I am an applied mathematician who is interested in using methods from the calculus of variations, applied analysis, and asymptotic analysis to answer questions about ordinary and partial differential equations (ODEs and PDEs) that arises in physical and life sciences. The primary focus of my dissertation investigates how the presence of surface tension on the fluid free surface influences the (natural) sloshing frequencies of an incompressible, inviscid fluid with irrotational flow in rigid containers.

Overview

Sloshing dynamics refers to the study of liquid free surface (fluid-air interface) motion inside partially filled containers or tanks. Liquid sloshing has attracted considerable attention from engineers and mathematicians as it is an inevitable phenomenon in many practical engineering applications, causing detrimental impacts on the dynamics and stability of marine, rail, road, and space transportation systems [11, 7]. For example, violent fuel sloshing in liquid propellant spacecraft produces highly localized pressure on tank walls possibly leading to deviation from the planned flight path or compromise structural integrity.

In a variety of engineering applications, it is important to know the natural sloshing frequencies of the liquid in partially filled containers of arbitrary shape since large amplitude sloshing tends to occur in the vicinity of resonance, *i.e.,* when the external excitation frequency of the container is close to one of these natural frequencies. It is therefore desirable to avoid external excitation at these natural frequencies in the analysis and design of liquid tanks. However, except for a few simple geometries (such as cylindrical and rectangular containers) with a flat free surface the best we can hope to do is to obtain bounds for the sloshing frequencies. The simplest model for liquid sloshing considers only the gravitational force, but in a microgravity environment or in small-scale engineering processes surface tension effects should be considered as well. I have been working on the following four projects, the common theme being the consideration of surface tension effects in the sloshing model:

1. I derived a **new variational characterization** of these sloshing frequencies which are eigenvalues of a **coupled generalized mixed-Steklov problem**, where the spectral parameter appears in the boundary conditions on the free surface. In particular, we established a **domain monotonicity** result for the fundamental sloshing frequency and gave the **first-order perturbation formula** for a simple fundamental sloshing frequency [22].

2. I investigated the **isoperimetric sloshing problem** in shallow containers. Among all shallow canals or radially symmetric containers with fixed free surface width and cross-sectional area, we are able to **determine explicit container shapes that furnish the largest fundamental sloshing frequency**. This problem is equivalent to maximizing the principal eigenvalue of a **coupled system of singular Sturm-Liouville problems** [21].

3. Together with fellow graduate student Nathan Willis (Utah), we derived an equivalent **integro-differential formulation** for the ice-fishing problem (fluid sloshing in a half-space) and numerically determined the corresponding sloshing frequencies. These frequencies are **universal upper bounds for sloshing frequencies in an arbitrary container having the same free surface**.

4. I am collaborating with Eva Kanso's group (University of Southern California) on a combined experimental-theoretical study to **measure the fluid surface tension from the dispersion relation measurement of Faraday waves generated in a narrow-width rectangular container, called a Hele-Shaw cell**.

To analyze these problems, I employ a variety of techniques from calculus of variations, applied analysis, classical ODEs and PDEs, potential flow theory, asymptotic methods, and spectral theory.

Project 1: A Variational Characterization of Fluid Sloshing with Surface Tension [22]

Surface tension is the intermolecular force required to contract the liquid surface to its minimal surface area. This is why small fluid bodies tend to evolve to spheres, because a sphere has the smallest surface area to volume ratio. A consequence of surface tension is that it causes the fluid free surface to climb the container walls, resulting in a meniscus. This implies that the behavior of the contact line, *i.e.,* the intersection between the free surface and the container walls, plays an important role in determining the sloshing frequencies and modes of capillary-gravity waves. Experimental evidence suggests that the contact line behavior depends on the contact angle (*i.e.,* the angle where the free surface meets the container walls) and the contact line velocity [4, 3]. Examples of contact line boundary condition (CLBC) include: (i) *Free-edge*: the contact line can freely slip on the vertical walls while intersecting it orthogonally; (ii) *Pinned-edge*: the contact line remains at rest but the contact angle changes [1, 8]; (iii) *Wetting*: the contact angle is assumed to be a linear function of the contact line velocity [10, 19].

