Contents

1 Metric Spaces

1.1 Metric Space.

Definition 1.1.1.

1. *A **metric**, d on X is a function defined on $X \times X$ such that for all $x, y, z \in X$, we have:*

   (M1) d is real-valued, finite and nonnegative.

   (M2) $d(x, y) = 0$ if and only if $x = y$.

   (M3) $d(x, y) = d(y, x)$. *(Symmetry).*

   (M4) $d(x, y) \leq d(x, z) + d(z, y)$. *(**Triangle Inequality**).*

2. *A **metric subspace** $(Y, \tilde{d})$ of (X, d) is obtained if we take a subset $Y \subset X$ and restrict d to $Y \times Y$; thus the metric on Y is the restriction*

   $$\tilde{d} = d|_{Y \times Y}.$$

   *$\tilde{d}$ is called the metric **induced** on Y by d.*

3. *We take any set X and on it the so-called **discrete metric** for X, defined by*

   $$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

   *This space (X, d) is called a **discrete metric space**.*

   - *Discrete metric space is often used as (extremely useful) counterexamples to illustrate certain concepts.*

1. Show that the real line is a metric space.

Solution: For any $x, y \in X = \mathbb{R}$, the function $d(x, y) = |x - y|$ defines a metric on $X = \mathbb{R}$. It can be easily verified that the absolute value function satisfies the axioms of a metric.

2. Does $d(x, y) = (x - y)^2$ define a metric on the set of all real numbers?

Solution: No, it doesn't satisfy the triangle inequality. Choose $x = 3$, $y = 1$ and $z = 2$, then

$$d(3, 1) = (3 - 1)^2 = 2^2 = 4$$

but

$$d(3, 2) + d(2, 1) = (3 - 2)^2 + (2 - 1)^2 = 2.$$

3. Show that $d(x,y) = \sqrt{|x-y|}$ defines a metric on the set of all real numbers.

Solution: Fix $x, y, z \in X = \mathbb{R}$, we need to verify the axioms of a metric. (M1) to (M3) follows easily from properties of absolute value. To verify (M4), for any $x, y, z \in \mathbb{R}$ we have

$$\begin{aligned}\Big[d(x,y)\Big]^2 = |x-y| &\le |x-z| + |z-y| \\ &\le |x-z| + |z-y| + 2\sqrt{|x-z|}\sqrt{|z-y|} \\ &= (\sqrt{|x-z|} + \sqrt{|z-y|})^2 \\ &= \Big[[d(x,z) + d(z,y)\Big]^2.\end{aligned}$$

Taking square root on both sides yields the triangle inequality.

4. Find all metrics on a set X consisting of two points. Consisting of one point only.

Solution: If X has only two points, then the triangle inequality property is a consequence of (M1) to (M3). Thus, any functions satisfy (M1) to (M3) is a metric on X. If X has only one point, say, x_0, then the symmetry and triangle inequality property are both trivial. However, since we require $d(x_0, x_0) = 0$, any nonnegative function $f(x,y)$ such that $f(x_0, x_0) = 0$ is a metric on X.

5. Let d be a metric on X. Determine all constants k such that the following is a metric on X

   (a) kd,

   Solution: First, note that if X has more than one point, then the zero function cannot be a metric on X; this implies that $k \neq 0$. A simple calculation shows that any positive real numbers k lead to kd being a metric on X.

   (b) $d + k$.

   Solution: For $d + k$ to be a metric on X, it must satisfy (M2). More precisely, if $x = y$, then $d(x,y) + k$ must equal to 0; but since d is a metric on X, we have that $d(x,y) = 0$. This implies that $d(x,y) + k = k = 0$. Thus, k must be 0.

6. Show that $d(x,y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|$ satisfies the triangle inequality for any x, y in l^∞.

Solution: Fix $x = (\xi_j)$, $y = (\eta_j)$ and $z = (\zeta_j)$ in l^∞. Usual triangle inequality on real numbers yields

$$\begin{aligned}|\xi_j - \eta_j| &\le |\xi_j - \zeta_j| + |\zeta_j - \eta_j| \\ &\le \sup_{j\in\mathbb{N}} |\xi_j - \zeta_j| + \sup_{j\in\mathbb{N}} |\zeta_j - \eta_j| \\ &= d(x,z) + d(z,y).\end{aligned}$$

Taking supremum over $j \in \mathbb{N}$ on both sides gives the desired inequality.

7. If A is the subspace of l^∞ consisting of all sequences of zeros and ones, what is the induced metric on A?

Solution: For any distinct $x, y \in A$, $d(x,y) = 1$ since they are sequences of zeros and ones. Thus, the induced metric on A is the discrete metric.

8. Show that another metric $\tilde{d}$ on $C[a,b]$ is defined by

$$\tilde{d}(x,y) = \int_a^b |x(t) - y(t)|\, dt.$$

Solution: (M1) and (M3) are satisfied, as we readily see. For (M4),

$$\begin{aligned}d(x,y) = \int_a^b |x(t) - y(t)|\, dt &\le \int_a^b |x(t) - z(t)| + |z(t) - y(t)|\, dt \\ &= d(x,z) + d(z,y)\end{aligned}$$

For (M2), the if statement is obvious. For the only if statement, suppose $d(x,y) = 0$. Then

$$\int_a^b |x(t) - y(t)|\, dt = 0 \implies |x(t) - y(t)| = 0 \text{ for all } t \in [a,b]$$

since the integrand $|x - y|$ is a continuous function on $[a,b]$.

9. Show that the **discrete metric** is in fact a metric.

Solution: (M1) to (M4) can be checked easily using definition of the discrete metric.

10. **(Hamming distance)** Let X be the set of all ordered triples of zeros and ones. Show that X consists of eight elements and a metric d on X is defined by $d(x,y) =$

number of places where x and y have different entries. (This space and similar spaces of n-tuples play a role in switching and automata theory and coding. $d(x, y)$ is called the *Hamming distance* between x and y.

Solution: X has $2^3 = 8$ elements. Consider the function d defined above. (M1) to (M3) follows easily by definition. Verifying (M4) is a little tricky, but still doable.

