1 Limits

Let me start with a confession: This is my first time teaching calculus, so you are in for a treat. Before we begin our 16-weeks-full-of-uncertainty-Calculus-journey, I have a few requests for you.

1. **Be attentive**: Put away your phone once you step into the classroom and give me your 50 minutes of undivided attention.

2. **Question authority**: I want you to feel empowered to question me when something is wrong, don't just sit around and let me shove definitions and theorems onto you.

1.1 Introduction to Limits

Among all the forthcoming sections, this section is the most important one in this course. Chapter 1 is all about limits and continuous functions but it really means nothing to you now. I spent the whole summer (ok, maybe just the last few weeks) thinking how to introduce limits to you, until recently when I read this on a blog: Why on Earth would anyone care about the following definition

To say that $\lim_{x \to c} f(x) = L$ means that $f(x)$ approaches L as x approaches c.

without any proper motivation? The reason is actually pretty simple: There is a rich set of practical problems that can only be solved using differential and integral calculus, and these branches of mathematics originated from the understanding of limits itself.

Example 1.1. Consider the function $f(x) = 5x + 2$.

1. What is the value of $f(x)$ at $x = 1$? This is simply asking you to find what is $f(1)$:

   $$f(1) =$$

2. What is happening to $f(x)$ as x approaches 1? This is asking about the behaviour of $f(x)$ when x is getting close to or near the number 1.

The previous example seems to suggest that we may simply plug in the x-value into the function and arrive at the correct limit. This is only true in certain cases so quickly snap yourself out of this toxic thought!

Example 1.2. Evaluate the following limits.

(a) $\lim_{x \to -3} (x^2 + 1)$

(b) $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$

Example 1.3. Suppose we want to find $\lim_{x\to 0}\left(x^2 - \frac{\cos(x)}{10000}\right)$. It is not straightforward to graph this function by hand, and clearly it does not simplify like the previous example. Well, we can use a calculator to guess the limit.

If there is no one value that $f(x)$ approaches as x gets closer to c, then we say that the limit of $f(x)$, as x approaches c, does not exist. We can write this as

$$\lim_{x\to c} f(x) \text{ does not exist or “DNE” for short.}$$

There are different ways in which a limit does not exist.

Example 1.4. Explain why the following limits do not exist.

(a) $\lim_{x\to 0} \frac{1}{x^2}$

(b) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

One-sided limits

Let us examine the definition of limit again:

To say that $\lim_{x \to c} f(x) = L$ means that $f(x)$ approaches L as x approaches c.

1. Do we need $f(x)$ to be defined at $x = c$?

2. Suppose $f(x)$ is defined at $x = c$, *i.e.* we can actually compute $f(c)$. Would knowing $f(c)$ be helpful in evaluating the limit?

3. We need to look at values of x near c, but this includes both values of $x < c$ and $x > c$.

   ♣ We write $\lim_{x \to c^-} f(x) = L$ and say "the **left-hand limit** of $f(x)$, as x approaches c, is L" if $f(x)$ is close to the value L whenever x approaches c from the left.

   ♣ We write $\lim_{x \to c^+} f(x) = L$ and say "the **right-hand limit** of $f(x)$, as x approaches c, is L" if $f(x)$ is close to the value L whenever x approaches c from the right.

   Thus, in order for the limit to exist and equal L, $f(x)$ must be getting closer to L as x approaches c both from the left and from the right:

