Teaching Material

Chee Han Tan

This document contains course materials from Math 1210: Calculus 1 in Fall 2018, my most recent in-person teaching, and Math 2250: Differential Equations and Linear Algebra in Spring 2021 (and Fall 2020), my current flipped classroom.

Contents

1 Math 1210 – Calculus 1

For a better understanding of the context of this course material, my syllabus for Math 1210 is presented here.

1.1 Syllabus

Math 1210-018 : Calculus I, Fall 2018

Time and Location	MTWF 12:55PM-1:45PM, LCB (LeRoy E. Cowles Building) 219
Instructor	Chee Han Tan (My first name is actually Chee Han)
Contact	Office: JWB 129 Email: tan@math.utah.edu (Please include "Math 1210" in the subject line) Office Hours: MW 12:25PM-12:55PM & 1:45PM-2:15PM or **always available by appointment**
Lab Information	Learning Assistant: Hayden Strikwerda Email: strikhayden@gmail.com Section 19: Thursday 12:55PM-01:45PM in AEB (Alfred C. Emery Building) 360 Section 20: Thursday 2:00PM-2:50PM in AEB 360
Textbook	ISBN-10: 0132306336, ISBN-13: 9780132306331. *Calculus with Differential Equations*, by Varberg, Purcell, and Rigdon (9th edition). For information on purchasing the textbook, go to `http://www.math.utah.edu/schedule/bookInfo/CalcBookInfo-3.pdf`.
Course Website	Canvas will be used heavily for posting announcements, homework assignments, grades, files and any relevant supplementary material. I will hold you accountable for receiving these information. If you do not check Canvas regularly, you should have announcements forwarded to an email address that you do check regularly. Either sign in through CIS or go to `https://utah.instructure.com/courses/511502`.
Prerequisites	"C" or better in ((MATH 1050 AND 1060) OR MATH 1080 OR (MATH 1060 AND Accuplacer CLM score of 80+)) OR AP Calc AB score of 3+ OR Accuplacer CLM score of 90+ OR ACT Math score of 28+ OR SAT Math score of 630+. *Important note*: The mathematics department DOES enforce prerequisites for all undergraduate courses. If you were able to register for this class based on your enrollment in the prerequisite course last semester and you did not receive the minimum grade in that course to enter this class, then you will be dropped from this class on Friday of the first week of classes. If you are in this situation, it is in your best interest to drop yourself from this class and enroll in a class for which you have the prerequisites before you are forcibly dropped.
Course Information	Math 1210 Calculus I is a 4-credit course. This will be a homework intensive class. According to the University of Utah, a 4-credit course should have about 4 hours of lecture and 8 hours of outside study/homework time. This means that in our class, it will take the average student about 7 hours per week for homework and studying plus 1 hour in the lab each week. Some students will be able to get by on less, and some students will need more depending on their math background and desired grade. Please note that if you miss a lecture this time will go up considerably.
Course Description	Functions and their graphs, differentiation of polynomial, rational and trigonometric functions. Velocity and acceleration. Geometric applications of the derivative, minimization and maximization problems, the indefinite integral, and an introduction to differential equations. The definite integral and the Fundamental Theorem of Calculus.
Reading	You are strongly encouraged to have read the chapters before the corresponding class. You do not have to understand everything that you read the first time! Even if you spend as little as 10 minutes on this, it makes the discussion in class much clearer, and overall you will save time.
Attendance	Like any college course, attendance is not mandatory. However, concepts will be thoroughly explained and reviewed in class, thus it is to your absolute benefit to attend all classes. Students who regularly attend score on average 30% higher on exams than those who do not.

Expected Learning Outcomes

Upon successful completion of this course, a student should be able to:

1. Take limits of algebraic and trigonometric expressions of the form $0/0$ (that simplify), non-zero number over 0, including limits that go to (positive or negative) infinity, limits that don't exist and limits that are finite.

2. Use and understand the limit definitions of derivative for polynomial, rational and some trigonometric functions; understand the definition of continuity and consequences.

3. Differentiate all polynomial, rational, radical, and trigonometric functions and compositions of those functions; perform implicit differentiation and compute higher order derivatives.

4. Use differentiation to find critical points and inflection points, the signs of the first and second derivatives, and domain and limit information to determine vertical and horizontal asymptotes. Then use all of that information to sketch the graph of $y = f(x)$.

5. Apply differentiation to optimization, related rates, linear approximation, and problems involving differentials.

6. Compute indefinite integrals and find antiderivatives, including finding constants of integration given initial conditions.

7. Compute definite integrals using the definition for simple polynomial functions. Compute definite integrals using the power rule, basic u-substitution, and the Fundamental Theorems of Calculus.

8. Apply the definite integral to compute area between two curves, volumes of solids of revolutions, arc length, surface area for surfaces of revolution, and work problems.

Calculators

Calculators will not be allowed on exams. They may be used on homework, but you should still write out the details of your computation. It is in your best interest not to become too dependent on your calculator since they will not be allowed on exams.

Cheating

If a student is caught cheating on any homework, quizzes or exams, they will automatically receive a "0" for that assignment. Depending on the severity of the cheating, they may fail the class. Please note that the use (or even just pulling it out of your pocket) of a cellphone or any other electronic device is considered cheating and cause for receiving an automatic zero on any exams. If you exhibit any other behaviors that are unethical, I will not hesitate to report your behavior to the Dean of Students.

Letter Grades

Semester letter grades will be converted from the numerical semester scores N as follows:

$93 \leq N \leq 100$	A	$73 \leq N < 78$	C
$90 \leq N < 93$	A−	$70 \leq N < 73$	C−
$88 \leq N < 90$	B+	$68 \leq N < 70$	D+
$83 \leq N < 88$	B	$63 \leq N < 68$	D
$80 \leq N < 83$	B−	$60 \leq N < 63$	D−
$78 \leq N < 80$	C+	$N < 60$	E

Grading

Grades for each student will be calculated using the following formula:

Homework (15%) + Quiz (5%) + Lab (10%) + Midterms ($3 \times 15 = 45\%$) + Final (25%)

There will be no make-up homework assignments, quizzes, lab worksheets and exams. Students who miss an exam will receive a "0" on the missed exam.

1. **Homework:** Roughly three textbook sections are due most Fridays at the beginning of class (including days of exams, but not the week following). The homework will typically cover material covered up to and including the preceding Monday (with possibly a little spill-over to Wednesday). See the "Assignments" tab in Canvas for the list of assigned problems. Three of the problems will be selected for grading by the grader, each graded out of 5 points; completion and submission counts for 5 points. Two lowest homework scores will be dropped. Homework will only be accepted in class, no electronic copies. **No late homework will be accepted, unless accompanied by a doctor's note or other verification of extenuating circumstance.**

2. **Quizzes:** There will be roughly 10 weekly quizzes (Fridays when there is no midterm). You must be in attendance to take the quiz. They will be approximately 10-15 minutes and given near the end of class. Two lowest quiz scores will be dropped.

3. **Lab:** Every Thursday a Learning Assistant (LA)-directed lab section will be held. These lab sections will have smaller class sizes, consisting of working on lab worksheets in groups. The LA will be there to help guide students through the problems. The worksheets will typically be due at the end of the lab period. **Attendance to the lab section is required**, and will count for half of the lab grade (5% of the total course grade); the remaining grade (5% of the total course grade) will be based on both completeness and correctness of the lab worksheets. To receive attendance points you must arrive in lab within 5 minutes of the start time and must stay until the end of class. The lowest lab score will be dropped.

4. **Midterm Exams:** There will be 3 in-class midterm exams on select Fridays. The content will be determined based on the pace of the course. A review sheet and/or practice exam will be posted a week prior to the midterm that will cover the same material. Please note the time:

   MIDTERMS: Sept 14, Oct 19 & Nov 16, all on Fridays

5. **Final Exam:** All students are expected to take the two-hour comprehensive final exam. The room will be announced during the last week of classes. As with the midterms, a review sheet and/or practice exam will be posted a week prior. All students are expected to arrange their personal schedule to allow them to take the exam. Students with conflicts should speak to the instructor as soon as possible but unless it is an absolute emergency no student will be allowed to take the final exam early. Please note the time:

   FINAL: Wednesday, December 12, 2018, 1:00-3:00PM

Important Dates

Last day to add without a permission code: Friday, August 24
Last day to add, drop, audit, elect CR/NC: Friday, August 31
Last day to withdraw from classes: Friday, October 19
Last day to reverse CR/NC option: Friday, November 30
Midterm 1: Friday, September 14
Midterm 2: Friday, October 19
Midterm 3: Friday, November 16
Final exam: Wednesday, December 12

Additional Resources

Your Classmates: You can learn a great deal from discussing mathematics with your classmates, and you are encouraged to work on your homework together (solutions, however, must be written up independently). That said, it is important that these mathematical discussions not be one-sided: the only real way to learn mathematics is to struggle through it, and not simply to accept the fruit of someone else's understanding. Be honest with yourself about this when working with classmates.

Mathematics Tutoring Center: The math department offers free drop-in tutoring for students, at the T. Benny Rushing Mathematics Student Center. The center is located underneath the walkway between LCB (LeRoy Cowles Building) and JWB (John Widtsoe Building), and can be accessed by entering either building.
Opening hours: Monday - Thursday 8AM-8PM and Friday 8AM-6PM.
Website: http://www.math.utah.edu/ugrad/mathcenter.html

Mathematics Department Video Lectures: Video lectures are available at http://www.math.utah.edu/lectures/math1210.html By combining the textbook, the lectures, and the videos, you will have an abundance of perspectives to complete your understanding of our course material. Also, if you are shaky on some of the prerequisites, I encourage you to review the departmental videos from the College Algebra and Trigonometry courses.

Private Tutoring: University Tutoring Services, 330 SSB. There is also a list of tutors at the math department office JWB 233.

Student Responsibilities

All students are expected to maintain professional behavior in the classroom setting, according to the Student Code, spelled out in the Student Handbook. You have specific rights in the classroom as detailed in Article III of the Code. The Code also specifies proscribed conduct (Article XI) that involves cheating on tests, collusion, fraud, theft, etc. Students should read the Code carefully and know you are responsible for the content. According to Faculty Rules and Regulations, it is the faculty responsibility to enforce responsible classroom behaviors, beginning with verbal warnings and progressing to dismissal from class and a failing grade. Students have the right to appeal such action to the Student Behavior Committee. See http://regulations.utah.edu/academics/6-400.php

ADA Statement

The University of Utah seeks to provide equal access to its programs, services and activities for people with disabilities. If you will need accommodations in the class, reasonable prior notice needs to be given to the Center for Disability & Access (CDA), located at 162 Olpin Union Building. To do so, contact CDA at 801-581-5020 (V/TDD) to set up an appointment. CDA will work with you and the instructor to make arrangements for accommodations. All information in this course can be made available in alternative format with prior notification to CDA.

Addressing Sexual Misconduct

Title IX makes it clear that violence and harassment based on sex and gender (which Includes sexual orientation and gender identity/expression) is a civil rights offense subject to the same kinds of accountability and the same kinds of support applied to offenses against other protected categories such as race, national origin, color, religion, age, status as a person with a disability, veterans status or genetic information. If you or someone you know has been harassed or assaulted, you are encouraged to report it to the Title IX Coordinator in the Office of Equal Opportunity and Affirmative Action, 135 Park Building, 801-581-8365, or the Office of the Dean of Students, 270 Union Building, 801-581-7066. For support and confidential consultation, contact the Center for Student Wellness, 426 SSB, 801-581-7776. To report to the police, contact the Department of Public Safety, 801-585-2677(COPS).