In joint work with Christel Hohenegger and Braxton Osting, we considered the **linear sloshing problem with free-edge CLBC for arbitrary three-dimensional containers with vertical walls near the free surface** and looked for time-harmonic solutions. The crucial dimensionless parameter measuring the relative magnitudes of gravitational and capillary forces is the Bond number Bo. The sloshing problem without surface tension corresponds to $\mathrm{Bo} = \infty$ and it has been investigated extensively, see [17] for a recent survey. Let $\mathcal{D}$ and $\mathcal{F}$ be the equilibrium fluid domain and flat free surface, respectively and define the Dirichlet (kinetic) energy of $\Phi \in H^1(\mathcal{D})$ and free surface (potential) energy of $\xi \in H^1(\mathcal{F})$ by:

$$D[\Phi] = \frac{1}{2}\int_{\mathcal{D}} |\nabla\Phi|^2 \, dV \quad \text{and} \quad S[\xi] = \frac{1}{2}\int_{\mathcal{F}} \left(\xi^2 + \frac{1}{\mathrm{Bo}}|\nabla_{\mathcal{F}}\xi|^2\right) dA,$$

where $H^1(\mathcal{D})$ is the standard Sobolev space and $\nabla_{\mathcal{F}} := (\partial_x, \partial_y)$. **Our main results are as follows:**

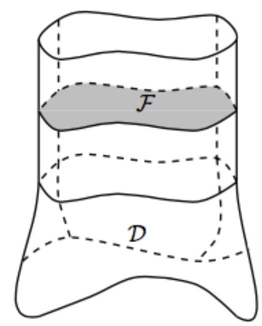

Figure 1: Example of a three-dimensional container $\mathcal{D}$ with vertical walls near the free surface $\mathcal{F}$. Taken from [22].

1. Let ω_1 be the fundamental sloshing frequency. Kopachevsky and Krein [12] stated that ω_1^2 can be found by minimizing the ratio $S[\xi]/D[\Phi]$ subject to certain PDE constraints and boundary conditions; call this (KKVF). We derived a **new variational formulation** for ω_1:

$$\omega_1 = \inf_{(\Phi,\xi)\in H^1(\mathcal{D})\times H^1(\mathcal{F})} D[\Phi] + S[\xi] \quad \text{subject to} \quad \int_{\mathcal{F}} \Phi\xi \, dA = 1, \int_{\mathcal{F}} \Phi \, dA = 0 = \int_{\mathcal{F}} \xi \, dA, \qquad (1)$$

   and **proved the existence of a minimizer of** (1) using the direct method of calculus of variations. Our variational formulation differs from (KKVF) in the following way: (i) The energy functional in (1) is the Lagrangian as it is **the sum of kinetic and potential energies**; (ii) (1) is a **free boundary condition variational principle** in the sense that any minimizer of (1) automatically satisfies the sloshing equation and all boundary conditions; (iii) we proved that (KKVF) follows directly from (1) but it is not obvious how to deduce (1) from (KKVF).

2. We proved a Rayleigh-Ritz generalization of (1) for higher sloshing frequencies ω_m, where in addition to the integral constraint in (1), we also require the admissible functions (Φ, ξ) to be biorthogonal to all the lower sloshing modes (Φ_j, ξ_j) over $\mathcal{F}$, *i.e.,* $\int_{\mathcal{F}} \Phi\xi_j \, dA = 0 = \int_{\mathcal{F}} \xi\Phi_j \, dA, j = 1, \ldots, m-1$.

3. An immediate consequence of (1) is the property of **domain monotonicity**, stating that if we have two containers $\mathcal{D}, \tilde{\mathcal{D}}$ with identical free surface $\mathcal{F}$ and both container walls being vertical near

$\mathcal{F}$, and such that $\mathcal{D}$ envelops $\tilde{\mathcal{D}}$, then we have $\omega_1(\tilde{\mathcal{D}}) \leq \omega_1(\mathcal{D})$. This result is **useful in providing upper bounds for the fundamental sloshing frequency containers of arbitrary shape** since we may choose $\mathcal{D}$ to be a larger container with a simple geometry.