$$\boxed{\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.}$$

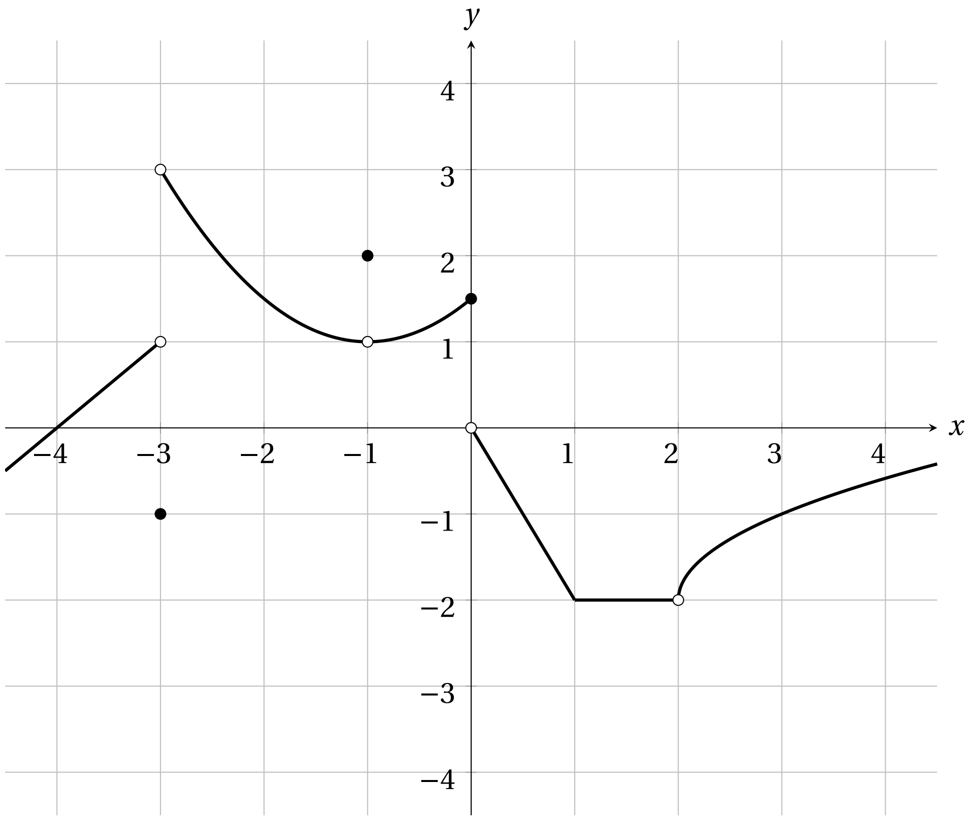
Example 1.5. Compute the following limits, if they exist.

(a) $\displaystyle \lim_{x \to 0^+} \frac{|x|}{x}$

(b) $\displaystyle \lim_{x \to 0^-} \frac{|x|}{x}$

(c) $\displaystyle \lim_{x \to 0} \frac{|x|}{x}$

Example 1.6. For the function f graphed below, find the indicated limit or function value, or state that it does not exist.



(a) $f(-3) =$

(b) $\lim_{x\to -3^-} f(x) =$

(c) $\lim_{x\to -3^+} f(x) =$

(d) $\lim_{x\to -3} f(x) =$

(e) $f(-1) =$

(f) $\lim_{x\to -1^-} f(x) =$

(g) $\lim_{x\to -1^+} f(x) =$

(h) $\lim_{x\to -1} f(x) =$

(i) $f(0) =$

(j) $\lim_{x\to 0^-} f(x) =$

(k) $\lim_{x\to 0^+} f(x) =$

(l) $\lim_{x\to 0} f(x) =$

(m) $f(2) =$

(n) $\lim_{x\to 2^-} f(x) =$

(o) $\lim_{x\to 2^+} f(x) =$

(p) $\lim_{x\to 2} f(x) =$

1.2 Rigorous Study of Limits

We state the ε-δ definition of limit: To say that $\lim_{x\to c} f(x) = L$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending on ε) such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Example 1.7. Prove using the ε-δ definition that $\lim_{x\to 3}(4x - 5) = 7$.