Student Names and Personal Pronouns

Class rosters are provided to the instructor with the students legal name as well as preferred first name (if previously entered by you in the Student Profile section of your CIS account). While CIS refers to this as merely a preference, I will honor you by referring to you with the name and pronoun that feels best for you in class, on papers, exams, group projects, etc. Please advise me of any name or pronoun changes (and update CIS) so I can help create a learning environment in which you, your name, and your pronoun will be respected. If you need assistance getting your preferred name on your UIDcard, please visit the LGBT Resource Center Room 409 in the Olpin Union Building, or email bpeacock@sa.utah.edu to schedule a time to drop by. The LGBT Resource Center hours are M-F 8AM-5PM, and 8AM-6PM on Tuesdays.

Wellness Statement

Personal concerns such as stress, anxiety, relationship difficulties, depression, cross-cultural differences, etc., can interfere with a student's ability to succeed and thrive at the University of Utah. For helpful resources contact the Center for Student Wellness at `https://www.wellness.utah.edu` or 801-581-7776.

Additional Policies

Due to experience, I have decided to make some additional policies regarding my classroom administration and grading.

1. I will demand respectful behavior in my classroom. Examples of disrespect include, but are not limited to, reading a newspaper or magazine in class, social chatting with your friend in class, text-messaging your buddies during class, or cuddling with your girl/boyfriend in class. If you choose to be disrespectful with distracting behavior during our class, please keep in mind that you put me in a position of choosing between protecting/taking a stand for you OR for the other students or myself whom you are disrupting. I can guarantee I will choose to stand for the students who are there to learn without disruptions and I will thus take action to terminate your distracting behavior, and that action may not be desirable for you.

2. Cellphones and laptops are prohibited in the classroom. If you need to use your phone during class, please leave the classroom. It is almost impossible to take notes for a math class on a laptop in real time. However, if you are using a tablet or iPad or some similar device to take notes and the screen lies parallel to your desk, that is fine.

3. There will be no retakes of exams, for any reason.

4. There will be no cursing nor negative ranting (for example, "math sucks") on any written work turned in. The penalty for such things on your written work will be a zero score on that assignment or test.

5. If there are any emergencies that prevents you from attending the exam or turning in homework and lab worksheet, it is 100% your responsibility to notify me before any of these events. I will try my best to accomodate and help you in some manner, which I am truly happy to do; but the longer you wait to communicate me, the less I can and am willing to do to help. The best way to contact me is by email or in office hours. Please keep in mind that I do not check my email regularly during the weekend.

6. If you have questions about any exam/quiz/homework grade, or you want to appeal the grading, you must bring it to me within one week of the return of the exam/quiz/homework. I am happy to look over your appeal and/or questions and give my feedback to benefit your learning.

7. Please make sure you do your best throughout the semester, knowing the grading scheme and what's expected of you, and come talk to me if you need further study strategies. I will be happy to brainstrom ideas to help you maximize your study strategies and improve your mathematical understanding. I will NOT offer any additional extra credit at the end of the semester or any other way for you to improve your grade at that time. No exceptions. Please respect this and do not ask for special favors or extra credit when you realize you do not like your grade. Most likely, I just will not respond to such emails or questions in person.

8. **Don't be afraid to ask questions!** Most of the time, there might be at least 8 other students who have the same questions as you. You are encouraged to speak to me immediately after the class about any questions concerning the course materials, although I very much prefer you to do that during the class, as this will benefit the entire classroom.

Disclaimer

This syllabus is not a binding legal contract. I reserve the right to make changes as I see fit at any time, but all adjustments will be announced.

Tentative Course Schedule, Fall 2018

Week	Section	Topic
Week 1 (August 20 - 24)	1.1	Introduction to Limits
	1.2	Rigorous Study of Limits
	1.3	Limit Theorems
		Lab: Algebra Review
Week 2 (August 27 - 31)	0.7	Trigonometric Functions
	1.4	Limits Involving Trigonometric Functions
	1.5	Limits at Infinity; Infinite Limits
		Lab: Basics of Limits
Week 3 (September 3 - 7)		NO CLASS Monday September 3
	1.6	Continuity of Functions
	2.1	Two Problems with One Theme
	2.2	The Derivative
		Lab: Limits and Infinities
Week 4 (September 10 - 14)	2.3	Rules for Finding Derivatives
		REVIEW in class
		Lab: Midterm 1 Review
		******Midterm 1 (Friday, September 14)******
Week 5 (September 17 - 21)	2.4	Derivatives of Trigonometric Functions
	2.5	The Chain Rule
	2.6	Higher-Order Derivatives
		Lab: Derivative as a Limit
Week 6 (September 24 - 28)	2.7	Implicit Differentiation
	2.8	Related Rates
	2.9	Differentials and Approximations
		Lab: Derivative Rules
Week 7 (October 1 - 5)	3.1	Maxima and Minima
	3.6	The Mean Value Theorem for Derivatives
	3.2	Monotonicity and Concavity
		Lab: Linearization and Differentials
(October 8 - 12)		FALL BREAK
Week 8 (October 15 - 19)	3.3	Local Extrema and Extrema on Open Intervals
		REVIEW in class
		Lab: Midterm 2 Review
		******Midterm 2 (Friday, October 19)******

7

Week	Section	Topic
Week 9 (October 22 - 26)	3.5	Graphing Functions Using Calculus
	3.4	Practical Problems
	3.7	Solving Equations Numerically
		Lab: Optimization
Week 10 (October 29 - November 2)	3.8	Antiderivatives
	3.9	Introduction to Differential Equations
	4.1	Introduction to Area
		Lab: Graphing Functions and Mean Value Theorem
Week 11 (November 5 - 9)	4.2	The Definite Integral
	4.3	The First Fundamental Theorem of Calculus
	4.4	The Second Fundamental Theorem of Calculus and the Method of Substitution
		Lab: Antiderivatives and Applications
Week 12 (November 12 - 16)	4.5	The Mean Value Theorem for Integrals and the Use of Symmetry
	4.6	Numerical Integration
		REVIEW in class
		Lab: Midterm 3 Review
		******Midterm 3 (Friday, November 16)******
Week 13 (November 19 - 23)	5.1	The Area of a Plane Region
	5.2	Volumes of Solids: Slabs, Disks, Washers
		NO CLASS Friday November 23
		Lab: None
Week 14 (November 26 - 30)	5.3	Volumes of Solids of Revolutions: Shells
	5.4	Length of a Plane Curve
		Lab: Applications of Integration
Week 15 (December 3 - 6)	5.5	Work and Fluid Force
		REVIEW in class
		Lab: Final Exam Review
Week 17		******Final Exam (Wednesday, December 12, 1:00-3:00PM)******

1.2 Lecture notes

I provide my students partial lecture notes outlining key concepts and information, and blank spaces so they can fill in the details during lecture. This saves students time from writing definitions and theorems and prompts active student listening and participation. It is not a guided lecture notes, rather it is a narrative lecture notes: I write the lecture notes with the intention of taking my students through the journey of learning calculus from my perspective when they revisit the notes for future studying. Included here is the lecture note about Limits.

1 Limits

Let me start with a confession: This is my first time teaching calculus, so you are in for a treat. Before we begin our 16-weeks-full-of-uncertainty-Calculus-journey, I have a few requests for you.

1. **Be attentive**: Put away your phone once you step into the classroom and give me your 50 minutes of undivided attention.

2. **Question authority**: I want you to feel empowered to question me when something is wrong, don't just sit around and let me shove definitions and theorems onto you.

1.1 Introduction to Limits

Among all the forthcoming sections, this section is the most important one in this course. Chapter 1 is all about limits and continuous functions but it really means nothing to you now. I spent the whole summer (ok, maybe just the last few weeks) thinking how to introduce limits to you, until recently when I read this on a blog: Why on Earth would anyone care about the following definition

$$\text{To say that } \lim_{x\to c} f(x) = L \text{ means that}$$
$$f(x) \text{ approaches } L \text{ as } x \text{ approaches } c.$$

without any proper motivation? The reason is actually pretty simple: There is a rich set of practical problems that can only be solved using differential and integral calculus, and these branches of mathematics originated from the understanding of limits itself.

Example 1.1. Consider the function $f(x) = 5x + 2$.

1. What is the value of $f(x)$ at $x = 1$? This is simply asking you to find what is $f(1)$:

   $f(1) =$

2. What is happening to $f(x)$ as x approaches 1? This is asking about the behaviour of $f(x)$ when x is getting close to or near the number 1.

The previous example seems to suggest that we may simply plug in the x-value into the function and arrive at the correct limit. This is only true in certain cases so quickly snap yourself out of this toxic thought!

Example 1.2. Evaluate the following limits.

(a) $\lim_{x \to -3} (x^2 + 1)$

(b) $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$

Example 1.3. Suppose we want to find $\lim_{x \to 0} \left(x^2 - \frac{\cos(x)}{10000} \right)$. It is not straightforward to graph this function by hand, and clearly it does not simplify like the previous example. Well, we can use a calculator to guess the limit.

If there is no one value that $f(x)$ approaches as x gets closer to c, then we say that the limit of $f(x)$, as x approaches c, does not exist. We can write this as

$$\lim_{x \to c} f(x) \text{ does not exist or “DNE” for short.}$$

There are different ways in which a limit does not exist.

Example 1.4. Explain why the following limits do not exist.

(a) $\lim_{x \to 0} \frac{1}{x^2}$

(b) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

One-sided limits

Let us examine the definition of limit again:

To say that $\lim_{x\to c} f(x) = L$ means that $f(x)$ approaches L as x approaches c.

1. Do we need $f(x)$ to be defined at $x = c$?
2. Suppose $f(x)$ is defined at $x = c$, *i.e.* we can actually compute $f(c)$. Would knowing $f(c)$ be helpful in evaluating the limit?
3. We need to look at values of x near c, but this includes both values of $x < c$ and $x > c$.

♣ We write $\lim_{x\to c^-} f(x) = L$ and say "the **left-hand limit** of $f(x)$, as x approaches c, is L" if $f(x)$ is close to the value L whenever x approaches c from the left.

♣ We write $\lim_{x\to c^+} f(x) = L$ and say "the **right-hand limit** of $f(x)$, as x approaches c, is L" if $f(x)$ is close to the value L whenever x approaches c from the right.

Thus, in order for the limit to exist and equal L, $f(x)$ must be getting closer to L as x approaches c both from the left and from the right:

$$\boxed{\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L.}$$

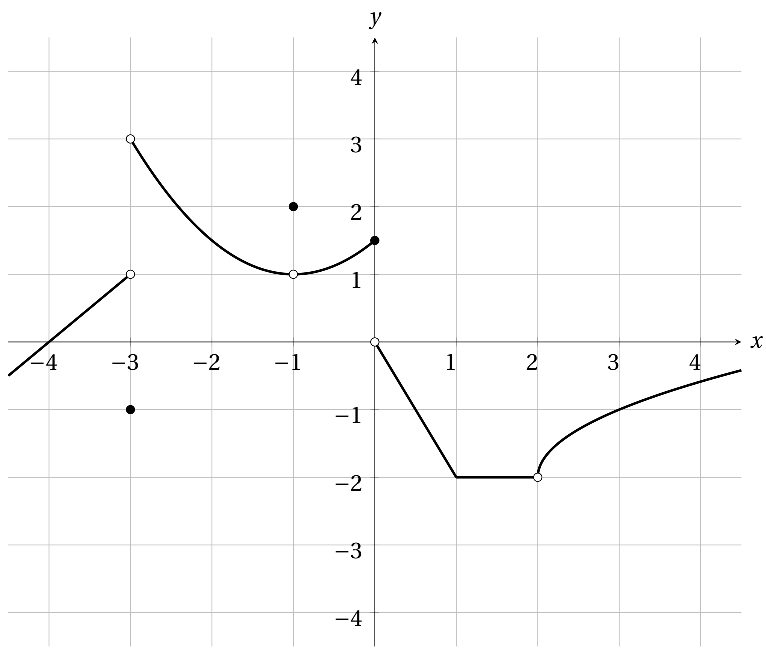
Example 1.5. Compute the following limits, if they exist.