4. We investigated the asymptotic behavior of a simple sloshing frequency in the limit of zero surface tension. We obtained a **first-order perturbation formula for any simple sloshing frequencies** $\omega(\varepsilon)$ for $\varepsilon := \mathrm{Bo}^{-1} \ll 1$:
$$\left.\frac{d\omega}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\omega_0}{2}\left(\frac{\|\nabla_{\mathcal{F}}\Phi_0\|^2_{L^2(\mathcal{F})}}{\|\Phi_0\|^2_{L^2(\mathcal{F})}}\right),$$
where (ω_0, Φ_0) is a solution of the sloshing problem without surface tension.

Project 2: An Isoperimetric Sloshing Problem in a Shallow Container with Pinned Contact Line [21]

An isoperimetric problem is the problem of optimizing a certain geometrical quantity (such as perimeter or area) given another geometrical constraint. The canonical example is the problem of determining the shape with a given perimeter that encloses the largest area. In 1965, Troesch [23] studied the isoperimetric sloshing problem (ISP) of maximizing the fundamental sloshing frequency $\lambda_1 = \omega_1^2$ for the following two families of shallow symmetric containers: (i) *Canals (CT)* (uniform horizontal cylinders of arbitrary cross-section) with given free surface width x_0 and cross-sectional area A; (ii) *Radially symmetric containers (RS)* with given rim radius r_0 and volume V.

In joint work with Christel Hohenegger and Braxton Osting, we generalize Troesch's result by including surface tension effects coupled with the pinned-edge CLBC. One application for the result of this investigation is in the field of aerospace engineering, where it is desirable to design tanks that have a large spectral gap to avoid the lowest natural frequencies; an example of a shallow container would be a nearly-empty fuel tank. **Our main results are as follows:**

1. Exploiting the symmetry along the canal length and axial symmetry for CT and RS containers, respectively, we reduced the sloshing problem to a two-dimensional problem on the container cross-section and derived a **similar variational formulation for ω_1 as in** (1) for pinned-edge CLBC. An application of shallow water theory gave us the "shallow" version of a variational formulation (call this SVF) where the container shape function now appears explicitly in the integrand of the kinetic energy functional. We obtained the **linear shallow sloshing equations** with pinned-edge CLBC by **computing the Euler-Lagrange equation of (SVF)**.

2. Using (SVF), we formulated ISP as a one-dimensional spectral optimization problem with an area/volume constraint, in the sense that the free surface is fixed and only the shape function h is allowed to vary while preserving the given cross-sectional area. We derived the **first-order optimality condition** for the optimal solution, h^*, **solved for the critical points** of the isoperimetric problem, and proved that **the critical point is indeed the unique maximizer** using the extremal property of the fundamental sloshing frequency. The largest λ_1 was found by imposing the area/volume constraint.

3. For RS containers, we looked for solutions of the form $\Phi(r,\theta,z) = \varphi(r)\cos(m\theta), \xi(r,\theta) = \zeta(r)\cos(m\theta)$, where $m \geq 0$ represents the number of azimuthal nodal lines and characterizes the symmetry class. With zero surface tension, Troesch proved that $\lambda_1 \leq 4V/\pi$ for $m = 1$ and $\lambda_1 \leq 18V/\pi$ for $m = 0$. Including surface tension, we showed that the upper bounds for both $m = 1$ and $m = 0$ are Troesch's upper bounds multiplied by some constant $C_m = C_m(\mathrm{Bo})$ such that $C_m \to 1$ as $\mathrm{Bo} \to +\infty$.