1.3 Limit Theorems

This section contains important theorems that are practical in evaluating limits, in the sense that no graphing is required and only algebraic manipulation is involved. These are going to be your best friends for the next few weeks. A wise man named Franco once said:

You need to recite these theorem every night before you go to bed

(A) Main Limit Theorem (Limit Laws)

Let n be a positive integer, K a constant and f and g be functions that have limits at c. Then

1. $\lim\limits_{x\to c} K = K$. [The limit of a constant is just the constant itself]
2. $\lim\limits_{x\to c} x = c$. [The limit of x, as x tends to c, is just c]
3. $\lim\limits_{x\to c} kf(x) = k\lim\limits_{x\to c} f(x)$. [Constants pull out of limits]
4. $\lim\limits_{x\to c} [f(x)+g(x)] = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$. [The limit of a sum is the sum of the limits]
5. $\lim\limits_{x\to c} [f(x)-g(x)] = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x)$. [The limit of a difference is the difference of the limits]
6. $\lim\limits_{x\to c} [f(x)g(x)] = \left(\lim\limits_{x\to c} f(x)\right)\left(\lim\limits_{x\to c} g(x)\right)$. [The limit of a product is the product of the limits]
7. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$, provided $\lim\limits_{x\to c} g(x) \neq 0$. [The limit of a quotient is the quotient of the limits, provided we are not dividing by zero]
8. $\lim\limits_{x\to c} \left[f(x)\right]^n = \left[\lim\limits_{x\to c} f(x)\right]^n$. [The limit of a power is the power of the limit]
9. $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to c} f(x)}$, provided $\lim\limits_{x\to c} f(x) > 0$ when n is even. [The limit of the nth root is the nth root of the limit, provided the latter is defined]

Before you even think about using any of these limit laws, always remind yourself to check and see if both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist.

Example 1.8. Use the limit laws to evaluate the following limits. Be sure to indicate which limit laws you are using at each step.

(a) $\lim_{x \to 1} (5x + 2)$

(b) $\lim_{x \to 3} (x^3 - 9x)$

(c) $\lim_{x \to 0} \sqrt{x^2 + 9}$

(d) $\lim_{x \to 2} \left(\frac{x^2 + 4}{x - 1} \right)^3$

Albeit sophisticated and abstract, it is important that you are shown the proof of the limit laws, with the hope that you can truly appreciate the practical nature of the theorem.

Proof of Limit Law 4. Since f and g both have limits at $x = c$, this means that there exist finite numbers L and M such that

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M.$$

From the ε-δ definition, this means that given an $\varepsilon > 0$, we can find $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} 0 < |x-c| < \delta_1 &\implies \left|f(x) - L\right| < \frac{\varepsilon}{2} \\ 0 < |x-c| < \delta_2 &\implies \left|g(x) - M\right| < \frac{\varepsilon}{2} \end{aligned}$$

Choose $\delta = \min\{\delta_1, \delta_2\} > 0$. This means that $0 < |x-c| < \delta$ implies both $0 < |x-c| < \delta_1$ and $0 < |x-c| < \delta_2$. Consequently,

$$\begin{aligned} \left|f(x) + g(x) - (L+M)\right| &= \left|\left[f(x) - L\right] + \left[g(x) - M\right]\right| \\ &\le \left|f(x) - L\right| + \left|g(x) - M\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

In particular, we have the following implication:

$$0 < |x-c| < \delta \implies \left|(f(x) + g(x)) - (L+M)\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that

$$\lim_{x\to c}\left[f(x) + g(x)\right] = L + M = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

□

B Direct Substitution

If f is a polynomial or rational function, then

$$\lim_{x\to c} f(x) = f(c)$$

provided $f(c)$ is defined. In the case of a rational function, this means that the value of the denominator at c is NOT ZERO.

C "Cancellation is Fine"

If $f(x) = g(x)$ for all x in an open interval containing the number c, except possibly at the number c itself, and if $\lim_{x\to c} g(x)$ exists, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

Example 1.9. Find the following limits, if they exist.

(a) $\displaystyle\lim_{h\to 0}\frac{(3+h)^2-3^2}{h}$

(b) $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x-3}$

(c) $\displaystyle\lim_{x\to -3}\frac{x^2+5x+6}{x^2-9}$

(d) $\displaystyle\lim_{x\to 2}\frac{x^2-9}{(x-2)^2}$

Example 1.10. Consider the piecewise function

$$f(x) = \begin{cases} 1-x & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 1 \\ 2-x & \text{if } x > 1 \end{cases}.$$

Find the following limits, if they exist.