- Note that (M4) is trivial if $x, y, z \in X$ are not distinct, so suppose they are distinct; this assumption together with definiton of d both imply that $d(x, y), d(x, z), d(z, y)$ has 1 as their mininum and 3 as their maximum.
- (M4) is trivial if $d(x, y) = 1$ or $d(x, y) = 2$, so consider the case when $d(x, y) = 3$. It can then be shown that for any $z \neq x, y$, we have that $d(x, z) + d(z, y) = 3$.

Thus, (M4) is satisfied for any $x, y, z \in X$ and we conclude that d is a metric on X.

11. Prove the **generalised triangle inequality**.

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n).$$

Solution: We prove the generalised triangle inequality by induction. The case $n = 3$ follows from definition of a metric. Suppose the statement is true for $n = k$. For $n = k + 1$,

$$\begin{aligned} d(x_1, x_{k+1}) &\leq d(x_1, x_k) + d(x_k, x_{k+1}) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{k-1}, x_k) + d(x_k, x_{k+1}) \end{aligned}$$

where the last inequality follows from the induction hypothesis. Since $k \geq 3$ is arbitrary, the statement follows from induction.

12. **(Triangle inequality)** The triangle inequality has several useful consequences. For instance, using the generalised triangle inequality, show that

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w).$$

Solution: Suppose (X, d) is a metric space. For any x, y, z, w in X, the **generalised triangle inequality** yields

$$\begin{aligned} & d(x, y) \leq d(x, z) + d(z, w) + d(w, y) \\ \implies & d(x, y) - d(z, w) \leq d(x, z) + d(w, y) \end{aligned}$$

$$
\begin{aligned}
&= d(x,z) + d(y,w) \qquad \Big[\text{ by (M3) }\Big]. \\
d(z,w) &\le d(z,x) + d(x,y) + d(y,w) \\
\implies d(z,w) - d(x,y) &\le d(z,x) + d(y,w) \\
&= d(x,z) + d(y,w) \qquad \Big[\text{ by (M3) }\Big].
\end{aligned}
$$

Combining these two inequalities yields the desired statement.

13. Using the triangle inequality, show that

$$|d(x,z) - d(y,z)| \le d(x,y).$$

Solution: Suppose (X,d) is a metric space. For any x, y, z in X, (M4) yields:

$$
\begin{aligned}
d(x,z) &\le d(x,y) + d(y,z) \\
\implies d(x,z) - d(y,z) &\le d(x,y) \\
d(y,z) &\le d(y,x) + d(x,z) \\
\implies d(y,z) - d(x,z) &\le d(y,x) = d(x,y) \quad \text{ by (M3) }.
\end{aligned}
$$

Combining these two inequalities yields the desired statement.

14. **(Axioms of a metric)** (M1) to (M4) could be replaced by other axioms without changing the definition. For instance, show that (M3) and (M4) could be obtained from (M2) and

$$d(x,y) \le d(z,x) + d(z,y). \tag{\dagger}$$

Solution: We first prove (M3). Fix $x, y \in X$. Choose $z = y$, then

$$
\begin{aligned}
d(x,y) - d(y,x) &\le d(z,x) + d(z,y) - d(y,x) \\
&= d(y,x) + d(y,y) - d(y,x) = 0 \text{ from (M2)}.
\end{aligned}
$$

Choose $z = x$, then

$$
\begin{aligned}
d(y,x) - d(x,y) &\le d(z,y) + d(z,x) - d(x,y) \\
&= d(x,y) + d(x,x) - d(x,y) = 0 \text{ from (M2)}.
\end{aligned}
$$

Combining these two inequalities gives $|d(x,y) - d(y,x)| \le 0 \implies d(x,y) = d(y,x)$ for any $x, y \in X$.

To prove (M4), we apply (†) twice. More precisely, for any $x, y, z \in X$,

$$
\begin{aligned}
d(x,y) &\le d(z,x) + d(z,y) \\
&\le d(w,z) + d(w,x) + d(z,y)
\end{aligned}
$$

(M4) follows from (M2) and choosing $w = x$.

15. Show that the nonnegativity of a metric follows from (M2) to (M4).

Solution: The only inequality we have is (M4), so we start from (M4). Choose any $x \in X$. If $z = x$, then for any $y \in X$,

$$\begin{aligned} & d(x,z) \leq d(x,y) + d(y,z) && \left[\text{ from (M4) } \right] \\ \implies & d(x,x) \leq d(x,y) + d(y,x) = 2d(x,y) && \left[\text{ from (M3) } \right] \\ \implies & d(x,y) \geq 0 && \left[\text{ from (M2) } \right] \end{aligned}$$

Since $x, y \in X$ were arbitrary, this shows the nonnegativity of a metric.

1.2 Further Examples of Metric Spaces.

We begin by stating three important inequalities that are indispensable in various theoretical and practical problems.

$$\textbf{Holder inequality}: \sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{\frac{1}{q}},$$

$$\text{where } p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

$$\textbf{Cauchy-Schwarz inequality}: \sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^2\right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} |\eta_m|^2\right)^{\frac{1}{2}}.$$

$$\textbf{Minkowski inequality}: \left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^{\infty} |\eta_m|^p\right)^{\frac{1}{p}},$$

$$\text{where } p > 1.$$

1. For $x = (\xi_j)$ and $y = (\eta_j)$, the function

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

defines a metric on the sequence space s. Show that we can obtain another metric by replacing $1/2^j$ with $\mu_j > 0$ such that $\sum \mu_j$ converges.

Solution: The proof for triangle inequality is identical. To ensure finiteness of d, we require that $\sum \mu_j$ converges since

$$d(x,y) = \sum_{j=1}^{\infty} \mu_j \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} < \sum_{j=1}^{\infty} \mu_j < \infty.$$

2. Suppose we have that for any α, β positive numbers,

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

where p, q are conjugate exponents. Show that the geometric mean of two positive numbers does not exceed the arithmetic mean.

Solution: Choose $p = q = 2$, which are conjugate exponents since $\dfrac{1}{2} + \dfrac{1}{2} = 1$; we then have $ab \leq \dfrac{a^2}{2} + \dfrac{b^2}{2}$. Multiplying by 2 and adding $2ab$ to both sides yield:

$$2ab + 2ab \leq a^2 + b^2 + 2ab$$

$$
\begin{aligned}
4ab &\leq (a+b)^2 \\
ab &\leq \left(\frac{a+b}{2}\right)^2.
\end{aligned}
$$

Since ab is a positive quantity, the desired statement follows from taking square root of both sides.