(a) $\lim_{x\to 0^+} \frac{|x|}{x}$

(b) $\lim_{x\to 0^-} \frac{|x|}{x}$

(c) $\lim_{x\to 0} \frac{|x|}{x}$

Example 1.6. For the function f graphed below, find the indicated limit or function value, or state that it does not exist.



(a) $f(-3) =$

(b) $\lim\limits_{x\to-3^-} f(x) =$

(c) $\lim\limits_{x\to-3^+} f(x) =$

(d) $\lim\limits_{x\to-3} f(x) =$

(e) $f(-1) =$

(f) $\lim\limits_{x\to-1^-} f(x) =$

(g) $\lim\limits_{x\to-1^+} f(x) =$

(h) $\lim\limits_{x\to-1} f(x) =$

(i) $f(0) =$

(j) $\lim\limits_{x\to0^-} f(x) =$

(k) $\lim\limits_{x\to0^+} f(x) =$

(l) $\lim\limits_{x\to0} f(x) =$

(m) $f(2) =$

(n) $\lim\limits_{x\to2^-} f(x) =$

(o) $\lim\limits_{x\to2^+} f(x) =$

(p) $\lim\limits_{x\to2} f(x) =$

1.2 Rigorous Study of Limits

We state the ε-δ definition of limit: To say that $\lim_{x\to c} f(x) = L$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending on ε) such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Example 1.7. Prove using the ε-δ definition that $\lim_{x\to 3}(4x - 5) = 7$.

1.3 Limit Theorems

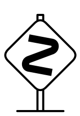
This section contains important theorems that are practical in evaluating limits, in the sense that no graphing is required and only algebraic manipulation is involved. These are going to be your best friends for the next few weeks. A wise man named Franco once said:

You need to recite these theorem every night before you go to bed

(A) Main Limit Theorem (Limit Laws)

Let n be a positive integer, K a constant and f and g be functions that have limits at c. Then

1. $\lim\limits_{x\to c} K = K$. [The limit of a constant is just the constant itself]
2. $\lim\limits_{x\to c} x = c$. [The limit of x, as x tends to c, is just c]
3. $\lim\limits_{x\to c} kf(x) = k\lim\limits_{x\to c} f(x)$. [Constants pull out of limits]
4. $\lim\limits_{x\to c} \big[f(x)+g(x)\big] = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$. [The limit of a sum is the sum of the limits]
5. $\lim\limits_{x\to c} \big[f(x)-g(x)\big] = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x)$. [The limit of a difference is the difference of the limits]
6. $\lim\limits_{x\to c} \big[f(x)g(x)\big] = \Big(\lim\limits_{x\to c} f(x)\Big)\Big(\lim\limits_{x\to c} g(x)\Big)$. [The limit of a product is the product of the limits]
7. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$, provided $\lim\limits_{x\to c} g(x) \neq 0$. [The limit of a quotient is the quotient of the limits, provided we are not dividing by zero]
8. $\lim\limits_{x\to c} \big[f(x)\big]^n = \Big[\lim\limits_{x\to c} f(x)\Big]^n$. [The limit of a power is the power of the limit]
9. $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to c} f(x)}$, provided $\lim\limits_{x\to c} f(x) > 0$ when n is even. [The limit of the nth root is the nth root of the limit, provided the latter is defined]



Before you even think about using any of these limit laws, always remind yourself to check and see if both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist.

Example 1.8. Use the limit laws to evaluate the following limits. Be sure to indicate which limit laws you are using at each step.

(a) $\lim_{x\to 1}(5x+2)$

(b) $\lim_{x\to 3}(x^3-9x)$

(c) $\lim_{x\to 0}\sqrt{x^2+9}$

(d) $\lim_{x\to 2}\left(\frac{x^2+4}{x-1}\right)^3$

Albeit sophisticated and abstract, it is important that you are shown the proof of the limit laws, with the hope that you can truly appreciate the practical nature of the theorem.

Proof of Limit Law 4. Since f and g both have limits at $x = c$, this means that there exist finite numbers L and M such that

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M.$$

From the ε-δ definition, this means that given an $\varepsilon > 0$, we can find $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} 0 < |x-c| < \delta_1 &\implies \left|f(x) - L\right| < \frac{\varepsilon}{2} \\ 0 < |x-c| < \delta_2 &\implies \left|g(x) - M\right| < \frac{\varepsilon}{2} \end{aligned}$$

Choose $\delta = \min\{\delta_1, \delta_2\} > 0$. This means that $0 < |x-c| < \delta$ implies both $0 < |x-c| < \delta_1$ and $0 < |x-c| < \delta_2$. Consequently,

$$\begin{aligned} \left|f(x) + g(x) - (L+M)\right| &= \left|\left[f(x) - L\right] + \left[g(x) - M\right]\right| \\ &\le \left|f(x) - L\right| + \left|g(x) - M\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

In particular, we have the following implication:

$$0 < |x-c| < \delta \implies \left|(f(x) + g(x)) - (L+M)\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that

$$\lim_{x\to c}\left[f(x) + g(x)\right] = L + M = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

$\square$

B Direct Substitution

If f is a polynomial or rational function, then

$$\lim_{x\to c} f(x) = f(c)$$

provided $f(c)$ is defined. In the case of a rational function, this means that the value of the denominator at c is NOT ZERO.

C "Cancellation is Fine"

If $f(x) = g(x)$ for all x in an open interval containing the number c, except possibly at the number c itself, and if $\lim_{x\to c} g(x)$ exists, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

Example 1.9. Find the following limits, if they exist.

(a) $\lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h}$

(b) $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

(c) $\lim_{x \to -3} \frac{x^2 + 5x + 6}{x^2 - 9}$

(d) $\lim_{x \to 2} \frac{x^2 - 9}{(x-2)^2}$

Example 1.10. Consider the piecewise function

$$f(x) = \begin{cases} 1-x & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 1 \\ 2-x & \text{if } x > 1 \end{cases}.$$

Find the following limits, if they exist.

(a) $\lim\limits_{x\to 0^-} f(x)$

(b) $\lim\limits_{x\to 0^+} f(x)$

(c) $\lim\limits_{x\to 0} f(x)$

(d) $\lim\limits_{x\to 1^-} f(x)$

(e) $\lim\limits_{x\to 1^+} f(x)$

(f) $\lim\limits_{x\to 1} f(x)$

D Squeeze Theorem

Let f, g, and h be functions satisfying

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \text{ near } c \text{, except possibly at } c.$$

If $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Example 1.11. Use the Squeeze Theorem to compute $\lim\limits_{x\to 0} x^2\cos(x)$.

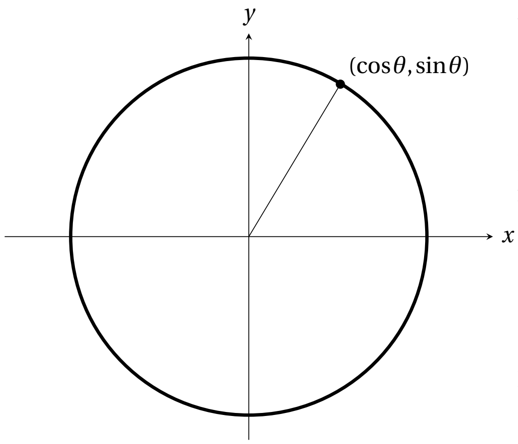
Example 1.12. Assume we know that

$$1 - \frac{x^2}{6} \leq \frac{\sin(x)}{x} \leq 1$$

for all x near but different from 0. Use the Squeeze Theorem to compute $\lim_{x\to 0} \frac{\sin(x)}{x}$.

1.4 Limits Involving Trigonometric Functions

Unit circle and right triangles



It follows from the equation of the unit circle that $x^2 + y^2 = 1$. Substituting $x = \cos\theta$ and $y = \sin\theta$ then yields

$$\cos^2\theta + \sin^2\theta = 1.$$

Let us introduce four additional trigonometric functions:

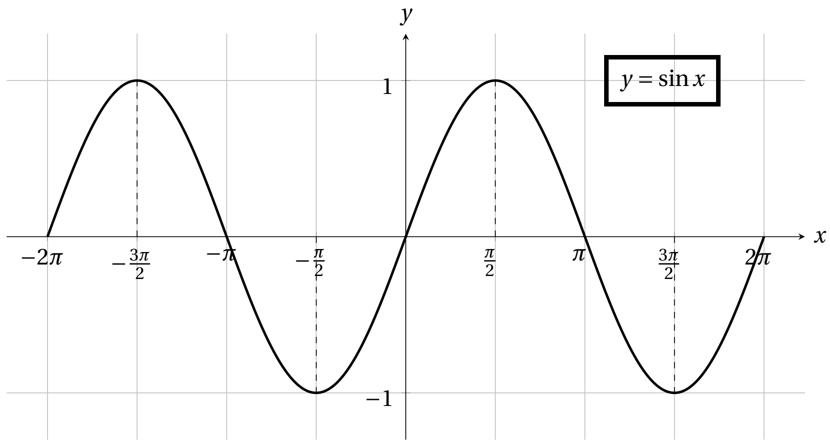
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

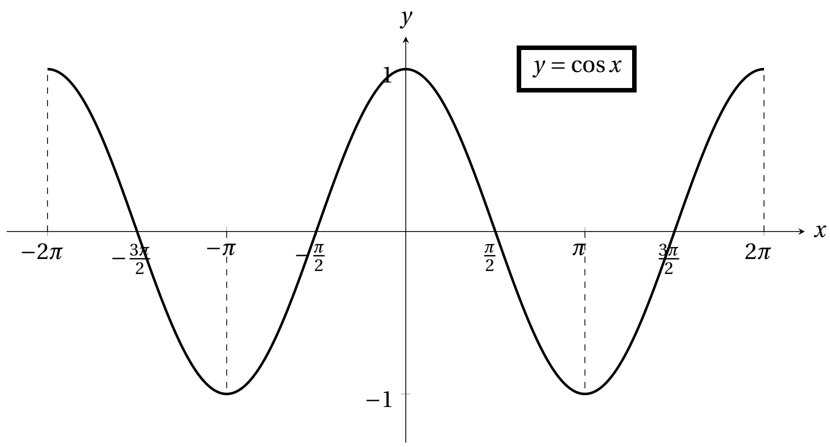
Dividing the identity $\cos^2\theta + \sin^2\theta = 1$ by either $\cos^2\theta \neq 0$ or $\sin^2\theta \neq 0$ then yields

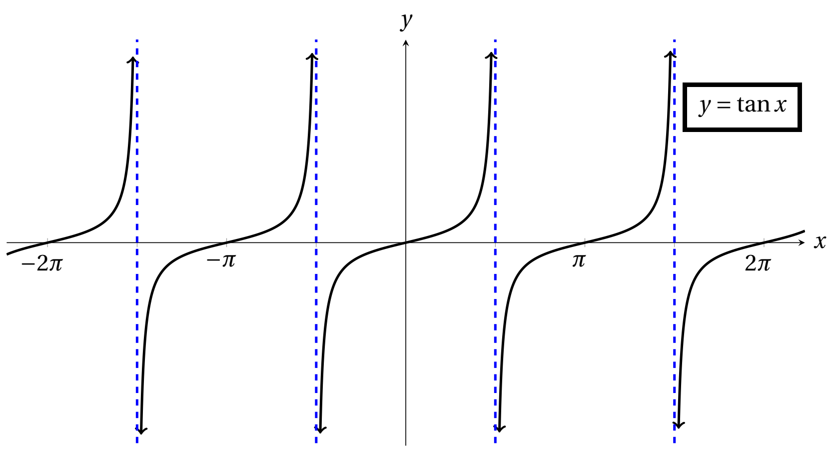
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			
$\frac{\pi}{4}$			
$\frac{\pi}{6}$			
$\frac{\pi}{3}$			

Graphs of sine, cosine and tangent







Example 1.13. Find the periods and amplitudes and sketch the graphs of the following trigonometric functions.