(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to 1^-} f(x)$

(e) $\lim_{x\to 1^+} f(x)$

(f) $\lim_{x\to 1} f(x)$

D Squeeze Theorem

Let f, g, and h be functions satisfying

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \text{ near } c \text{, except possibly at } c.$$

If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Example 1.11. Use the Squeeze Theorem to compute $\lim_{x\to 0} x^2 \cos(x)$.

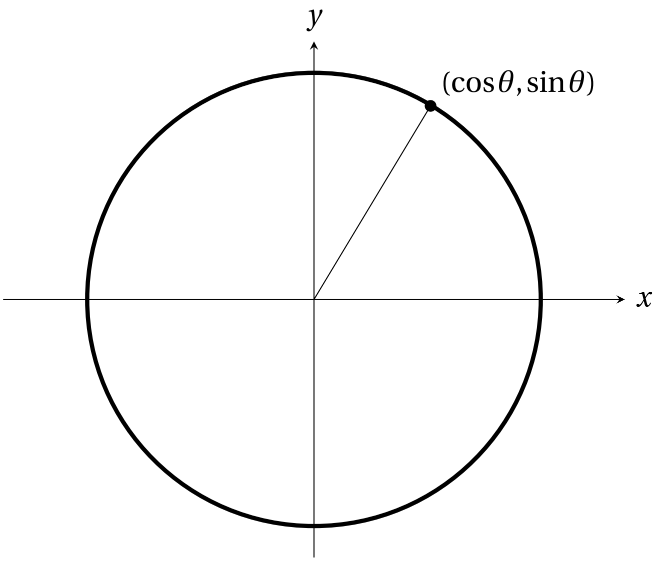
Example 1.12. Assume we know that

$$1 - \frac{x^2}{6} \le \frac{\sin(x)}{x} \le 1$$

for all x near but different from 0. Use the Squeeze Theorem to compute $\lim_{x \to 0} \frac{\sin(x)}{x}$.

1.4 Limits Involving Trigonometric Functions

Unit circle and right triangles



It follows from the equation of the unit circle that $x^2 + y^2 = 1$. Substituting $x = \cos\theta$ and $y = \sin\theta$ then yields

$$\cos^2\theta + \sin^2\theta = 1.$$

Let us introduce four additional trigonometric functions:

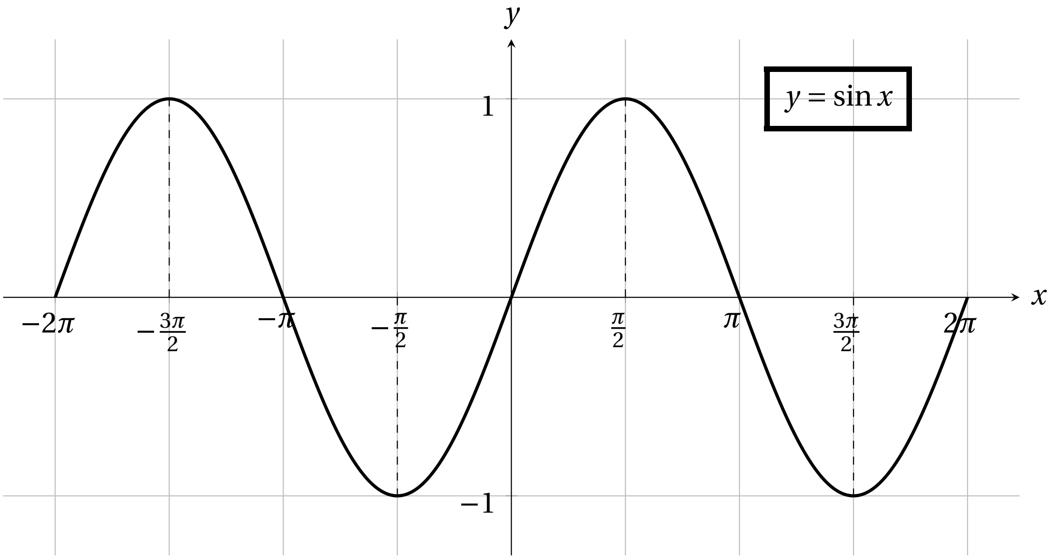
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