3. Show that the **Cauchy-Schwarz inequality** for sums implies

$$(|\xi_1| + \cdots + |\xi_n|)^2 \leq n(|\xi_1|^2 + \ldots + |\xi_n|^2).$$

Solution: An equivalent formulation of the **Cauchy-Schwarz inequality** for (finite) sums is

$$\left(\sum_{j=1}^{n} |\xi_j \eta_j|\right)^2 \leq \left(\sum_{k=1}^{n} |\xi_k|^2\right)\left(\sum_{m=1}^{n} |\eta_m|^2\right).$$

Choosing $\eta_j = 1$ for all $j \geq 1$ yields the desired inequality.

4. **(Space l^p)** Find a sequence which converges to 0, but is not in any space l^p, where $1 \leq p < +\infty$.

Solution: Consider the sequence (b_j) with numbers $a(k)$, $N(k)$ times, where for $k \geq 1$, $a(k) = \dfrac{1}{k}$ and $N(k) = 2^k$, i.e.

$$(b_j) = \left(\underbrace{1, 1,}_{\text{2 times}} \underbrace{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2},}_{\text{4 times}} \underbrace{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3},}_{\text{8 times}} \ldots\ldots\right).$$

By construction, $(b_j) \longrightarrow 0$ as $j \longrightarrow \infty$ and $\displaystyle\sum_{j=1}^{\infty} |b_j|^p = \sum_{j=1}^{\infty} 2^j \left(\frac{1}{j}\right)^p$. However, since for all $p \geq 1$, $\dfrac{2^j}{j^p} \not\longrightarrow 0$ as $j \longrightarrow \infty$, **Divergence Test for Series** implies that the series $\displaystyle\sum_{j=1}^{\infty} |b_j|^p$ diverges for all $p \geq 1$. By definition, this means that $(b_j) \notin l^p$ for all $p \geq 1$.

5. Find a sequence x which is in l^p with $p > 1$ but $x \notin l^1$.

Solution: The sequence $(a_n) = \left(\frac{1}{n}\right)$ belongs to l^p with $p > 1$ but not l^1.

6. **(Diameter, bounded set)** The *diameter* $\delta(A)$ of a nonempty set A in a metric space (X, d) is defined to be

$$\delta(A) = \sup_{x,y \in A} d(x, y).$$

A is said to be *bounded* if $\delta(A) < \infty$. Show that $A \subset B$ implies $\delta(A) \leq \delta(B)$.

Solution: This follows from property of least upper bound.

7. Show that $\delta(A) = 0$ if and only if A consists of a single point.

Solution: Suppose $\delta(A) = 0$, this means that $d(x, y) = 0$ for all $x, y \in A$; (M2) then implies $x = y$, i.e. A has only one element. Conversely, suppose that A consists of a single point, say x; (M2) implies that $\delta(A) = 0$ since $d(x, x) = 0$.

8. **(Distance between sets)** The *distance* $D(A, B)$ between two nonempty subsets A and B of a metric space (X, d) is defined to be

$$D(A, B) = \inf_{\substack{a \in A \\ b \in B}} d(a, b).$$

Show that D *does not* define a metric on the power set of X. (For this reason we use another symbol, D, but one that still reminds us of d.)

Solution: Consider $X = \{1, 2, 3\}$ with d being the absolute value function, and consider its power set $A = \{1\}$ and $B = \{1, 2\}$. By construction, $D(A, B) = 0$ but $A \neq B$.

9. If $A \cap B \neq \varnothing$, show that $D(A, B) = 0$ in Problem 8. What about the converse?

Solution: If $A \cap B \neq \varnothing$, then for any $x \in A \cap B$,

$$0 \leq D(A, B) \leq d(x, x) = 0 \implies D(A, B) = 0.$$

The converse does not hold. Consider $X = \mathbb{Q}$, with $A = \{0\}$ and $B = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. Then $D(A, B) = \lim_{n \to \infty} d(0, 1/n) = \lim_{n \to \infty} \frac{1}{n} = 0$, but $A \cap B = \varnothing$.

10. The distance $D(x, B)$ from a point x to a non-empty subset B of (X, d) is defined to be

$$D(x, B) = \inf_{b \in B} d(x, b)$$

in agreement with Problem 8. Show that for any $x, y \in X$,

$$|D(x, B) - D(y, B)| \leq d(x, y).$$

Solution: Let $x, y \in X$. For any $z \in B$, we have

$$\begin{aligned} D(x, B) &\leq d(x, z) \leq d(x, y) + d(y, z). \\ D(y, B) &\leq d(y, z) \leq d(y, x) + d(x, z). \end{aligned}$$

Taking infimum over all $z \in B$ on the RHS of both inequalities yields

$$\begin{aligned} D(x, B) &\leq d(x, y) + D(y, B). \\ D(y, B) &\leq d(x, y) + D(x, B). \end{aligned}$$

Rearranging and combining these two together gives the desired inequality.

Remark: *This result says that for any nonempty set $B \subset X$, the function $x \mapsto D(x, B)$ is Lipschitz with Lipschitz constant 1.*

11. If (X, d) is any metric space, show that another metric on X is defined by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

and X is bounded in the metric $\tilde{d}$.

Solution: Note that X is bounded in the metric $\tilde{d}$ since $\tilde{d}(x, y) \leq 1 < \infty$. (M1) to (M3) are satisfied, as we readily see. To show that $\tilde{d}$ satisfies (M4), consider the auxiliary function f defined on $\mathbb{R}$ by $f(t) = \dfrac{t}{1+t}$. Differentiation gives $f'(t) = \dfrac{1}{(1+t)^2}$, which is positive for $t > 0$. Hence f is monotone increasing. Consequently, $d(x, y) \leq d(x, z) + d(z, y)$ implies

$$\begin{aligned} \tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} &\leq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} \\ &= \frac{d(x, z)}{1 + d(x, z) + d(z, y)} + \frac{d(z, y)}{1 + d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} \\ &= \tilde{d}(x, z) + \tilde{d}(z, y). \end{aligned}$$

12. Show that the union of two bounded sets A and B in a metric space is a bounded set. (Definition in Problem 6.)

Solution: Let $X = A \cup B$, we need to show $\delta(X) = \sup_{x,y\in X} d(x,y) < \infty$. Observe that if x, y are both in A or B, then $d(x,y) < \infty$ by assumption, so WLOG it suffices to prove that $\sup_{x\in A, y\in B} d(x,y) < \infty$.