(a) $y = 3\sin(2\pi x)$

(b) $y = 5\cos(x+\pi) - 2$

Limits of trigonometric functions

If $t = c$ is in the domain of the trigonometric function, then we have

1. $\lim_{t\to c} \sin(t) = \sin(c)$
2. $\lim_{t\to c} \cos(t) = \cos(c)$
3. $\lim_{t\to c} \tan(t) = \tan(c)$
4. $\lim_{t\to c} \cot(t) = \cot(c)$
5. $\lim_{t\to c} \sec(t) = \sec(c)$
6. $\lim_{t\to c} \csc(t) = \csc(c)$

Example 1.14. Evaluate the following limits.

(a) $\lim_{t\to\pi} (t \sec t - \tan t)$

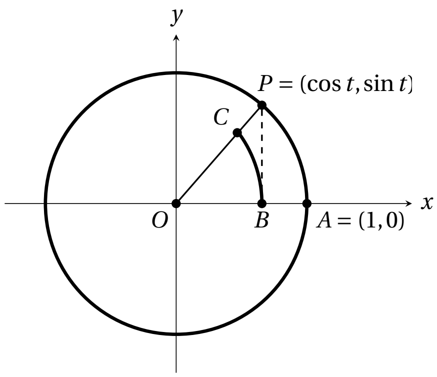
(b) $\lim_{t\to\pi/2} \dfrac{t + t^2 \cos t}{\sin t}$

(c) $\lim_{t\to 0} \dfrac{\sin^2 t}{1 - \cos t}$

Special trigonometric limits

$$\boxed{\lim_{t\to 0}\frac{\sin t}{t} = 1} \quad \text{and} \quad \boxed{\lim_{t\to 0}\frac{1-\cos t}{t} = 0}$$

Proof:



Area of sector $OBC =$

Area of triangle $OBP =$

Area of sector $OAP =$

$$\lim_{t\to 0}\frac{1-\cos t}{t} =$$

Example 1.15. Show that $\lim_{x\to 0} \dfrac{\tan x}{x} = 1$.

Example 1.16. Use the special trigonometric limits, along with some algebraic manipulation, to compute the following limits.

(a) $\lim_{t\to 0} \dfrac{\cos(3t) - \cos^2(3t)}{t}$

(b) $\lim_{t \to 0} \dfrac{\sin(2t)}{t}$

(c) $\lim_{t \to 0} \dfrac{\tan(4t)}{\sin(3t)}$

1.5 Limits at Infinity, Infinite Limits

End behaviour

In the language of College Algebra, limits at infinity means **end behaviour**. Recall that finding end behaviour of a function $f(x)$ means the following two questions:

1. What happens to $f(x)$ as x gets larger and larger in the positive direction, *i.e.* as $x \longrightarrow \infty$?
2. What happens to $f(x)$ as x gets larger and larger in the negative direction, *i.e.* as $x \longrightarrow -\infty$?

In limit notation, the first question translates to

$$\text{What is } \lim_{x\to\infty} f(x)?$$

and the second question translates to

$$\text{What is } \lim_{x\to-\infty} f(x)?$$

The crucial question is, what could possibly happen to $f(x)$ as $x \longrightarrow \infty$ (similarly, as $x \longrightarrow -\infty$)?

Example 1.17 (Almost obvious, but important)**.** Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for the following functions.

(a) $f(x) = \dfrac{1}{x}$

(b) $f(x) = \dfrac{1}{x^2}$

(c) $f(x) = \dfrac{1}{x^3}$

For positive integer $n = 1, 2, 3, \ldots$, we have the following limits:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Example 1.18. Compute the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - x + 1}{2x^2 + x}$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x+4}{x^2+1}$

(c) $\displaystyle\lim_{x\to-\infty} \frac{x^3+2}{(x^2+1)^2}$

(d) $\displaystyle\lim_{x\to\infty} \sqrt{\frac{5x^3 - x}{x^3+2}}$

(e) $\lim_{x\to\infty} \dfrac{3x^5 - x^2 - x + 1}{x^4 - 1}$

(f) $\lim_{x\to-\infty} \dfrac{3x^5 - x^2 - x + 1}{x^4 - 1}$

(g) $\lim_{x\to\infty} \dfrac{x^4 + 3x^3 - x - 1}{x^2 + 1}$

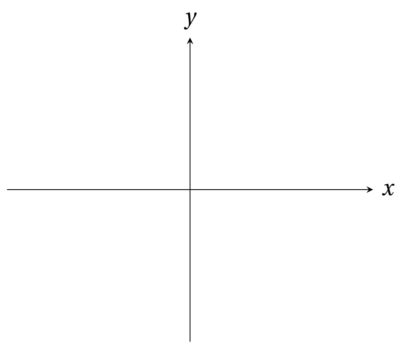
(h) $\lim_{x\to-\infty} \dfrac{x^4 + 3x^3 - x - 1}{x^2 + 1}$

One has to be extra cautious when dealing with indeterminate forms:

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0\times\pm\infty,\ \infty-\infty,\ 0^0,\ \infty^0.$$

Infinite Limits

The reciprocal function $f(x) = \dfrac{1}{x}$ is truly exquisite, simply because one can learn so much about limits just by sketching its graph. In lament terms, infinite limits mean the function grows without bound as x approaches a particular number c. To visualise what this means, let us graph the reciprocal function and examine what happens to $f(x)$ as x approaches 0:



$$\lim_{x\to 0^-}\frac{1}{x} = \qquad\qquad \lim_{x\to 0^+}\frac{1}{x} =$$

The crucial thing to observe is that it makes sense to talk about one-sided limits in this case, because x is approaching a real number c, not $\pm\infty$. More importantly, the upshot of this example is that even in the case of infinite limits, if the one-sided limits do not agree, *i.e.*

$$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x),$$

then we say that the limit does not exist. Thus in this case

$$\lim_{x\to 0}\frac{1}{x}\ \text{DNE}.$$

- ♣ $\lim_{x\to c} f(x) = \infty$ means we can make values of $f(x)$ arbitrarily large by choosing values of x sufficiently near c, but not equal to c.
- ♣ $\lim_{x\to c} f(x) = -\infty$ means we can make values of $f(x)$ arbitrarily negative by choosing values of x sufficiently near c, but not equal to c.

Example 1.19. Evaluate the following limits, answering either a number, ∞, $-\infty$, or DNE.

(a) $\displaystyle\lim_{x\to 2}\frac{x^2-9}{(x-2)^2}$

(b) $\displaystyle\lim_{x\to 0}\frac{3}{1-\cos(2x)}$

(c) $\lim_{x\to 1} \frac{x-1}{x^4-2x^3+x^2}$

(d) $\lim_{x\to 0} \frac{x-1}{x^4-2x^3+x^2}$

Definition 1.20. A function has a vertical asymptote at $x = c$ if

$$\lim_{x \to c^-} f(x) = \pm\infty \quad \text{and/or} \quad \lim_{x \to c^+} f(x) = \pm\infty.$$

Example 1.21. Find all the vertical asymptotes of $f(x) = \dfrac{x^3 + x}{x^2 - x}$.

1.6 Continuity of Functions

Continuity at a point

Definition 1.22. Let f be defined on an open interval containing $x = c$. We say that f is **continuous** at $x = c$ if

$$\lim_{x \to c} f(x) = f(c).$$

Specifically, we say that f is continuous at $x = c$ if and only if the following conditions are satisfied:

1. $\lim_{x \to c} f(x)$ exists;
2. f is defined at $x = c$, *i.e.* $f(c)$ exists;
3. $\lim_{x \to c} f(x) = f(c)$.

If any of these three conditions fails, then we say that f is **discontinuous** at $x = c$, or that f has a **discontinuity** at $x = c$.

Types of discontinuities

Let us examine cases where a function fails to satisfy one of the three conditions above.

Continuity of Polynomial, Rational and Trigonometric Functions

Polynomials are continuous everywhere. Rational functions and trigonometric functions are continuous wherever they are defined.

Example 1.23. Consider the following piecewise-defined function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ x & \text{if } 1 \leq x < 2, \\ 4 & \text{if } x = 2, \\ x & \text{if } 2 < x < 3, \\ \dfrac{1}{x-5} & \text{if } x \geq 3. \end{cases}$$

Find all points of discontinuity of f and classify each of these as removable, jump or infinite.

Example 1.24. Consider the following piecewise-defined function

$$g(x) = \begin{cases} 0 & \text{if } x < -\pi, \\ \dfrac{\sin x}{x} & \text{if } -\pi \le x < \pi, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \ge \pi. \end{cases}$$

Find all points of discontinuity of g and classify each of these as removable, jump or infinite.

Below we summarise types of discontinuities.

1. We say that f has a **removable discontinuity** at $x = c$ if f is discontinuous at $x = c$ but can be made to be continuous at $x = c$ by simply redefining $f(c)$.
2. We say that f has a **jump discontinuity** at $x = c$ if one sided limits exist but
$$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x).$$
3. We say that f has an **infinite discontinuity** at $x = c$ if f has a vertical asymptote at $x = c$.

Jump discontinuities and infinite discontinuities are together called **non-removable** discontinuities because we cannot simply fix the discontinuity by redefining the function there.

Continuity on an interval

Continuity on an interval should mean continuity at each point of that interval, but this is problematic if we consider a closed interval, say $[a,b]$. If we consider the square root function $f(x) = \sqrt{x}$ over the interval $[0,1]$, we see that $\lim_{x\to 0} f(x)$ DNE because f is not even defined to the left of $x = 0$. We circumvent this issue by imposing "one-sided continuity" at the endpoints.

Definition 1.25. A function f is

1. **right-continuous** at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$;
2. **left-continuous** at $x = b$ if $\lim_{x\to b^-} f(x) = f(b)$.

We say that f is continuous on $[a,b]$ if it is continuous everywhere on (a,b), right-continuous at $x = a$ and left-continuous at $x = b$.

Example 1.26. State the intervals (open, closed, half open) on which the following function is continuous.

(a) $$f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ x & \text{if } 1 \le x < 2, \\ 4 & \text{if } x = 2, \\ x & \text{if } 2 < x < 3, \\ \dfrac{1}{x-5} & \text{if } x \ge 3. \end{cases}$$

(b) $$g(x) = \begin{cases} 0 & \text{if } x < -\pi, \\ \dfrac{\sin x}{x} & \text{if } -\pi \le x < \pi, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \ge \pi. \end{cases}$$

Intermediate value theorem

Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$. Suppose M is any number between $f(a)$ and $f(b)$. Then there exists a point c with $a < c < b$ such that $f(c) = M$.

Example 1.27. Use the IVT to show that the polynomial $f(x) = x^5 + x - 1$ must have at least one root in the interval $(0, 1)$.

Example 1.28. Show that the equation $x\cos x = -2$ has at least one solution in the interval $\left(\frac{\pi}{2}, \pi\right)$.

1.3 Quiz solution

Immediately after students submit their quizzes, I give everyone a copy of the detailed solutions so they can verify their answer and ask me questions on the spot to clear up any confusion they might have about the relevant topics.