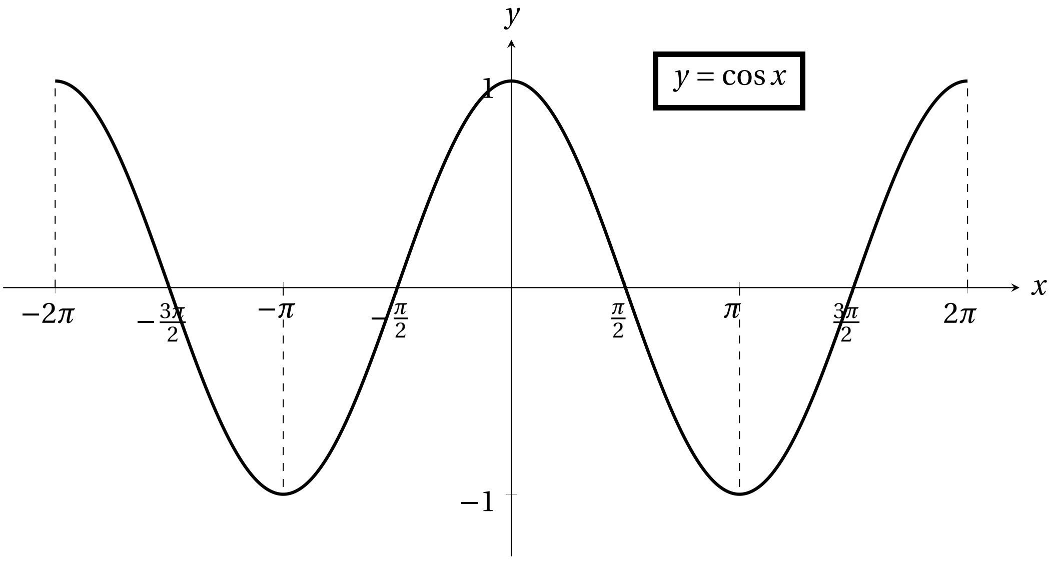
Dividing the identity $\cos^2\theta + \sin^2\theta = 1$ by either $\cos^2\theta \neq 0$ or $\sin^2\theta \neq 0$ then yields

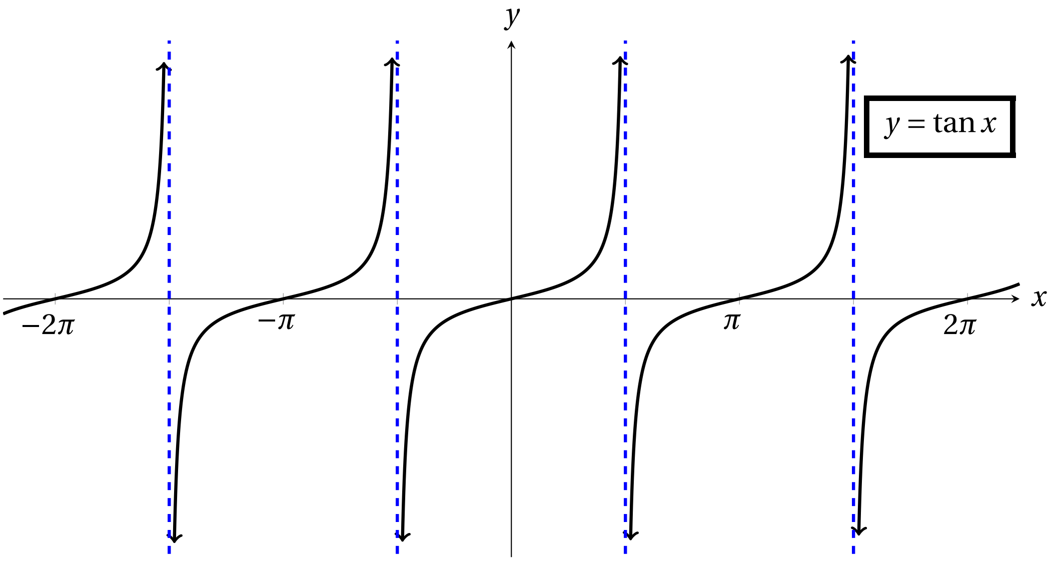
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			
$\frac{\pi}{4}$			
$\frac{\pi}{6}$			
$\frac{\pi}{3}$			

Graphs of sine, cosine and tangent







Example 1.13. Find the periods and amplitudes and sketch the graphs of the following trigonometric functions.