- Consider the first case where $A \cap B \neq \varnothing$. For any fixed $z \in A \cap B$,

$$d(x,y) \leq d(x,z) + d(z,y) \leq \delta(A) + \delta(B) < \infty.$$

The claim follows by taking supremum over $x \in A, y \in B$ in both sides of the inequality.

- Consider the second case where $A \cap B = \varnothing$. For every $\varepsilon > 0$, there exists $x^* \in A$ and $y^* \in B$ such that $d(x^*, y^*) \leq D(A,B) + \varepsilon$. For any $x \in A$ and $y \in B$,

$$\begin{aligned} d(x,y) &\leq d(x,x^*) + d(x^*,y^*) + d(y^*,y) \\ &\leq \delta(A) + D(A,B) + \varepsilon + \delta(B). \end{aligned}$$

Letting $\varepsilon \longrightarrow 0$, and taking supremum over $x \in A, y \in B$, we obtain the desired result.

13. **(Product of metric spaces)** The Cartesian product $X = X_1 \times X_2$ of two metric spaces (X_1, d_1) and (X_2, d_2) can be made into a metric space (X, d) in many ways. For instance, show that a metric d is defined by

$$d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2),$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$.

Solution:

- (M1) is satisfied since we are summing two real-valued, finite and nonnegative functions.

- Suppose $d(x,y) = 0$, this is equivalent to $d_1(x_1,y_1) = d_2(x_2,y_2) = 0$ since d_1 and d_2 are both nonnegative functions. This implies $x_1 = y_1$ and $x_2 = y_2$ or equivalently $x = y$. Conversely, suppose $x = y$, then

$$x_1 = y_1 \implies d_1(x_1,y_1) = 0 \quad \text{and} \quad x_2 = y_2 \implies d_2(x_2,y_2) = 0.$$

Consequently, $d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2) = 0$.

- (M3) is satisfied since for any $x, y \in X_1 \times X_2$,

$$d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2) = d_1(y_1,x_1) + d_2(y_2,x_2) = d(y,x).$$

- (M4) follows from combining triangle inequalities of d_1 and d_2. More precisely, let $z = (z_1, z_2) \in X_1 \times X_2$, then we have from (M4) of d_1 and d_2:

$$\begin{aligned}
d_1(x_1, y_1) &\leq d_1(x_1, z_1) + d_1(z_1, y_1). \\
d_2(x_2, y_2) &\leq d_2(x_2, z_2) + d_2(z_2, y_2). \\
\implies d(x, y) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\
&\leq d_1(x_1, z_1) + d_1(z_1, y_1) + d_2(x_2, z_2) + d_2(z_2, y_2) \\
&= \Big[d_1(x_1, z_1) + d_2(x_2, z_2)\Big] + \Big[d_1(z_1, y_1) + d_2(z_2, y_2)\Big] \\
&= d(x, z) + d(z, y).
\end{aligned}$$

14. Show that another metric on X in Problem 13 is defined by

$$\tilde{d}(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}.$$

Solution: A similar argument in Problem 13 shows that (M1) to (M3) are satisfied. Let $z = (z_1, z_2) \in X_1 \times X_2$, then we have from (M4) of d_1 and d_2:

$$\begin{aligned}
d_1(x_1, y_1) &\leq d_1(x_1, z_1) + d_1(z_1, y_1). \\
d_2(x_2, y_2) &\leq d_2(x_2, z_2) + d_2(z_2, y_2).
\end{aligned}$$

Squaring both sides yields:

$$\begin{aligned}
d_1(x_1, y_1)^2 &\leq d_1(x_1, z_1)^2 + d_1(z_1, y_1)^2 + 2d_1(x_1, z_1)d_1(z_1, y_1) \\
d_2(x_2, y_2)^2 &\leq d_2(x_2, z_2)^2 + d_2(z_2, y_2)^2 + 2d_2(x_2, z_2)d_2(z_2, y_2)
\end{aligned}$$

Summing these two inequalities and applying definition of $\tilde{d}$, we obtain:

$$\begin{aligned}
\tilde{d}(x, y)^2 &\leq \tilde{d}(x, z)^2 + \tilde{d}(z, y)^2 + 2\Big[d_1(x_1, z_1)d_1(z_1, y_1) + d_2(x_2, z_2)d_2(z_2, y_2)\Big] \\
&= \tilde{d}(x, z)^2 + \tilde{d}(z, y)^2 + 2\sum_{j=1}^{2} d_j(x_j, z_j)d_j(z_j, y_j) \\
&\leq \tilde{d}(x, z)^2 + \tilde{d}(z, y)^2 + 2\left(\sum_{j=1}^{2} d_j(x_j, z_j)^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{2} d_j(z_j, y_j)^2\right)^{\frac{1}{2}} \\
&= \tilde{d}(x, z)^2 + \tilde{d}(z, y)^2 + 2\tilde{d}(x, z)\tilde{d}(z, y) \\
&= \Big[\tilde{d}(x, z) + \tilde{d}(z, y)\Big]^2
\end{aligned}$$

where the inequality follows from **Cauchy-Schwarz inequality** for sums. (M4) follows from taking square root of both sides.

15. Show that a third metric on X in Problem 13 is defined by

$$\hat{d}(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

Solution: A similar argument in Problem 13 shows that (M1) to (M3) are satisfied. Let $z = (z_1, z_2) \in X_1 \times X_2$, then

$$\begin{aligned}
\hat{d}(x, y) &= \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\
&\leq \max\{d_1(x_1, z_1) + d_1(z_1, y_1), d_2(x_2, z_2) + d_2(z_2, y_2)\} \\
&\leq \max\{d_1(x_1, z_1), d_2(z_2, z_2)\} + \max\{d_1(z_1, y_1), d_2(z_2, y_2)\} \\
&= \hat{d}(x, z) + \hat{d}(z, y).
\end{aligned}$$

where we repeatedly used the fact that $|a| \leq \max\{|a|, |b|\}$ for any $a, b \in \mathbb{R}$.