Math 1210-018: Calculus I, Quiz #5, Week 6, Fall 2018

SOLUTION

1. Find the equation of the tangent line at the point $(1, -1)$ to the curve

$$y^2 + 5xy = x^2 - 5.$$

Solution: We use implicit differentiation, specifically, we differentiate implicitly with respect to x and use the Chain Rule. Together with the Product Rule, we obtain

$$\begin{aligned} D_x\left[y^2\right] + 5D_x[xy] &= D_x\left[x^2 - 5\right] \\ 2y\frac{dy}{dx} + 5\Big(xD_x[y] + D_x[x]\,y\Big) &= 2x \\ 2y\frac{dy}{dx} + 5\left(x\frac{dy}{dx} + y\right) &= 2x. \end{aligned}$$

At the point $(1, -1)$,

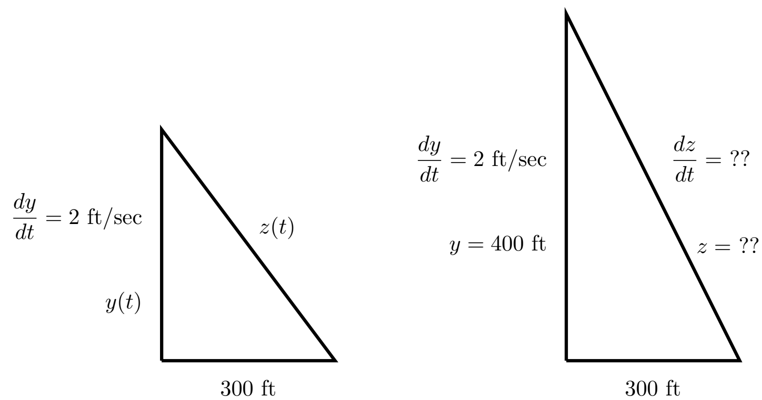
$$\begin{aligned} -2\frac{dy}{dx} + 5\left(\frac{dy}{dx} - 1\right) &= 2 \\ 3\frac{dy}{dx} - 5 &= 2 \\ \frac{dy}{dx} &= \frac{7}{3}. \end{aligned}$$

Hence the equation of the tangent line to the given curve at the point $(1, -1)$ is

$$y + 1 = \frac{7}{3}(x - 1).$$

2. A hot air balloon is rising vertically at a rate of 2 ft/sec. An observer is located 300 ft from a point on the ground directly below the balloon. At what rate is the distance between the balloon and the observer changing when the height of the balloon is 400 ft?

Solution: **We always start by drawing a diagram describing the scenario!**



Let

$$\begin{aligned} y(t) &= \text{ the height of the balloon at any time } t \\ z(t) &= \text{ the distance between the balloon and the observer at any time } t \end{aligned}$$

We were told that $\frac{dy}{dt} = 2$ ft/sec at any time $t > 0$. The equation relating $y(t)$ and $z(t)$ comes from the Pythagorean theorem:

$$(300)^2 + [y(t)]^2 = [z(t)]^2 .$$

Differentiating implicitly with respect to t and using the Chain Rule, we obtain

$$2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

It remains to find z at the particular instant when the height of the balloon is $y = 400$ ft. From the Pythagorean theorem,

$$300^2 + 400^2 = z^2 \implies z^2 = 250000 \implies z = 500 \text{ ft}.$$

Finally, we substitute all the known values $y, \frac{dy}{dt}, z$ and solve for $\frac{dz}{dt}$:

$$\begin{aligned} 2\,(400)\,(2) &= 2\,(500)\left(\frac{dz}{dt}\right) \\ 1600 &= 1000\frac{dz}{dt} \\ \frac{dz}{dt} &= \frac{1600}{1000} = \frac{16}{10} = 1.6 \text{ ft/sec}. \end{aligned}$$

Page 2

1.4 Practice exam solution and knowledge checklist

I usually provide my students a practice exam and detailed solutions so they have an expectation of what a "good" solution should be for the exam. I also give them an exam knowledge checklist so they can identify their weak areas and work on it when they review for the exam.

Math 1210-018: Calculus I

Midterm 2 - PRACTICE EXAM, Version 2

Read this instructions carefully before you begin.

1. This is a closed-book, closed-note exam. No phones or calculators are allowed.
2. Answer each question completely and write legibly; otherwise you place yourself at a grave disadvantage.
3. Show all your work and explain your reasoning when necessary, as partial credit will be given where appropriate. No credit will be given if the answer has no supporting work or if the work is at all ambiguous.
4. Point values are in the square to the left of the question.
5. You may ask for scratch paper, but please transfer all finished work onto the proper page in the test. I will not grade the work on the scratch paper.
6. You have 55 minutes to complete the exam. Think clear, stay calm, and **circle your final answer**.

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name : ______________________________

uID : ______________________________

Signature : ______________________________

Do not turn this page until you are told to do so.

1. Compute the following derivatives. **You don't need to simplify your answer.**

   (a) $D_x \left[x \tan x \right]$

   (b) $D_x \left[\dfrac{x^3 - 1}{x^2 + 3x - 2} \right]$

   (c) $D_x \left[x^2 - \dfrac{5}{x} \right]$

   (d) $D_x \left[\cos(\sin(2x)) \right]$

2. Consider the curve defined by the following equation

$$x^{1/3} + y - y^2 = 2.$$

   (a) Find $\dfrac{dy}{dx}$ by means of implicit differentiation.

   (b) Find the equation of the tangent line to the given curve at the point $(8, 1)$.

3. A ladder 20 feet long leans against a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 2 ft/sec, how fast is the ladder sliding down the building when the top of the ladder is 12 feet above the ground?

4. Consider the function $f(x) = \sin^2(x)$.

   (a) Find the linear approximation to $f(x)$ at $x = \dfrac{\pi}{4}$.

   (b) Use part (a) to approximate $\sin^2\left(\dfrac{\pi}{3}\right)$.

5. Find the value of c guaranteed by the Mean Value Theorem for the function $f(x) = (x+1)^3$ on the interval $[-1, 1]$.

6. The strength of gravity on the moon is about 1/6 of that on the Earth. An astronaut jumps on the moon. His height (in feet off of the moon's surface) after t seconds is given by the equation

$$h(t) = -2.5t^2 + 10t.$$

At what time does the astronaut reach his maximum height?

7. Consider the function $f(x) = \dfrac{x^2}{x^2+3}$.

   (a) Find the critical points of $f(x)$.

   Critical points: ____________________

   (b) Determine the intervals where $f(x)$ is increasing or decreasing.

   Increasing: ____________________

   Decreasing: ____________________

(c) Determine the intervals where $f(x)$ is concave up or concave down.

Concave up: ______________________________

Concave down: ______________________________

(d) Find all inflection points of $f(x)$. Explain your answer.

Points of inflection: ______________________________

Math 1210, Fall 2018
Midterm 2 Review

Please read this carefully before you proceed

Midterm 2 will cover Sections 2.3-2.9, 3.1, 3.6, 3.2. You are expected to know materials from these sections covered in class, covered in the textbook and from the homeworks. Below you will find a rough outline of concepts and topics from each section. This review is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. If something is not on this outline, that does not mean it will not be on the exam!

Section 2.3 (Rules for Finding Derivatives)

- Know the basic rules for taking derivatives, including Power Rule, Product Rule, Quotient Rule.

Section 2.4 (Derivatives of Trigonometric Functions)

- Know the derivatives of $\sin x$, $\cos x$, and $\tan x$.

Section 2.5 (The Chain Rule)

- Know how to apply the Chain Rule and when to apply it.
- Know that the Chain Rule can be applied multiple times. Specifically, if we have composition of more than two functions, then we can still apply the Chain Rule.

Section 2.6 (Higher-Order Derivatives)

- Know how to take multiple derivatives of a function.
- Understand the meaning of the notation $f''(x)$ or $\dfrac{d^2y}{dx^2}$, $f'''(x)$ or $\dfrac{d^3y}{dx^3}$, and so on.
- Be able to distinguish the difference between $\dfrac{d^2y}{dx^2}$ and $\left(\dfrac{dy}{dx}\right)^2$, for example.
- Know how the physics of moving objects relates to derivatives. Two important applications are "Velocity and Acceleration" and "Falling-Body Problems".

Section 2.7 (Implicit Differentiation)

- Understand what it means to differentiate a given equation of x and y implicitly.
- Be able to algebraically solve for $\dfrac{dy}{dx}$ after applying implicit differentiation.
- Be able to find the equation of the tangent line to a given curve (which is an implicit equation) at a given point.

Section 2.8 (Related Rates)

- Understand that related rates are an application of implicit differentiation, and it concerns about time rate of change, *i.e.* derivative with respect to time.
- Be able to solve related rates problems. Take a look at the plan of attack from the lecture.

Section 2.9 (Differentials and Approximations)

- Know how to find a linear approximation to a given function at a given value $x = c$.
- Know how to estimate values of a function near a given input $x = c$ using linear approximation.
- Know the differential expression $dy = f'(x)dx$ and how to find them.
- Know how to estimate changes of a function using differentials.
- Understand the difference between Δy (exact change of outputs) and dy (approximate change of outputs).
- Be able to recognise when to use linear approximation or differentials to solve a problem.
- Know how to find absolute error.

Section 3.1 (Maxima and Minima)

- Know what a global maximum value and global minimum value (these values are called extreme values) are and how to find them.
- Know what a critical point is and how to find critical points. There are three kinds of critical points:
  - endpoints;
  - stationary points, *i.e.* points where $f'(c) = 0$;
  - singular points, *i.e.* points where $f'(c)$ DNE.
- Understand the difference between extreme values (y-values)$andcriticalpoints$(x$-values$).

Section 3.6 (The Mean Value Theorem for Derivatives)

- Know what the Mean Value Theorem is and how and when to apply it.

Section 3.2 (Monotonicity and Concavity)

- Know how to determine and find intervals of increasing, decreasing, concave up, and concave down for a given function.
- Understand how the question above relates to the first derivative and the second derivative of the function.
- Know what an inflection point is (points where a function changes concavity), and how to find such points.
- Know how to handle points where $f'(x) = 0$ or DNE, and points where $f''(x) = 0$ or DNE.

Review Problems

Below are suggested problems for you as you prepare for the exam. I recommend re-doing all the assigned homework problems and quizzes if you need more practice. Some of them are quite difficult (too difficult to even put on an exam) so don't worry if they are a struggle.

Chapter 2 review

- Section 2.3: 1-46 even, 50, 52
- Section 2.4: 2, 4, 10, 12, 14, 16, 20, 26, 28
- Section 2.5: 1-40 even, 48, 50, 52, 62, 64, 66
- Section 2.6: 1-16 even, 24, 26, 30, 34, 36
- Section 2.7: 1-32 even
- Section 2.8: 2, 4, 6, 8, 10, 12, 20
- Section 2.9: 2, 4, 6, 8, 22, 37-44 even (only the linear approximation)
- Concepts Test Problems (pp 147-148): 17, 18, 19, 20, 21, 22, 23, 25, 27, 28, 29, 32, 33, 36
- Sample Test Problems (pp 148-149): 1f, 1h, 5-29, 35-41, 43, 44, 46, 47, 50

Chapter 3 review

- Section 3.1: 1-26 even
- Section 3.6: 1-18 even
- Section 3.2: 1-36 even
- Concepts Test Problems (pp 209-210): 1-10, 17, 18, 22, 24, 26, 27
- Sample Test Problems (pp 210-212): 1-19 odd, 45

2 Math 2250 – Differential Equations and Linear Algebra

Last semester I decided to collaborate with one of my peers and flip our Math 2250 classroom. For each section a short video, quiz, and worksheet will be posted prior to the Zoom lectures. During lecture, students will be able to ask questions on the work they have done at home, work on the day?s assignment using the breakout rooms feature in Zoom, and prepare for the next topic. For each lecture video, student will complete a short multiple choice quiz on Canvas. Students will also turn in weekly homework electronically via Gradescope.

