the nullclines and plot the nullclines on the phase-plane
2. Find equilibrium points by solving the equation algebraically or if it is obvious, by looking at plots
3. Draw directional arrows on the nullclines. Also indicate the flow direction in each of the region separated by the nullclines.
4. Draw trajectories starting from each region and predict where they go.

2 Epidemic Model

In Lab 3, you worked on a model of an epidemic (originally developed by Kermack and McKendrick in 1927) by reducing it to a first order system. In this problem, you will see how the analysis is easier when done in the phase-plane.

$$\frac{dI}{dt} = \alpha IS - \mu I \tag{3}$$

$$\frac{dS}{dt} = \mu I - \alpha IS \tag{4}$$

where $\mu, \alpha > 0$.

1. Do the equations above remind you of any other system we worked on before?
2. Find all the equilibrium of the system. Write them down in terms of the parameters α and μ
3. Using $\alpha = 2.0$ and $\mu = 1.0$, do a phase-plane analysis.

3 Epidemic Model Yet Again

Modify the epidemic model above to account for this: Suppose that all individuals become susceptible upon recovery (just like in the basic model above). In addition all individuals give birth at rate b. The offspring of all individuals are susceptible.

1. Find all the equilibrium of the system. Write them down in terms of the parameters α, μ, and b
2. Using $\alpha = 2.0$, $\mu = 1.0$, and $b = 1.0$, do a phase-plane analysis.

4 Harmonic Oscillator

The motion $x(t)$ of an object attached to a spring can be described by the equation

$$\frac{d^2x}{dt^2} = -kx \tag{5}$$

See page 226-229 for the explanation. k is the spring constant. The equation above describe the acceleration in terms of the position of the object. For simplicity, take $k = 1$.
This one equation for the second derivative can be written as a system of coupled two autonomous differential equations. Let $v(t)$ represent the derivative of $x(t)$.

$$\frac{dx}{dt} = v \tag{6}$$

Then, the derivative of $v(t)$ is exactly the acceleration.

$$\frac{dv}{dt} = -kx \tag{7}$$

1. Do the phase-plane analysis for this system.
2. Why do you think this system is called a harmonic oscillator? Briefly describe how you expect the object will move.

5 Damped Harmonic Oscillator

We can include the effect of friction on the mass-spring system above. The new equation will be

$$\frac{d^2x}{dt^2} = -kx - \gamma v \tag{8}$$

where v is the velocity of the system. Take $k = 2$ and $\gamma = 1$.

1. Write the equation above as a system of two autonomous differential equation by following the steps given above.
2. Do the phase-plane analysis for this system.
3. Describe the effect of including friction to the system. Try to explain what this means physically.