(The metrics in Problem 13 to 15 are of practical importance, and other metrics on X are possible.)

1.3 Open Set, Closed Set, Neighbourhood.

Definition 1.3.1.

1. *Given a point $x_0 \in X$ and a real number $r > 0$, we define three types of sets:*

$$B_r(x_0) = \{x \in X \colon d(x, x_0) < r\} \qquad \textbf{\textit{(Open ball)}}.$$
$$\tilde{B}_r(x_0) = \{x \in X \colon d(x, x_0) \leq r\} \qquad \textbf{\textit{(Closed ball)}}.$$
$$S_r(x_0) = \{x \in X \colon d(x, x_0) = r\} \qquad \textbf{\textit{(Sphere)}}.$$

In all three cases, x_0 is called the center and r the radius.

2. *A subset M of a metric space X is said to be* ***open*** *if it contains a ball about each of its points. A subset K of X is said to be* ***closed*** *if its complement (in X) is open, that is, $K^C = X \setminus K$ is open.*

3. *We call x_0 an* ***interior point*** *of a set $M \subset X$ if M is a neighbourhood of x_0. The* ***interior*** *of M is the set of all interior points of M and may be denoted by Int(M).*
   - *By neighbourhood of x_0 we mean any subset of X which contains an ε-neighbourhood of x_0.*
   - *Int(M) is open and is the largest open set contained in M.*

Definition 1.3.2. *A* ***topological space*** *(X, τ) is a set X together with a collection τ of subsets of X such that τ satisfies the following properties:*
(a) $\varnothing \in \tau$, $X \in \tau$.

(b) The union of any members of τ is a member of τ.

(c) The intersection of finitely many members of τ is a member of τ.

- *From this definition, we have that* ***a metric space is a topological space****.*

Definition 1.3.3 (Continuous mapping)**.** *Let $X = (X, d)$ and $Y = (Y, \tilde{d})$ be metric spaces. A mapping $T \colon X \longrightarrow Y$ is said to be* ***continuous*** *at a point $x_0 \in X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\tilde{d}(Tx, Tx_0) < \varepsilon \qquad \text{for all } x \text{ satisfying } d(x, x_0) < \delta.$$

T is said to be ***continuous*** *if it is continuous at every point of X.*

Theorem 1.3.4 (Continuous mapping)**.** *A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.*

Definition 1.3.5. *Let M be a subset of a metric space X. A point x_0 of X (which may or may not be a point of M) is called an* ***accumulation point*** *of M (or* ***limit point*** *of M) if every neighbourhood of x_0 contains at least one point $y \in M$ distinct from x_0. The set consisting of the points of M and the accumulation points of M is called the* ***closure*** *of M and is denoted by $\bar{M}$. It is the smallest closed set containing M.*

Definition 1.3.6 (Dense set, separable space)**.**

1. *A subset M of a metric space X is said to be* ***dense*** *in X if $\bar{M} = X$.*
   - *Hence, if M is dense in X, then every ball in X, no matter how small, will contain points of M; in other words, in this case there is no point $x \in X$ which has a neighbourhood that does not contain points of M.*
2. *X is said to be* ***separable*** *if it has a countable dense subset of X.*

1. Justify the terms "open ball" and "closed ball" by proving that

   (a) any open ball is an open set.

   Solution: Let (X, d) be a metric space. Consider an open ball $B_r(x_0)$ with both center $x_0 \in X$ and radius $r > 0$ fixed. For any $x \in B_r(x_0)$, we have $d(x, x_0) < r$. We claim that $B_\varepsilon(x)$ with $\varepsilon = r - d(x, x_0) > 0$ is contained in $B_r(x_0)$. Indeed, for any $y \in B_\varepsilon(x)$,

$$\begin{aligned} d(y, x_0) &\leq d(y, x) + d(x, x_0) \\ &< \varepsilon + d(x, x_0) \\ &= \varepsilon + r - \varepsilon = r. \end{aligned}$$

   Since $x \in B_r(x_0)$ was arbitrary, this shows that $B_r(x_0)$ contains a ball about each of its points, and thus is an open set in X. Since $x_0 \in X$ and $r > 0$ were arbitrary, this shows that any open ball in X is an open set in X.

   (b) any closed ball is a closed set.

   Solution: Let (X, d) be a metric space. Consider a closed ball $\tilde{B}_r(x_0)$ with both center $x_0 \in X$ and radius $r > 0$ fixed. To show that it is closed in X, we need to show that $\tilde{B}_r(x_0)^C = X \setminus \tilde{B}_r(x_0)$ is open in X. For any $x \in \tilde{B}_r(x_0)^C$, we have $d(x, x_0) > r$. We claim that $B_\varepsilon(x)$ with $\varepsilon = d(x, x_0) - r > 0$ is contained in $\tilde{B}_r(x_0)^C$. Indeed, for any $y \in B_\varepsilon(x)$, triangle inequality of a metric gives:

$$\begin{aligned} d(x, x_0) &\leq d(x, y) + d(y, x_0) \\ \implies d(y, x_0) &\geq d(x, x_0) - d(x, y) \\ &= d(x, x_0) - d(y, x) \\ &> d(x, x_0) - \varepsilon = r. \end{aligned}$$

   Since $x \in \tilde{B}_r(x_0)^C$ was arbitrary, this shows that $\tilde{B}_r(x_0)^C$ contains a ball about each of its points, and thus is an open set in X or equivalently $\tilde{B}_r(x_0)$ is a closed set in X. Since $x_0 \in X$ and $r > 0$ were arbitrary, this shows that any closed ball in X is a closed set in X.

2. What is an open ball $B_1(x_0)$ in $\mathbb{R}$? In $\mathbb{C}$? In $C[a, b]$?

Solution:

- An open ball $B_1(x_0)$ in $\mathbb{R}$ is the open interval $(x_0 - 1, x_0 + 1)$.
- An open ball $B_1(x_0)$ in $\mathbb{C}$ is the open disk $\mathcal{D} = \{z \in \mathbb{C}\colon |z - x_0| < 1\}$.
- Given $x_0 \in C[a,b]$, an open ball $B_1(x_0)$ in $C[a,b]$ is any continuous function $x \in C[a,b]$ satisfying $\sup\limits_{t\in[a,b]} |x(t) - x_0(t)| < 1$.