2.1 Lecture Notes

We provide students partial lecture notes so they can fill in the details while watching the video lectures. Included here is the lecture notes about first-order differential equations.

§1.1: 1st-Order Differential Equations

What is a differential equation?

- Algebraic Equation

  Ex) $2x + 3 = 5$

- Differential Equation

  Ex) $2y'' + y' - 3y + x = 0$

General Form of Differential Equations

An ***n*th-order** differential equation with independent variable x and dependent variable $y = y(x)$ takes the form

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0. \qquad (\bigstar)$$

1. The **order** of a DE is the order of the highest derivative that appears in it.
2. We say that $y(x)$ is a solution of the DE $(\bigstar)$ if it satisfies $(\bigstar)$ on some interval I.

Example 1.1. Which of the following are differential equations? If it is a differential equation, find its order.

(a) $x^2 + 2x = 4$

(b) $x'' + t(x')^2 + x = \sin(t)$

(c) $P' = kP$

(d) $y'' + 4y' - 2$

1

Why study differential equations?

- ♣ Trajectory of a ball
- ♣ Chemical reactions
- ♣ Models of infected populations
- ♣ Motions of bodies in space or in fluid

The goal is to formulate a mathematical model of real-world problems, analyze the mathematical model, interpret the results in the context of the real world problem, and then update the model if needed.

Remark 1.1. When modeling something that changes with more than one independent variable (*i.e.* space AND time), we may use a PDE. A classical PDE is the *heat equation* which models the temperature $u(x,t)$ in a perfectly insulated rod.

However, this is beyond the scope of this class. In this class, we will deal with ODEs only, therefore solutions can be $u(x)$ OR $u(t)$, but not $u(x,t)$.

Example 1.2 (Newton's Law of Cooling)**.** The time rate of change of the temperature $T(t)$ of an object is proportional to the difference between its own temperature T and the surrounding temperature A.

Example 1.3 (Population Model).

- Suppose the time rate of change of population $P(t)$ is proportional to the population size.

- Verify that $P(t) = Ce^{kt}$ is a solution to the differential equation for any constant C.

- *Can we constrain to only one solution?* This leads to the **Initial Value Problem (IVP)**, which is a differential equation coupled with an **initial condition**. In this case, if we impose the condition $P(0) = 1000$ we can solve the following IVP which has a unique solution!

$$\begin{cases} \dfrac{dP}{dt} = kP \\ P(0) = 1000 \end{cases}$$

The input (dependent variable) for an initial condition doesn't need to be 0. For example, $P(10) = 1000$ is also an initial condition.

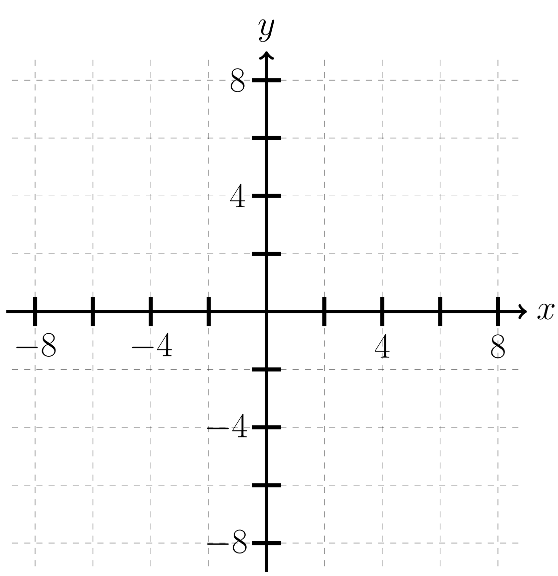
Remark 1.2. We still have the parameter k to solve for. To solve for k we need additional information, such as $P(1) = 3000$ for example. Then

§1.2: Integrals as General and Particular Solutions

Example 1.4. Solve the following initial value problem (IVP).

$$\begin{cases} \dfrac{dy}{dx} = 3x - 1 \\ y(2) = 6 \end{cases}$$

Plot the general solution and the particular solution of the given IVP.



4

Directly Integrable 1st-Order DEs

In general, if the right hand side of the differential equation only depends on x (or the independent variable) then we can integrate both sides to obtain the general solution. That is

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad y(x) = \int f(x)\,dx + C.$$

Example 1.5. Solve the following differential equation.

$$\frac{d^2y}{dx^2} = x + 1.$$

Velocity and Acceleration

The motion of a particle along a straight line has

- position
$$x(t)$$
- (instantaneous) velocity
$$x'(t) = v(t)$$
- acceleration
$$x''(t) = v'(t) = a(t)$$

Newton's second law of motion states that

$$F = ma \quad \text{or} \quad F(t) = ma(t).$$

Constant Acceleration

Suppose F is constant, therefore, $a = \dfrac{F}{m}$ is constant. We set the initial velocity as $v(0) = v_0$ and solve the following IVP for $v(t)$.

$$\begin{cases} \dfrac{dv}{dt} = a \\ v(0) = v_0 \end{cases}$$

Now, we set the initial position as $x(0) = x_0$ and solve the following IVP for $x(t)$.

$$\begin{cases} \dfrac{dx}{dt} = v(t) \\ x(0) = x_0 \end{cases}$$

Be aware of sign and units when acceleration is gravity, since

$$g \approx 32 \text{ ft/s}^2 \approx 9.8 \text{ m/s}^2.$$

Example 1.6. A ball is thrown up from the ground ($x_0 = 0$) with initial velocity $v_0 = 96$ ft/s. What is the maximum height of the ball?

§1.3: Slope Fields and Solution Curves

Consider a differential equation of the form

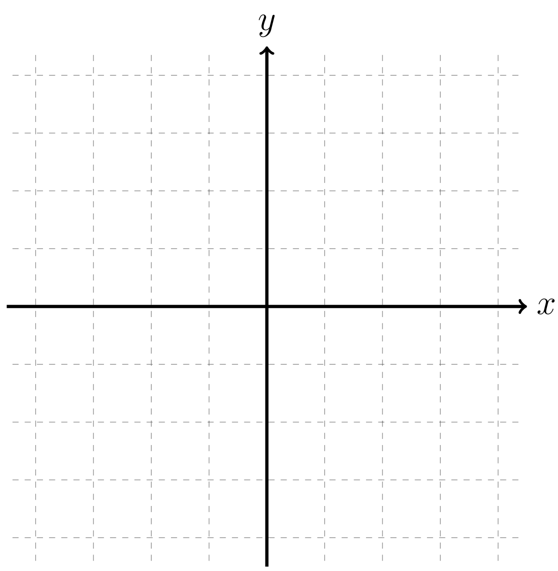
$$\frac{dy}{dx} = f(x, y).$$

Now, the right hand side depends on the unknown function $y(x)$ as well as the independent variable x, therefore we can't just integrate both sides with respect to x. What then?!

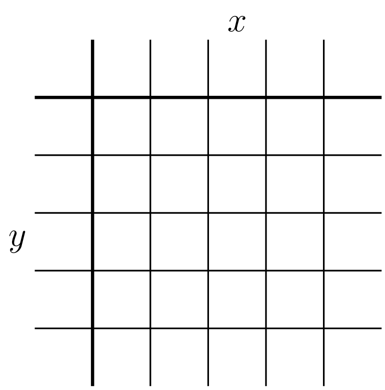
Slope Fields and Graphical Solutions

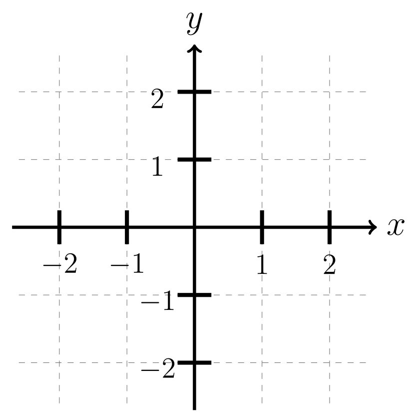
We can approach this problem graphically, by drawing a **slope field** or **direction field** and use this to plot various **solution curves**. The crucial observation is that the value of $f(x, y)$ gives us the slope of the solution $y = y(x)$ at every point (x, y).

Example 1.7. Consider the differential equation $\dfrac{dy}{dx} = -y$.



Example 1.8. Consider the differential equation $\dfrac{dy}{dx} = x - y$.

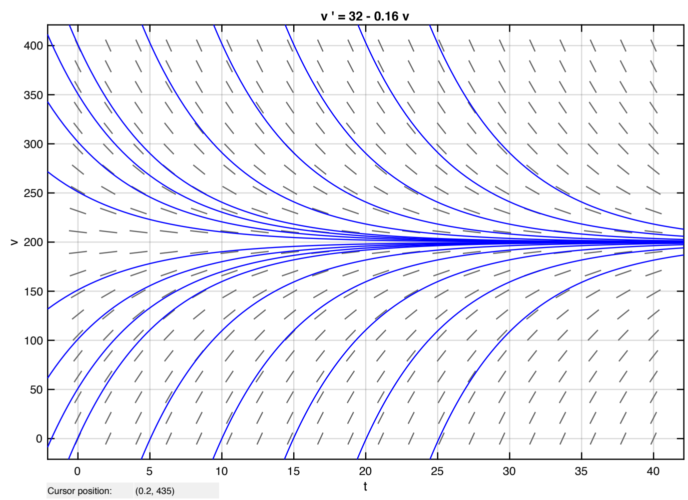




Example 1.9. Consider throwing a baseball straight down from a helicopter at a speed of v_0. We may assume that the acceleration due to air resistance is proportional to velocity.

- Write a differential equation describing the velocity of the ball.

- Since we don't yet know how to solve this, we will create a slope field, plot several solution curves, and use this to analyze the qualitative properties of the model.



- To plot this I used a MATLAB program called dfield.

Existence and Uniqueness of Solutions

In the theory of differential equations, it is important to know that solutions actually *exist* and whether there is a *unique* solution of the DE satisfying a given initial condition before we attempt to solve an IVP.

Example 1.10 (No Solution)**.** Consider the following IVP.

$$\begin{cases} y'(x) = \dfrac{1}{x} \\ y(0) = 0 \end{cases}$$

Example 1.11 (No Unique Solutions)**.** Verify that for any constant C, the function $y(x) = Cx^2$ is a solution to the following IVP.

$$\begin{cases} x\dfrac{dy}{dx} = 2y \\ y(0) = 0 \end{cases}$$

Existence and Uniqueness of Solutions of 1st-Order ODEs

Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(a) = b \end{cases}$$

1. If $f(x,y)$ is continuous near the point (a,b), then there exists at least one solution on some open interval containing $x = a$.
2. If, in addition to Condition (1), $\dfrac{\partial}{\partial y} f(x,y)$ is also continuous near the point (a,b), then there exists a unique solution on some (possibly) smaller interval containing $x = a$.

Example 1.12. Consider the following IVP.

$$\begin{cases} \dfrac{dy}{dx} = x^2 - y^2 \\ y(0) = 1 \end{cases}$$

The Existence and Uniqueness Theorem tells us NOTHING if the two conditions are not met! We call these conditions *sufficient but not necessary.*

§1.4: Separable Equations and Applications

Separable DEs

A differential equation $\dfrac{dy}{dx} = f(x, y)$ is said to be **separable** if there exists some functions $g(x)$ and $h(y)$ such that

$$\frac{dy}{dx} = f(x, y) = g(x)h(y).$$

Example 1.13. For each of the following differential equation, determine whether it is separable and if so, find $g(x)$ and $h(y)$ such that the right hand side $f(x, y) = g(x)h(y)$.