4. For CT containers, we looked for **traveling sinusoidal solutions** along the canal $\Phi(x, y, z) = \psi(x)\cos(\alpha y), \xi(x, y) = \zeta(x)\cos(\alpha y)$, where $\alpha \geq 0$ is the wavenumber. Troesch only considered the limiting case $\alpha = 0$. With zero surface tension, Troesch proved that $\lambda_1 \leq 12A$. Including surface tension, we showed that $\lambda_1 \leq C(12A)$ for some constant $C = C(\text{Bo})$ such that $C \to 1$ as $\text{Bo} \to +\infty$. **For $\alpha > 0$, the result for the zero surface tension case is new**. We proved that optimal cross-sections exist only for certain values of α: For $\alpha \in (0, \pi)$, we obtained $\lambda_1 \leq 2\alpha^3 A/(\alpha - \sin\alpha)$ and for $\alpha = n\pi, n = 1, 2, \ldots$, we obtained $\lambda_1 \leq 2\alpha^2 A$. Including surface tension, we proved that optimal cross-sections exist for $\alpha \in (0, \pi)$ and $\alpha = \pi$. For $\alpha > \pi$, $\alpha \neq n\pi, n = 1, 2, \ldots$, we proved that optimal cross-sections exist if and only if α is not a solution to certain transcendental equations.

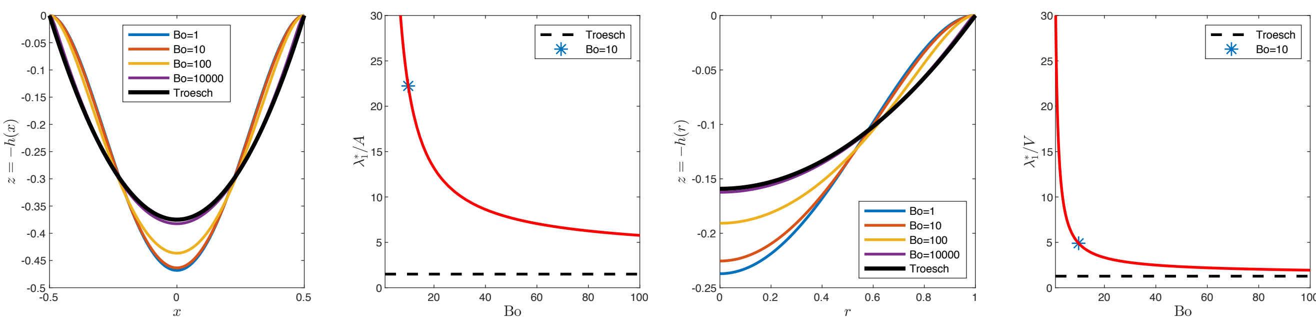

Figure 2: Optimal cross-section profiles and universal upper bounds for varying Bo. The first two plots are (CT) containers with $\alpha = 0$ and the last two plots are (RS) containers with $m = 1$. For both containers, we can see that both the optimal cross-section and universal upper bounds converge to the result without surface tension as $\text{Bo} \to \infty$.

Project 3: On the Two-Dimensional Ice-Fishing Problem with Pinned Contact Line

The ice-fishing problem (IFP) is the linear sloshing problem in a half-space bounded above by an infinite rigid plane that contains a circular or strip-like aperture. For the two-dimensional problem, the half-space is bounded above by the x-axis and the free surface is an interval on the x-axis. One motivation for studying IFP is the property of domain monotonicity: sloshing frequencies increase monotonically when the container is enlarged while maintaining a fixed free surface. Since every container is contained in a half-space, the sloshing frequencies for IFP thus furnish universal upper bounds for sloshing frequencies of arbitrary containers sharing the same free surface. Henrici et al. [9] presented an extensive treatment of (IFP), where they obtained both upper and lower bounds of the sloshing frequencies using the Rayleigh-Ritz method. Concurrently but independently, Davis [5] solved the two-dimensional IFP where he explicitly constructed the Green's function for the half-space with aperture and reformulated IFP as an explicit integral equation.