3. Consider $C[0, 2\pi]$ and determine the smallest r such that $y \in \tilde{B}(x;r)$, where $x(t) = \sin(t)$ and $y(t) = \cos(t)$.

Solution: We want to maximise $y(t) - x(t)$ over $t \in [0, 2\pi]$. Consider $z(t) = \cos(t) - \sin(t)$, differentiating gives $z'(t) = -\sin(t) - \cos(t)$, which is equal to 0 if and only if $\sin(t) + \cos(t) = 0$, or

$$\tan(t) = -1 \implies t_c = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Evaluating $z(t)$ at t_c gives $z(t_c) = \pm\sqrt{2}$. Thus, the smallest $r > 0$ such that $y \in \tilde{B}_r(x)$ is $r = \sqrt{2}$.

4. Show that any nonempty set $A \subset (X, d)$ is open if and only if it is a union of open balls.

Solution: Suppose A is a nonempty open subset of X. For any $x \in A$, there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subset A$. We claim that $\bigcup\limits_{x\in A} B_{\varepsilon_x}(x) = A$. It is clear that $A \subset \bigcup\limits_{x\in A} B_{\varepsilon_x}(x)$. Suppose $x_0 \in \bigcup\limits_{x\in A} B_{\varepsilon_x}(x)$, then $x_0 \in B_{\varepsilon_{x_0}}(x_0) \subset A \implies \bigcup\limits_{x\in A} B_{\varepsilon_x}(x) \subset A$. Consequently, A is a union of open balls.

Conversely, suppose $A \subset (X, d)$ is a union of open balls, which is also a union of open sets since open balls are open in X. Let Λ be an indexing set (which might be uncountable), we can write A as $A = \bigcup\limits_{n\in\Lambda} U_n$, where U_n is open. Fix any $x \in A$, there exists an $j \in \Lambda$ such that $x \in U_j$. Since U_j is open, there exists an $\varepsilon > 0$ such that

$$x \in B_\varepsilon(x) \subset U_j \subset \bigcup_{n>0} U_n = A.$$

Since $x \in A$ is arbitrary, $A \subset (X, d)$ is open.

5. It is important to realise that certain sets may be open and closed at the same time.

(a) Show that this is always the case for X and $\varnothing$.

> **Solution:** $\varnothing$ is open since $\varnothing$ contains no elements. For any $x \in X$, choose $\varepsilon = 1 > 0$, then $B_\varepsilon(x) \subset X$ by definition. This immediately implies that $\varnothing$ and X are both closed since $\varnothing^C = X$ and $X^C = \varnothing$ are both open.

(b) Show that in a discrete metric space X, every subset is open and closed.

> **Solution:** Consider any subset A in X. For any $x \in A$, there exists an open ball around x that is contained in A by the structure of the discrete metric. Indeed, with $0 < \varepsilon < 1$, $B_\varepsilon(x) = \{x\} \subset A$. Similarly, A is closed by the same argument. Indeed, for any $y \in A^C$, with $0 < \varepsilon < 1$, $B_\varepsilon(y) = \{y\} \subset A^C$.

6. If x_0 is an accumulation point of a set $A \subset (X, d)$, show that any neighbourhood of x_0 contains infinitely many points of A.

> **Solution:** Denote by N a neighbourhood of x_0, by definition it contains an ε-neighbourhood of x_0. Observe that for $\varepsilon_j = \varepsilon/2^j$, $\{B_{\varepsilon_j}(x_0)\}_{j=0}^{\infty}$ are also neighbourhoods of x_0. Since x_0 is an accumulation point of a set $A \subset (X, d)$, by definition each $B_{\varepsilon_j}(x_0)$ contains at least one point $y_j \in A$ distinct from x_0. By construction, $\{y_j\}_{j=0}^{\infty} \subset \bigcup_{j=0}^{\infty} B_{\varepsilon_j}(x_0) = B_\varepsilon(x_0) \subset N$. Since N was an arbitrary neighbourhood of x_0, the statement follows.

7. Describe the closure of each of the following subsets.
   (a) The integers on $\mathbb{R}$,
   (b) the rational numbers on $\mathbb{R}$,
   (c) the complex numbers with rational real and imaginary parts in $\mathbb{C}$,
   (d) the disk $\{z \mid |z| < 1\} \subset \mathbb{C}$.

> **Solution:** (a) $\mathbb{Z}$ (b) $\mathbb{R}$ (c) $\mathbb{C}$ (d) The closed unit disk $\mathcal{D} = \{z \in C\colon |z| \leq 1\}$. Note that (b) and (c) follows from the fact that $\mathbb{Q}$ are dense in $\mathbb{R}$.

8. Show that the closure $\overline{B(x_0; r)}$ of an open ball $B(x_0; r)$ in a metric space can differ from the closed ball $\tilde{B}(x_0; r)$.

> **Solution:** Consider a discrete metric space (X, d), and an open ball $B_1(x) = \{x\}$ with $x \in X$. Then $\overline{B_1(x)} = \{x\}$ but $\tilde{B}_1(x) = X$.

9. Show that

   (a) $A \subset \bar{A}$,

   Solution: This follows immediately from definition of $\bar{A}$.

   (b) $\bar{\bar{A}} = \bar{A}$,

   Solution: It is clear from definition of closure that $\bar{A} \subset \bar{\bar{A}}$. Now suppose $x \in \bar{\bar{A}}$. Either $x \in \bar{A}$ or x is an accumulation point of $\bar{A}$, but accumulation points of $\bar{A}$ are precisely those of A. Thus in both cases $x \in \bar{A}$ and $\bar{\bar{A}} \subset \bar{A}$. Combining these two set inequalities yields the desired set equality.