(a) $\dfrac{dy}{dx} = -x \sin y$

(b) $\dfrac{dy}{dx} = x - y$

(c) $\dfrac{dy}{dx} = e^{x-y}$

How to Solve Separable DEs?

Step 0: Check if the given differential equation $\dfrac{dy}{dx} = f(x,y)$ is separable.

Step 1: If it is separable, algebraically move all terms involving y (**including** $\boldsymbol{dy}$) to one side of the equation and all terms involving x (**including** $\boldsymbol{dx}$) to the other side of the equation.

Step 2: Integrate both sides of the equation. **Do NOT forget the constant of integration.**

Step 3: If the given problem is an IVP, solve for the constant of integration.

Step 4: If possible, solve explicitly for $y(x)$.

Note that Steps (3) and (4) are interchangeable.

Example 1.14. Solve the following IVP.

$$\begin{cases} \dfrac{dy}{dx} = -6xy \\ y(0) = -4 \end{cases}$$

Example 1.15. Solve the following IVP.

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{4-2x}{3y^2-5} \\ y(1) = 3 \end{cases}$$

Cautionary Remarks About Implicit Solutions

† **Solving an implicit solution for $y(x)$ may give you explicit functions that do NOT satisfies the initial condition of the IVP.** Consider for example the IVP

$$\begin{cases} x + y\dfrac{dy}{dx} = 0 \\ y(0) = -2 \end{cases}.$$

We verify that $x^2 + y^2 = 4$ is an implicit solution of the given differential equation:

Solving explicitly for y, we get

$$y(x) = \sqrt{4 - x^2} \quad \text{OR} \quad y(x) = -\sqrt{4 - x^2}$$

which both satisfies the differential equation. However, $y = -\sqrt{4 - x^2}$ is the only explicit solution satisfying the initial condition $y(0) = -2$.

† **It is possible to "lose" solutions when we solve an implicit solution.** To see this, consider the differential equation $\dfrac{dy}{dx} = 6x(y-1)^{2/3}$.

Note: The first version of the notes had $\dfrac{dy}{dx} = 6x(y-1)^{1/3}$, this was a typo. The exponent should indeed be 2/3 rather than 1/3.

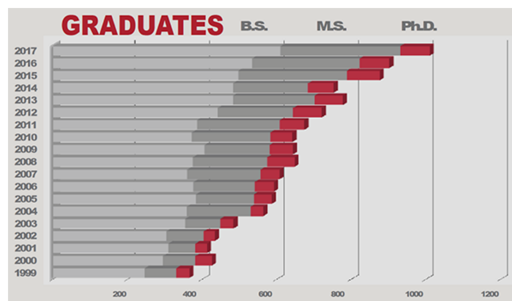
16

Exponential (Natural) Growth and Decay

Let $k > 0$ be a constant. A quantity $x(t)$ whose time rate of change is proportional to its current size satisfies

$$\begin{cases} \dfrac{dx}{dt} = kx & \text{if } x(t) \text{ is growing (increasing)}, \\ \dfrac{dx}{dt} = -kx & \text{if } x(t) \text{ is decaying (decreasing)}. \end{cases}$$

Example 1.16. In 2009 the College of Engineering at the University of Utah graduated (approximately) 400 students with Bachelor of Science, and in 2017 the College of Engineering graduated (approximately) 600 students. Assume the time rate of change of graduates is proportional to the current number of graduates. How many students will graduate from the University of Utah with Bachelor degrees in Engineering in 2021 and in 2050?



Bar graph from the College of Engineering (coe.utah.edu/college)

17

§1.5: Linear First-Order Equations

So far we have solved differential equations of the form

$$\underbrace{\frac{dy}{dx} = f(x)}_{\text{Directly Integrable}} \qquad \text{and} \qquad \underbrace{\frac{dy}{dx} = g(x)h(y)}_{\text{Separable}}.$$

The goal of this section is to solve linear first-order differential equations.

Linear 1st-Order DEs

A linear first-order differential equation has the general form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where the coefficient functions $P(x)$ and $Q(x)$ are continuous.

Example 1.17. Determine whether each of the following is a linear first-order differential equation and if so, compare it to the general form and find the corresponding $P(x)$ and $Q(x)$.

(a) $\dfrac{dy}{dx} = -6xy$

(b) $\dfrac{dy}{dx} + \cos(x)y = \sqrt{x}$

(c) $\left(\dfrac{dy}{dx}\right)^2 - y = 2$

To solve linear first-order DEs, we introduce an **integrating factor** which is the function

$$\rho(x) = e^{\int P(x)\,dx}.$$

We expand the following term:

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] =$$

Example 1.18. Let us try and use this idea to solve the following differential equation.

$$\frac{dy}{dx} = (1-y)\cos x.$$

Method of Integrating Factor

Step 0: Rewrite the first-order DE in general form $\dfrac{dy}{dx} + P(x)y = Q(x)$.

Step 1: Compute the integrating factor:

Step 2: Multiply both sides of the DE by the integrating factor:

Step 3: Recognize the left hand side of the DE as the derivative of a product:

Step 4: Integrate both sides with respect to x and solve for y to obtain the general solution:

Step 5: If the given problem is an IVP, solve for the constant of integration.

As Francis Su said, *every time you see a formula, remember it's the culmination of many repeated attempts to unlock a mystery, a triumph by each person who, like you, earnestly ask "why" and endeavors to understand its rightness for themselves.*

Example 1.19 (Mixture Problem)**.** A 120-gallon tank initially contains 90 pounds of salt dissolved in 90 gallons of water. Brine containing 2 pounds of salt per gallon of water flows into the tank at a rate of 4 gallons per minute and a well-stirred mixture flows out of the tank at a rate of 3 gallons per minute. How much salt does the tank contain when it is full?

§4.1 & §4.2: Vector Spaces and Subspaces

n-space and Vector Operations

The n-dimensional space $\mathbb{R}^n$ is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers.

1. The elements of $\mathbb{R}^n$ are called **points** or **vectors**.
2. A vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ has both **magnitude** and **direction**, and the ith entry of $\mathbf{x}$ is called its ith **component**.

Vector Addition

The sum $\mathbf{u} + \mathbf{v}$ of the two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

Example 4.1. Given $\mathbf{u} = (2, 6)$ and $\mathbf{v} = (3, -2)$, find $\mathbf{u} + \mathbf{v}$ and interpret the result geometrically.

Scalar Multiplication

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is a vector and c is a real number (**scalar**), then the **scalar multiple** $c\mathbf{u}$ is the vector

$$c\mathbf{u} = (cu_1, cu_2, \ldots, cu_n).$$

Example 4.2. Given $\mathbf{u} = (2, 1)$, find $5\mathbf{u}$ and $-3\mathbf{u}$ and interpret these results geometrically.

Length or Magnitude of a Vector

The length or magnitude of a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, denoted by $|\mathbf{u}|$, is the distance from the origin to the point $(u_1, u_2, \ldots, u_n)$.

Definition of a Vector Space

Let V be a set of elements called *vectors*. V is called a **vector space** if

- ♣ V is closed under vector addition and scalar multiplication, that is, given any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V and scalar c, the vector $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ are also inside V ("inside the vector space" means that **the result stays in the space**), and
- ♣ for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and scalars a, b, the following properties hold:

  1. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
  2. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$
  3. $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$
  4. $\mathbf{u}+(-\mathbf{u}) = (-\mathbf{u})+\mathbf{u} = \mathbf{0}$
  5. $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$
  6. $(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$
  7. $(ab)\mathbf{u} = a(b\mathbf{u})$
  8. $(1)\mathbf{u} = \mathbf{u}$

Subspaces

There are important vector spaces **inside** $\mathbb{R}^n$. Let W be a nonempty subset of the vector space V. Then W is a **subspace** of V provided W is itself a vector space with the operations of vector addition and scalar multiplication as defined in V.

Conditions for a Subspace

The nonempty subset W of a vector space V is a subspace of V if and only if it satisfies the following two conditions:

1. Closed under vector addition:
2. Closed under scalar multiplication:

Example 4.3. Examples of subspaces in $\mathbb{R}^3$ include:

- Trivial subspaces:

- Proper subspaces: all subspaces other than the trivial subspaces.
  - A line through the origin.
  - A plane through the origin.

Example 4.4. Let W be the set of all vectors (x_1, x_2) in $\mathbb{R}^2$ satisfying $x_1 = x_2$. Show that W is a subspace of $\mathbb{R}^2$.

Example 4.5. Determine if the following set W is a subspace of $\mathbb{R}^4$.

$$W = \left\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1, x_2, x_3, x_4 \geq 0\right\}$$

Consider the set of solutions to the matrix equation $\mathbf{Ax} = \mathbf{0}$, where the $m \times n$ matrix $\mathbf{A}$ can be square or rectangular. One solution is the trivial solution $\mathbf{x} = \mathbf{0}$. For invertible matrices this is the only solution but for other matrices, not invertible, there are nontrivial (nonzero) solutions to $\mathbf{Ax} = \mathbf{0}$. An important result in linear algebra is the following:

The set of all solutions to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ form a subspace of $\mathbb{R}^n$. We call this the solution space of $\mathbf{Ax} = \mathbf{0}$ (or the nullspace of $\mathbf{A}$) .

Linear Combination and Linear Independence

The two essential vector operations, vector addition and scalar multiplication, go on inside the vector space and produce *linear combinations*. More precisely, we define **linear combinations** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ as the set of all possible vectors

$$\{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n\}, \quad \text{where } a_1, a_2, \ldots, a_n \text{ are arbitrary scalars.}$$

Example 4.6. Consider the following linear system from Example 3.15, Section 3.4.

$$\begin{cases} x_1 + 3x_2 - 15x_3 + 7x_4 = 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 = 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 = 0 \end{cases}$$

5

One of the goals in Chapter 4 is to determine the *true size* of a subspace (which itself is a vector space). We will need to clarify what size means in this context, but roughly speaking,

The "dimension" is measured by counting the number of independent vectors.

Linear Dependence of 2 Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **linearly dependent** if and only if one is a scalar multiple of the other, that is, we can write $\mathbf{u} = c\mathbf{v}$ for some scalar c.

Equivalently, two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if there exists scalars a, b, not all zero, such that

$$a\mathbf{u} + b\mathbf{v} = \mathbf{0}.$$

In other words, there exists a nontrivial linear combination of $\mathbf{u}$ and $\mathbf{v}$ that produces the zero vector $\mathbf{0}$.

Linear Independence of 2 Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **linearly independent** if and only if

$$a\mathbf{u} + b\mathbf{v} = \mathbf{0} \Rightarrow a = 0 = b.$$

In other words, the only linear combination of $\mathbf{u}$ and $\mathbf{v}$ that produces the zero vector $\mathbf{0}$ is the trivial linear combination $0\mathbf{u} + 0\mathbf{v}$.

Example 4.7. Are the vectors $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (1, 3)$ linearly dependent?

Linear Dependence of 3 Vectors

Three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are **linearly dependent** if and only if there exists scalars a, b, c, not all zero, such that

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}.$$

In other words, there exists a nontrivial linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ that produces the zero vector $\mathbf{0}$.

Linear Independence of 3 Vectors

Three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are **linearly independent** if and only if

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0} \implies a = b = c = 0.$$

In other words, the only linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ that produces the zero vector $\mathbf{0}$ is the trivial linear combination $0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w}$.

Example 4.8. Show that the vectors $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, and $\mathbf{w} = (0, 0, 1)$ are linearly independent.

Bases of Vector Spaces

A **basis** of $\mathbb{R}^3$ is the set of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ such that every vector $\mathbf{x}$ in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. That is, for any $x \in \mathbb{R}^3$ there exists scalars a, b, c such that

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

3 Linearly Independent Vectors in $\mathbb{R}^3$ Form a Basis of $\mathbb{R}^3$

Any 3 linearly independent vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ constitute a basis of $\mathbb{R}^3$ and no more than 3 vectors can be linearly independent.