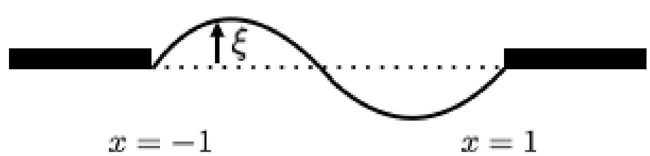

Figure 3: Fluid domain for the ice-fishing problem. Here, ξ is the sloshing height.

In joint work with Christel Hohenegger, Braxton Osting, and Nathan Willis, we consider the two-dimensional IFP with the pinned-edge CLBC. The domain monotonicity of sloshing frequencies follows from our variational formulation derived in Project 2 [21]. **Our main results are as follows:**

1. Following Henrici et al., we wrote the sloshing velocity potential as a single layer potential and use it to reformulate the problem as a **constrained Fredholm integro-differential eigenvalue problem** for the sloshing height $\xi(x)$ on the free surface $[-1, 1]$ and sloshing frequencies ω:

$$\xi - \frac{1}{\mathrm{Bo}}\xi_{xx} = -\frac{\omega^2}{\pi}\int_{-1}^{1} \ln|x-s|\xi(s)\,ds \quad \text{for } |x| < 1, \tag{2}$$

$$\int_{-1}^{1} \xi\,dx = 0, \quad \xi(\pm 1) = 0. \tag{3}$$

2. Because the aperture is symmetric about the y-axis, it follows that the solutions of (2) are either odd or even functions of x. Writing (2) in weak form and expanding solutions in suitably chosen Fourier basis satisfying both the zero mean condition and Dirichlet boundary condition reduces (2) to a generalized matrix eigenvalue problem of the form $\mathbf{B}\boldsymbol{\xi} = \omega^2\mathbf{C}\boldsymbol{\xi}$. The entries of $\mathbf{C}$ are integrals involving logarithmic kernel and we use two Gauss quadrature rules to evaluate these integrals.

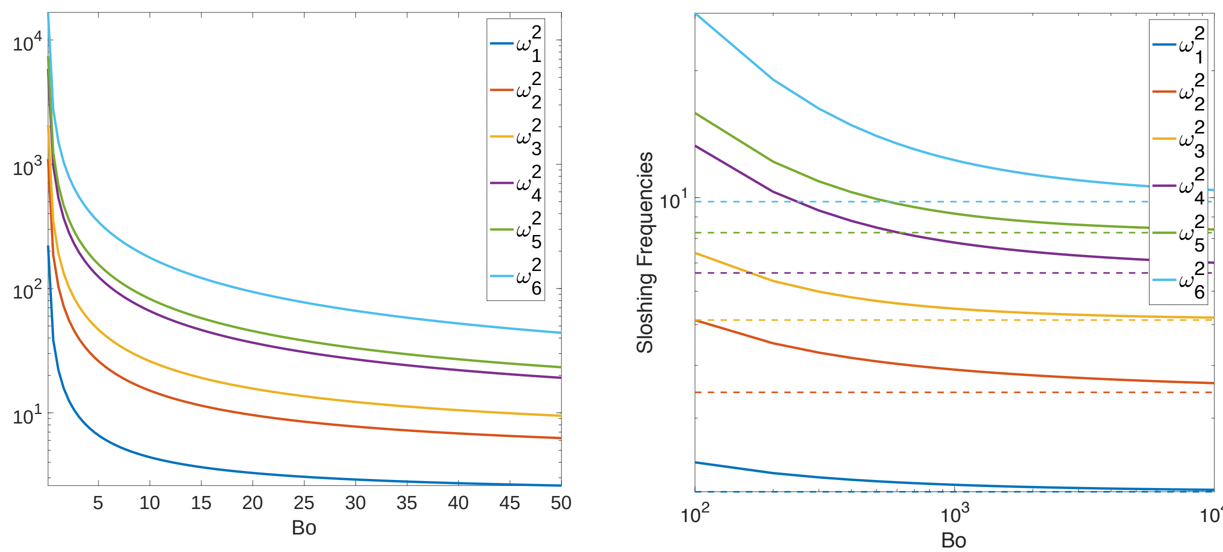

Figure 4: The square of the first six sloshing frequencies of (2) for varying Bo. The dashed lines on the right plot indicate the corresponding values without surface tension from Davis [5]. (Provided by N. Willis)