   (c) $\overline{A \cup B} = \bar{A} \cup \bar{B}$,

   Solution: Suppose $x \in \overline{A \cup B}$. Either $x \in A \cup B$, and either

   - $x \in A \subset \bar{A} \subset \bar{A} \cup \bar{B}$, or
   - $x \in B \subset \bar{B} \subset \bar{A} \cup \bar{B}$, or
   - $x \in A \cap B \subset \bar{A} \cap \bar{B} \subset \bar{A} \cup \bar{B}$, since $A \subset \bar{A}$ & $B \subset \bar{B} \implies A \cap B \subset \bar{A} \cap \bar{B}$.

   or x is an accumulation point of $A \cup B$, but it is contained in $\bar{A} \cup \bar{B}$ by the same reasoning as above. Thus, $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

   Now suppose $x \in \bar{A} \cup \bar{B}$. Either

   - $x \in \bar{A} \subset \overline{A \cup B}$, since $A \subset A \cup B$, or
   - $x \in \bar{B} \subset \overline{A \cup B}$, since $B \subset A \cup B$, or
   - $x \in \bar{A} \cap \bar{B} \subset \overline{A \cup B}$, since $A \cap B \subset A \cup B$.

   Thus, $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Combining these two set inequalities yields the desired set equality.

   (d) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

   Solution: Suppose $x \in \overline{A \cap B}$. Either $x \in A \cap B \implies x \in A \subset \bar{A}$ and $x \in B \subset \bar{B} \implies x \in \bar{A} \cap \bar{B}$, or x is an accumulation point of $A \cap B$. This means that x is an accumulation point of both A and B or equivalently $x \in \bar{A} \cap \bar{B}$.

10. A point x not belonging to a *closed* set $M \subset (X, d)$ always has a nonzero distance from M. To prove this, show that $x \in \bar{A}$ if and only if $D(x, A) = 0$; here A is any nonempty subset of X.

Solution: If $x \in A \subset \bar{A}$, then $D(x, A) = 0$ since $\inf_{y \in A} d(x, y)$ is attained at $y = x$, so suppose $x \in \bar{A} \setminus A$. By definition x is an accumulation point, so each neighbourhood $B_\varepsilon(x)$ of x contains at least one point $y_\varepsilon \in A$ distinct from x, with $d(y_\varepsilon, x) < \varepsilon$. Taking $\varepsilon \longrightarrow 0$ gives $D(x, A) = 0$.

Conversely, suppose $D(x, A) = \inf_{y \in A} d(x, y) = 0$. If this infimum is attained, then $x \in A \subset \bar{A}$, so suppose not. One property of infimum states that for every $\varepsilon > 0$, there exists an $y_\varepsilon \in A$ such that $d(y_\varepsilon, x) < 0 + \varepsilon = \varepsilon$. But this implies that x is an accumulation point of A.

11. **(Boundary)** A *boundary point* x of a set $A \subset (X, d)$ is a point of X (which may or may not belong to A) such that every neighbourhood of x contains points of A as well as points not belonging to A; and the *boundary* (or *frontier*) of A is the set of all boundary points of A. Describe the boundary of
   (a) the intervals (-1,1), [-1,1), [-1,1] on $\mathbb{R}$;
   (b) the set of all rational numbers $\mathbb{Q}$ on $\mathbb{R}$;
   (c) the disks $\{z \in \mathbb{C} \colon |z| < 1\} \subset \mathbb{C}$ and $\{z \in \mathbb{C} \colon |z| \leq 1\} \subset \mathbb{C}$.

Solution: (a) $\{-1, 1\}$. (c) The unit circle on the complex plane $\mathbb{C}$, $\{z \in \mathbb{C} \colon |z| = 1\}$. (b) Note that the interior of $\mathbb{Q} \subset \mathbb{R}$ is empty since for any $\varepsilon > 0$, the open ball $B_\varepsilon(x)$ with $x \in \mathbb{Q}$ is not contained in $\mathbb{Q}$; indeed, $B_\varepsilon(x)$ contains at least one irrational number. Since the closure of $\mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$ is $\mathbb{R}$, it follows that the boundary of $\mathbb{Q}$ on $\mathbb{R}$ is $\mathbb{R}$.

12. **(Space $\boldsymbol{B[a, b]}$)** Show that $B[a, b]$, $a < b$, is not separable.

Solution: Motivated by the proof of non-separability for the space l^∞, consider a subset A of $B[a, b]$ consisting of functions that are defined as follows: for every $c \in [a, b]$, define f_c as

$$f_c(t) = \begin{cases} 1 & \text{if } t = c, \\ 0 & \text{if } t \neq c. \end{cases}$$

It is clear that A is uncountable. Moreover, the metric on $B[a, b]$ shows that any distinct $f, g \in A$ must be of distance 1 apart. If we let each of these functions $f \in A$ be the center of a small ball, say, of radius $\frac{1}{3}$, these balls do not intersect and we have uncountably many of them. If M is any dense set in $B[a, b]$, each of these nonintersecting balls must contain an element of M. Hence M cannot be countable. Since M was an arbitrary dense subset of $B[a, b]$, this conclude that $B[a, b]$ cannot have countable dense subsets. Consequently, $B[a, b]$ is not separable by definition.

Remark: *A general approach to show non-separability is to construct an uncountable family of pairwise disjoint open balls.*

13. Show that a metric space X is separable if and only if X has a countable subset Y with the following property: For every $\varepsilon > 0$ and every $x \in X$ there is a $y \in Y$ such that $d(x, y) < \varepsilon$.

Solution: Suppose X is separable, by definition X has a countable dense subset Y, with $\bar{Y} = X$. Let $\varepsilon > 0$ and fix an $x \in X = \bar{Y}$. Definition of $\bar{Y}$ says that any ε-neighbourhood of x contains at least one $y \in Y$ distinct from x, with $d(y, x) < \varepsilon$. Since $x \in X$ was arbitrary, the statement follows.

Conversely, suppose X has a countable subset Y with the property given above. Then any $x \in X$ with that given property is either a point of Y (since then $d(x, x) = 0 < \varepsilon$) or an accumulation point of Y. Hence, $\bar{Y} = X$ and since Y is countable, X is separable by definition.

14. **(Continuous mapping)** Show that a mapping $T : X \longrightarrow Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.

Solution: The statement is a simple application of Theorem 1.5. It is useful to observe that if M is any subset of Y, and M_0 is the preimage (inverse image) of M under T, then the preimage of $Y \setminus M$ is precisely $X \setminus M_0$. More precisely,

$$M_0 = \{x \in X : f(x) \in M\}.$$
$$X \setminus M_0 = M_0^C = \{x \in X : f(x) \notin M\} = \{x \in X : f(x) \in Y \setminus M\}.$$

Suppose T is continuous. Let $M \subset Y$ be closed and M_0 be the preimage of M under T. Since $Y \setminus M$ is open in Y, theorem above implies that its preimage $X \setminus M_0$ is open in X, or equivalently M_0 is closed in X. Conversely, suppose the preimage of any closed set $M \subset U$ is a closed set in X. This is equivalent to saying that the preimage of any open set $N \subset U$ is an open set in X (refer to observation above). Thus, theorem above implies that T is continuous.