Example 4.9. Show that any four distinct vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbb{R}^3$ must be linearly dependent.

Example 4.10. Consider the sequence of vectors $\mathbf{u} = (1, 2, 3), \mathbf{v} = (5, 5, 7), \mathbf{w} = (1, 0, 1)$.

(a) Determine whether $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis of $\mathbb{R}^3$.

(b) Write $\mathbf{x} = (-1, 2, 4)$ as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

§4.3: Linear Combinations and Independence of Vectors

Recall that the vector $\mathbf{w}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if there exist scalars $a_1, a_2, \dots, a_n$ such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Example 4.11. Write $\mathbf{w} = (-7, 7, 11)$ as a linear combination of $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (-4, -1, 2)$, and $\mathbf{v}_3 = (-3, 1, 3)$.

Example 4.12. Find constants c_1 and c_2 such that

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}.$$

Linear Span of Vectors

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ **spans** a vector space V if **every** vector $\mathbf{v} \in V$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, i.e., linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ **fill the space**.

1. The set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a **spanning set** for V and we write $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.
2. More generally, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V (that do not span V). Then $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a subspace of V.

Example 4.13. Show that $\mathbb{R}^3$ is spanned by the unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$.

Linear Independence

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.

Linear Dependence

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to be **linearly dependent** if and only if there exists scalars $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Equivalently, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent if at least one vector is a linear combination of the other vectors.

Linear Independence of n vectors in $\mathbb{R}^n$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$ are linearly independent if and only if

$$\det\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \neq 0.$$

Example 4.14. Determine the linear dependence or independence of the following vectors.

(a) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 8 \\ 7 \end{bmatrix}$

(b) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$

Example 4.15. Decide the linear dependence or independence of the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 11 \\ 12 \\ -4 \end{bmatrix}.$$

§4.4: Bases and Dimension for Vector Spaces

Think of any two vectors in $\mathbb{R}^3$. Generally they span a plane. Your mind should fill in that plane by taking linear combinations. Mathematically you know other possibilities: two vectors could span a line, three vectors could span all of $\mathbb{R}^3$, or they could span only a plane or a line. Now suppose you are given the vector space $\mathbb{R}^3$. Two vectors can't span all of $\mathbb{R}^3$, even if they are linearly independent. Four vectors can't be independent, even if they span $\mathbb{R}^3$.

We want enough linearly independent vectors
to span the vector space (and not more).
A "basis" is just right.

Basis of a Vector Space

A finite set of vectors S in a vector space V is a **basis** for V if

1. the vectors in S are linearly independent, and
2. the vectors in S span V.

Example 4.16. The **standard** basis for $\mathbb{R}^n$ consists of the following unit vectors.

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0, \dots, 0) \\
\mathbf{e}_2 &= (0, 1, 0, \dots, 0) \\
\vdots \;&\;\vdots \qquad \vdots \quad \vdots \\
\mathbf{e}_n &= (0, 0, 0, \dots, 1)
\end{aligned}$$

n Linearly Independent Vectors in $\mathbb{R}^n$

Any set of n linearly independent vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

Example 4.17. Do the following vectors form a basis for $\mathbb{R}^4$?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 2 \end{bmatrix}$$

Important Fact about Bases of Vector Spaces

1. A basis for a vector space V is not unique.
2. Any two bases for a vector space V have the same number of vectors.

Dimension of a Vector Space

The **dimension** of a vector space V is the number of vectors in a basis for V.

Bases as Maximal Linearly Independent Sets
Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for the vector space V. Then any set of more than n vectors in V is linearly dependent.

Example 4.18. Find a basis for the solution space of the following linear system.

$$\begin{cases} x_1 - 3x_2 + 2x_3 - 4x_4 = 0 \\ 2x_1 - 5x_2 + 7x_3 - 3x_4 = 0 \end{cases}$$

Assume S is a basis for a vector space V, then

1.

2.

3.

2.2 In-class worksheet

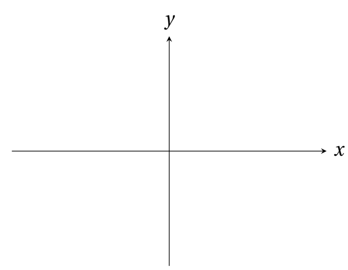
Students work together in groups during class time to complete the worksheets. These worksheets are designed to challenge students' understanding on key concepts and improve problem solving fluency. Included here is the worksheets about vector spaces.

4.1 & 4.2 (Part 1) Vector Spaces and Subspaces

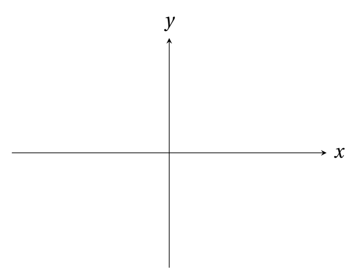
Discussion 1. We said that the proper subspaces in $\mathbb{R}^3$ are lines and planes through the origin. What is the significance that these subsets need to go through the origin to be subspaces of $\mathbb{R}^3$?

Example 1. This problem explores the concept of "closed under vector addition" (call this CVA) and "closed under scalar multiplication" (call this CSM) geometrically. For each of the following subsets of $\mathbb{R}^2$, sketch its graph and determine whether it satisfies the two conditions CVA and CSM by considering vector addition and scalar multiplication of vector graphically.

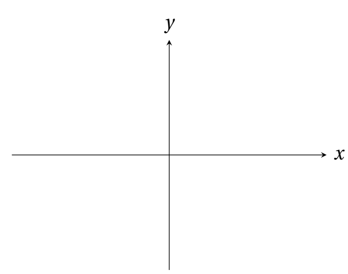
(a) $W = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$



(b) $W = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$



(c) $W = \{(x, 0) \in \mathbb{R}^2\}$



Example 2. Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$? If you think the subset is a subspace of $\mathbb{R}^3$, then you will need to show that they satisfy the two sufficient conditions to be a subspace. Otherwise, show that it is not a subspace by finding a set of vectors within the subset that fails one of the conditions.

(a) $W = \left\{(x, y, z) \in \mathbb{R}^3 \colon 7x + 2y = z\right\}$

(b) $W = \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 = z^2\right\}$

(c) $W = \left\{(x,y,z) \in \mathbb{R}^3 :\ y - z \geq 0\right\}$

(d) $W = \left\{(x,y,z) \in \mathbb{R}^3 :\ x + y - z = 0 \text{ and } 2y - 3z = 0\right\}$

4.1 & 4.2 (Part 2) Linear Combination and Linear Independence

Discussion 1. Geometrically, what does it mean for 2 vectors in $\mathbb{R}^3$ to be linearly dependent?

Example 1. Decide the linear dependence or independence of the following sets of vectors in $\mathbb{R}^2$. If you claim that the set of vectors is linearly dependent, write down a nontrivial linear combination of those vectors that produces the zero vector $\mathbf{0}$.

(a) $\{(1,0)\}$

(b) $\{(1,-3),(-5,4)\}$

(c) $\{(9,-6),(-3,2)\}$

(d) $\{(0,1),(1,2),(2,3)\}$

Example 2. Consider the vectors $\mathbf{u} = (1, -3, 2)$, $\mathbf{v} = (2, 1, -3)$, and $\mathbf{w} = (-3, 2, 1)$ in $\mathbb{R}^3$.

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

(b) Show that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

(c) Show that $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

(d) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

This example demonstrates the fact that proving linear independence of a smaller subset of vectors DOES NOT IMPLY linear independence of the entire set of vectors.

Example 3. Another way to describe linear dependence is this: "One vector is a combination of the other vectors." The definition we used is: "There exists a nontrivial linear combination that gives the zero vector $\mathbf{0}$." The former is clear while our definition is longer, so why not use the first definition? The answer is best explained in the "matrix language".

Consider the vectors $\mathbf{u} = (1,3,2)$, $\mathbf{v} = (2,1,3)$, and $\mathbf{w} = (3,2,1)$ in $\mathbb{R}^3$.

(a) Let $\mathbf{A}$ be the 3×3 matrix whose columns are given by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and consider the homogeneous system $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{x} = (a,b,c)$. How does linear dependence of the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ relate to $\det(\mathbf{A})$? Explain. *Hint: Consider the equivalent properties of nonsingular matrices from Section 3.5.*

(b) Find $\det(\mathbf{A})$ and decide the linear dependence or independence of the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

The truth is, all columns of $\mathbf{A}$ are treated the same; we look at $\mathbf{Ax} = \mathbf{0}$ and it either has a nonzero solution or it hasn't, simple as that. This is much better than trying to decide which vector is a combination of the other vectors!

4.1 & 4.2 (Part 3) Bases of Vector Spaces

Discussion 1. The vector space $\mathbb{R}^3$ requires 3 linearly independent vectors to form a basis. How many vectors do you think are necessary to form a basis for the vector space defined by a line in $\mathbb{R}^3$ which passes through the origin? How about a plane through the origin?

Example 1. For each of the following sets of vectors in $\mathbb{R}^3$, decide if it forms a basis of $\mathbb{R}^3$. Give a reason in one short sentence if it is not a basis.

(a) $\{(1,2,-1),(2,-1,1),(8,1,1)\}$

(b) $\{(2,-1,3),(-6,-3,-9)\}$

(c) $\left\{\left(1,-1,\sqrt{3}\right),(5,2,3),(-1,0,4),(\pi,e,-7)\right\}$

(d) $\{(1,1,0),(0,1,1),(1,0,1)\}$

Example 2. Consider the set of vectors $S = \{(1,2,0),(0,1,1),(1,0,3)\}$.

(a) Show that the set S is a basis of $\mathbb{R}^3$.

(b) Express $(7,7,8)$ as a linear combination of the vectors in S.

4.3 Linear Combinations and Independence of Vectors

Discussion 1. Explain why any set of more than n vectors in $\mathbb{R}^n$ is **always** linearly dependent.
Hint: Consider the reduced row echelon form of the augmented matrix corresponding to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}, \quad \textit{with } k > n.$$

Example 1. Find the largest possible number of linearly independent vectors among

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Example 2 (Minimal Spanning Set)**.** The following theorem tells us how to **remove unnecessary vectors from a spanning set of vectors**: Suppose that the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{w}$ span the vector space V and let $\mathbf{w}$ be a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ span V.

(a) Prove the above theorem.

(b) The theorem leads to an important process called **sifting**, which can be applied to any given set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V. We consider each vector $\mathbf{v}_i$ in turn. If it is a zero vector $\mathbf{0}$ or a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$, then we remove it from the list. What remains will be the minimal spanning set of span $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. Now, consider the following vectors in $\mathbb{R}^3$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$

Determine a linearly independent set of vectors that spans the same subspace of $\mathbb{R}^3$ as span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Chee Han Tan tan@math.utah.edu

4.4 Bases and Dimension for Vector Spaces

Discussion 1. In your own words, describe what it means for a basis for a vector space V to be a "minimal set spanning V".

Example 1. Determine whether the given sets of vectors are bases in $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

Example 2. Find the dimension and a basis for the solution space to the following homogeneous linear system.

$$\begin{cases} x_1 + 3x_2 + 8x_3 - x_4 = 0 \\ x_1 - 3x_2 - 10x_3 + 5x_4 = 0 \\ x_1 + 4x_2 + 11x_3 - 2x_4 = 0 \end{cases}$$

Example 3. Find the dimension and a basis for the following vector space.

$$V = \left\{(x, y, z, t) \in \mathbb{R}^4 : \ x + 2y - 3z = 0\right\}$$