(a) $y = 3\sin(2\pi x)$

(b) $y = 5\cos(x+\pi) - 2$

Limits of trigonometric functions

If $t = c$ is in the domain of the trigonometric function, then we have

1. $\lim_{t\to c} \sin(t) = \sin(c)$
2. $\lim_{t\to c} \cos(t) = \cos(c)$
3. $\lim_{t\to c} \tan(t) = \tan(c)$
4. $\lim_{t\to c} \cot(t) = \cot(c)$
5. $\lim_{t\to c} \sec(t) = \sec(c)$
6. $\lim_{t\to c} \csc(t) = \csc(c)$

Example 1.14. Evaluate the following limits.

(a) $\lim_{t\to\pi} (t \sec t - \tan t)$

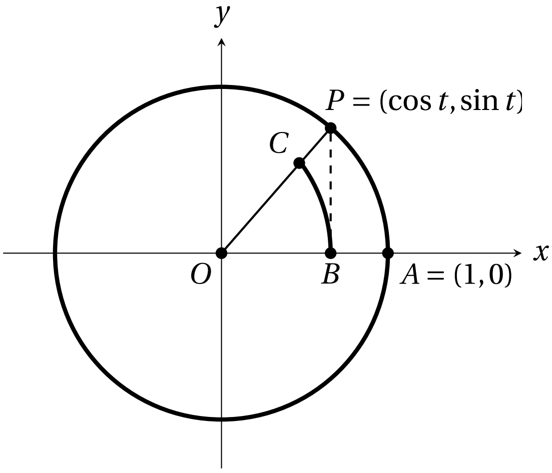
(b) $\lim_{t\to\pi/2} \dfrac{t + t^2 \cos t}{\sin t}$

(c) $\lim_{t\to 0} \dfrac{\sin^2 t}{1 - \cos t}$

Special trigonometric limits

$$\boxed{\lim_{t\to 0}\frac{\sin t}{t}=1} \quad \text{and} \quad \boxed{\lim_{t\to 0}\frac{1-\cos t}{t}=0}$$

Proof:



Area of sector $OBC =$

Area of triangle $OBP =$

Area of sector $OAP =$

$$\lim_{t\to 0}\frac{1-\cos t}{t}=$$

Example 1.15. Show that $\lim_{x \to 0} \frac{\tan x}{x} = 1$.

Example 1.16. Use the special trigonometric limits, along with some algebraic manipulation, to compute the following limits.

(a) $\lim_{t \to 0} \frac{\cos(3t) - \cos^2(3t)}{t}$

(b) $\lim_{t \to 0} \frac{\sin(2t)}{t}$

(c) $\lim_{t \to 0} \frac{\tan(4t)}{\sin(3t)}$

1.5 Limits at Infinity, Infinite Limits

End behaviour

In the language of College Algebra, limits at infinity means **end behaviour**. Recall that finding end behaviour of a function $f(x)$ means the following two questions:

1. What happens to $f(x)$ as x gets larger and larger in the positive direction, *i.e.* as $x \longrightarrow \infty$?
2. What happens to $f(x)$ as x gets larger and larger in the negative direction, *i.e.* as $x \longrightarrow -\infty$?

In limit notation, the first question translates to

$$\text{What is } \lim_{x \to \infty} f(x)?$$

and the second question translates to

$$\text{What is } \lim_{x \to -\infty} f(x)?$$

The crucial question is, what could possibly happen to $f(x)$ as $x \longrightarrow \infty$ (similarly, as $x \longrightarrow -\infty$)?

Example 1.17 (Almost obvious, but important)**.** Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for the following functions.