Project 4: Measuring Surface Tension in a Hele-Shaw cell (Ongoing Work)

When a liquid-filled container is undergoing a vertical sinusoidal excitation of the form $A\cos(2\pi f t)$, a pattern of standing waves known as *Faraday waves* is often observed at the liquid free surface for some combinations of excitation amplitude A and excitation frequency f. This phenomenon is known as the *Faraday instability*. Faraday waves can be generated in the laboratory and are often subharmonic oscillations, with frequency only half the excitation frequency.

A crucial component in the study of Faraday waves is the knowledge of the *Faraday dispersion relation*, *i.e.*, an equation relating the wavelength of Faraday waves to their frequency. Since the seminal work of

Benjamin and Ursell [2], the Faraday dispersion relation has been identified with that of natural sloshing modes, the latter depends only on the container geometry and particularly the fluid surface tension, σ. Douady exploited this property in his experimental study [6] to measure surface tension, where he plotted the measured wavelength versus the excitation frequency and fitted the curve against the dispersion relation, by adjusting the only unknown experimental parameter, σ. However, Rajchenbach and Clamond [20] recently refuted the result of Benjamin and Ursell, showing that the excitation amplitude A plays a major role in the dispersion relation, thereby establishing the first actual linear Faraday dispersion relation.

In collaboration with Eva Kanso's group (University of Southern California), we propose to measure the fluid surface tension from Faraday waves generated in a narrow-width rectangular container, called a Hele-Shaw cell. The cell is placed above a speaker and made to oscillate vertically. The excitation frequency is slowly varied until two-dimensional standing Faraday waves are generated and wavelength measurements are recorded; the macroscopic contact line behaviors are observed as well. Li et al. [18] recently proposed a novel hydrodynamic model to capture the variation of the dynamic contact line across the narrow gap and obtained a modified Faraday dispersion relation in Hele-Shaw cells. However, this dispersion relation did not make the prediction better but worse. We are currently **modifying known models for the dynamic contact line to match our experimental observation**. We will perform a **weakly nonlinear analysis on this modified model to derive a modified Faraday dispersion relation** and use it to measure fluid surface tension as in Douady. While there are many methods to measure fluid surface tension, they generally depend on the wetting properties and characteristics of the fluid. Our proposed experimental-theoretical study offers a simple graphical approach to measure fluid surface tension. The experimental setup is inexpensive relative to other methods requiring more sophisticated equipment.

Figure 5: Square cell half-filled with ethanol 70% undergoing vertical oscillation, above the instability threshold, at $f = 85$ Hz. (Provided by Lionel Vincent)

Future Directions

Our analysis from Projects 1–3 are all restricted to the case where the static free surface is flat. I plan to remove this assumption and generalize results from Projects 1–3 to the case of **static curved free surface**. For the ice-fishing problem, I would like to investigate the **dependence of fundamental sloshing frequency ω_1^2 on the shape of the free surface region**. To do so, I will start by extending my two-dimensional result to the three-dimensional case for arbitrary free-surface region and gain some insights by performing a **numerical study using finite-element method**.

For the isoperimetric problem, it is more difficult to obtain lower bounds for $\lambda_1 = \omega_1^2$ since λ_1 can be made arbitrarily small for a given free surface width and container volume. Troesch [25, 24] considered the **isoperimetric problem of minimizing λ_1 among shallow convex containers** and Kuzanek [15, 14] considered the similar isoperimetric sloshing problem on a symmetric shallow canal where he **replaced the area constraint by the arc length constraint**. I plan to explore these two problems with the consideration of surface tension effects. Finally, it would be of interest to extend the isoperimetric sloshing problem by removing the assumption of shallow container. This is a difficult shape optimization problem and **numerical approaches are required to find optimal container shapes**.

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