(a) $f(x) = \dfrac{1}{x}$

(b) $f(x) = \dfrac{1}{x^2}$

(c) $f(x) = \dfrac{1}{x^3}$

For positive integer $n = 1,2,3,\ldots$, we have the following limits:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Example 1.18. Compute the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - x + 1}{2x^2 + x}$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x+4}{x^2+1}$

(c) $\displaystyle\lim_{x\to-\infty} \frac{x^3+2}{(x^2+1)^2}$

(d) $\displaystyle\lim_{x\to\infty} \sqrt{\frac{5x^3 - x}{x^3+2}}$

(e) $\lim\limits_{x\to\infty} \dfrac{3x^5 - x^2 - x + 1}{x^4 - 1}$

(f) $\lim\limits_{x\to-\infty} \dfrac{3x^5 - x^2 - x + 1}{x^4 - 1}$

(g) $\lim\limits_{x\to\infty} \dfrac{x^4 + 3x^3 - x - 1}{x^2 + 1}$

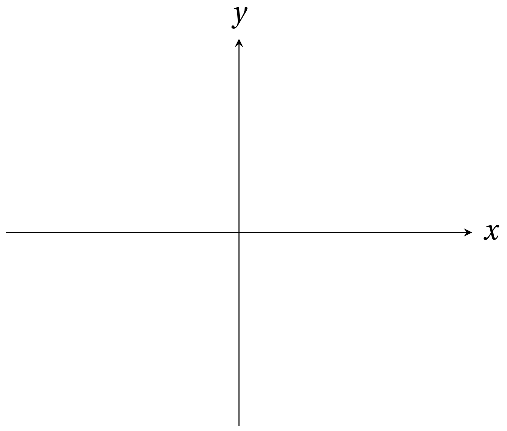
(h) $\lim\limits_{x\to-\infty} \dfrac{x^4 + 3x^3 - x - 1}{x^2 + 1}$

One has to be extra cautious when dealing with indeterminate forms:

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0\times\pm\infty,\ \infty-\infty,\ 0^0,\ \infty^0.$$

Infinite Limits

The reciprocal function $f(x) = \frac{1}{x}$ is truly exquisite, simply because one can learn so much about limits just by sketching its graph. In lament terms, infinite limits mean the function grows without bound as x approaches a particular number c. To visualise what this means, let us graph the reciprocal function and examine what happens to $f(x)$ as x approaches 0:



$$\lim_{x\to 0^-}\frac{1}{x} = \qquad \lim_{x\to 0^+}\frac{1}{x} =$$

The crucial thing to observe is that it makes sense to talk about one-sided limits in this case, because x is approaching a real number c, not $\pm\infty$. More importantly, the upshot of this example is that even in the case of infinite limits, if the one-sided limits do not agree, *i.e.*

$$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x),$$

then we say that the limit does not exist. Thus in this case

$$\lim_{x\to 0}\frac{1}{x}\ \text{DNE.}$$

- ♣ $\lim_{x\to c} f(x) = \infty$ means we can make values of $f(x)$ arbitrarily large by choosing values of x sufficiently near c, but not equal to c.
- ♣ $\lim_{x\to c} f(x) = -\infty$ means we can make values of $f(x)$ arbitrarily negative by choosing values of x sufficiently near c, but not equal to c.

Example 1.19. Evaluate the following limits, answering either a number, ∞, $-\infty$, or DNE.

(a) $\lim\limits_{x\to 2} \dfrac{x^2-9}{(x-2)^2}$

(b) $\lim\limits_{x\to 0} \dfrac{3}{1-\cos(2x)}$

(c) $\lim_{x \to 1} \frac{x-1}{x^4 - 2x^3 + x^2}$

(d) $\lim_{x \to 0} \frac{x-1}{x^4 - 2x^3 + x^2}$

Definition 1.20. A function has a vertical asymptote at $x = c$ if

$$\lim_{x \to c^-} f(x) = \pm\infty \quad \text{and/or} \quad \lim_{x \to c^+} f(x) = \pm\infty.$$

Example 1.21. Find all the vertical asymptotes of $f(x) = \dfrac{x^3 + x}{x^2 - x}$.

1.6 Continuity of Functions

Continuity at a point

Definition 1.22. Let f be defined on an open interval containing $x = c$. We say that f is **continuous** at $x = c$ if

$$\lim_{x \to c} f(x) = f(c).$$

Specifically, we say that f is continuous at $x = c$ if and only if the following conditions are satisfied:

1. $\lim_{x \to c} f(x)$ exists;
2. f is defined at $x = c$, *i.e.* $f(c)$ exists;
3. $\lim_{x \to c} f(x) = f(c)$.

If any of these three conditions fails, then we say that f is **discontinuous** at $x = c$, or that f has a **discontinuity** at $x = c$.

Types of discontinuities

Let us examine cases where a function fails to satisfy one of the three conditions above.

Continuity of Polynomial, Rational and Trigonometric Functions
Polynomials are continuous everywhere. Rational functions and trigonometric functions are continuous wherever they are defined.

Example 1.23. Consider the following piecewise-defined function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ x & \text{if } 1 \leq x < 2, \\ 4 & \text{if } x = 2, \\ x & \text{if } 2 < x < 3, \\ \dfrac{1}{x-5} & \text{if } x \geq 3. \end{cases}$$

Find all points of discontinuity of f and classify each of these as removable, jump or infinite.

Example 1.24. Consider the following piecewise-defined function

$$g(x) = \begin{cases} 0 & \text{if } x < -\pi, \\ \dfrac{\sin x}{x} & \text{if } -\pi \leq x < \pi, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \geq \pi. \end{cases}$$

Find all points of discontinuity of g and classify each of these as removable, jump or infinite.

Below we summarise types of discontinuities.

1. We say that f has a **removable discontinuity** at $x = c$ if f is discontinuous at $x = c$ but can be made to be continuous at $x = c$ by simply redefining $f(c)$.

2. We say that f has a **jump discontinuity** at $x = c$ if one sided limits exist but

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x).$$

3. We say that f has an **infinite discontinuity** at $x = c$ if f has a vertical asymptote at $x = c$.

Jump discontinuities and infinite discontinuities are together called **non-removable** discontinuities because we cannot simply fix the discontinuity by redefining the function there.

Continuity on an interval

Continuity on an interval should mean continuity at each point of that interval, but this is problematic if we consider a closed interval, say $[a,b]$. If we consider the square root function $f(x) = \sqrt{x}$ over the interval $[0,1]$, we see that $\lim_{x\to 0} f(x)$ DNE because f is not even defined to the left of $x = 0$. We circumvent this issue by imposing "one-sided continuity" at the endpoints.

Definition 1.25. A function f is

1. **right-continuous** at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$;
2. **left-continuous** at $x = b$ if $\lim_{x\to b^-} f(x) = f(b)$.

We say that f is continuous on $[a,b]$ if it is continuous everywhere on (a,b), right-continuous at $x = a$ and left-continuous at $x = b$.

Example 1.26. State the intervals (open, closed, half open) on which the following function is continuous.

(a) $$f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ x & \text{if } 1 \le x < 2, \\ 4 & \text{if } x = 2, \\ x & \text{if } 2 < x < 3, \\ \dfrac{1}{x-5} & \text{if } x \ge 3. \end{cases}$$

(b) $$g(x) = \begin{cases} 0 & \text{if } x < -\pi, \\ \dfrac{\sin x}{x} & \text{if } -\pi \le x < \pi, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \ge \pi. \end{cases}$$

Intermediate value theorem

Intermediate Value Theorem

Suppose f is continuous on the interval $[a, b]$. Suppose M is any number between $f(a)$ and $f(b)$. Then there exists a point c with $a < c < b$ such that $f(c) = M$.

Example 1.27. Use the IVT to show that the polynomial $f(x) = x^5 + x - 1$ must have at least one root in the interval $(0, 1)$.

Example 1.28. Show that the equation $x\cos x = -2$ has at least one solution in the interval $\left(\dfrac{\pi}{2}, \pi\right)$.