15. Show that the image of an open set under a continuous mapping need not be open.

Solution: Consider $x(t) = \sin(t)$, then x maps $(0, 2\pi)$ to $[-1, 1]$.

1.4 Convergence, Cauchy Sequence, Completeness.

Definition 1.4.1. *A sequence (x_n) in a metric space $X = (X, d)$ is said to* ***converge*** *or to be convergent if there exists an $x \in X$ such that*

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

x is called the ***limit*** *of (x_n).*

Lemma 1.4.2 (Boundedness, limit)**.** *Let $X = (X, d)$ be a metric space.*

(a) A convergent sequence in X is bounded and its limit is unique.

(b) If $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in X, then $d(x_n, y_n) \longrightarrow d(x, y)$.

Definition 1.4.3 (Cauchy sequence, completeness)**.** *A sequence (x_n) in a metric space $X = (X, d)$ is said to be* ***Cauchy*** *if for every $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that*

$$d(x_m, x_n) < \varepsilon \qquad \text{for every } m, n > N.$$

The space X is said to be ***complete*** *if every Cauchy sequence in X converges, that is, has a limit which is an element of X.*

Theorem 1.4.4. *Every convergent sequence in a metric space is a Cauchy sequence.*

Theorem 1.4.5 (Closure, closed set)**.** *Let M be a non-empty subset of a metric space (X, d) and $\bar{M}$ its closure.*

(a) $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \longrightarrow x$.

(b) M is closed if and only if the situation $x_n \in M$, $x_n \longrightarrow x$ implies that $x \in M$.

Theorem 1.4.6 (Complete subspace)**.** *A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.*

Theorem 1.4.7 (Continuous mapping)**.** *A mapping $T\colon X \longrightarrow Y$ of a metric space (X, d) into a metric space $(Y, \tilde{d})$ is continuous at a point $x_0 \in X$ if and only if*

$$x_n \longrightarrow x_0 \implies Tx_n \longrightarrow Tx_0.$$

- *The only if direction is proved using ε-δ definition of continuity, whereas the if direction is a proof by contradiction.*

1. **(Subsequence)** If a sequence (x_n) in a metric space X is convergent and has limit x, show that every subsequence (x_{n_k}) of (x_n) is convergent and has the same limit x.

> **Solution:** Suppose we have a convergent sequence (x_n) in a metric space (X, d), with limit $x \in X$. By definition, for any given $\varepsilon > 0$, there exists an $N_1 = N_1(\varepsilon)$ such that $d(x_n, x) < \varepsilon$ for all $n > N_1$. For any subsequence $(x_{n_k}) \subset (x_n)$ choose $N = N_1$, then for all $k > N$ (which implies that $n_k \geq k > N$ by definition of a subsequence), we have $d(x_{n_k}, x) < \varepsilon$. The statement follows.

2. If (x_n) is Cauchy and has a convergent subsequence, say, $x_{n_k} \longrightarrow x$, show that (x_n) is convergent with the limit x.

> **Solution:** Chhose any $\varepsilon > 0$. Since (x_n) is Cauchy, there exists an N_1 such that $d(x_m, x_n) < \dfrac{\varepsilon}{2}$ for all $m, n > N_1$. Since (x_n) has a convergent subsequence (x_{n_k}) (with limit $x \in X$), there exists an N_2 such that $d(x_{n_k}, x) < \dfrac{\varepsilon}{2}$ for all $k > N_2$. Choose $N = \max\{N_1, N_2\}$, then for all $n > N$ we have (by triangle inequality)
>
> $$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
>
> where we implicitly use the fact that $n_k \geq m > N_1$ for the first bound and $n_k \geq k > N_2$ for the second bound. Since $\varepsilon > 0$ was arbitrary, this shows that (x_n) is convergent with the limit $x \in X$.

3. Show that $x_n \longrightarrow x$ if and only if for every neighbourhood V of x there is an integer n_0 such that $x_n \in V$ for all $n > n_0$.

> **Solution:** Suppose $x_n \longrightarrow x$, then $\lim\limits_{n\to\infty} d(x_n, x) = \lim\limits_{n\to\infty} a_n = 0$, where (a_n) is a sequence of real numbers. Thus, given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - 0| = d(x_n, x) < \varepsilon$ for all $n > N$. In particular, we have that $x_n \in B_\varepsilon(x)$ for all $n > N$. Since $\varepsilon > 0$ was arbitrary, the statement follows by setting $V = B_\varepsilon(x)$ and $n_0 = N = N(\varepsilon)$. Conversely, suppose (x_n) is a sequence with the given property in the problem. For a fixed $\varepsilon > 0$, if we set $V = B_\varepsilon(x)$, there exists an $n_0 \in \mathbb{N}$ such that $x_n \in V$ for all $n > n_0$. In particular, we have that $d(x_n, x) < \varepsilon$ for all $n > n_0$. This shows that $x_n \longrightarrow x$ since $\varepsilon > 0$ was arbitrary.

4. **(Boundedness)** Show that a Cauchy sequence is bounded.

> **Solution:** Choose any Cauchy sequence (x_n) in a metric space (X, d). Given any $\varepsilon > 0$, there exists an N such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N$. Choose $\varepsilon = 1 > 0$, there exists N_ε such that $d(x_m, x_n) < 1$ for all $m, n > N_\varepsilon$. Let $\alpha = \max\limits_{j,k=1,\ldots,N_\varepsilon} d(x_j, x_k)$, and choose $A = \max\{\alpha, 1\}$. We see that $d(x_m, x_n) < A$ for all $m, n \geq 1$. This shows that (x_n) is bounded.

5. Is boundedness of a sequence in a metric space sufficient for the sequence to be Cauchy? Convergent?

Solution: It turns out that boundedness of a sequence in a metric space does not imply Cauchy or convergent. Consider the sequence (x_n), where $x_n = (-1)^n$; it is clear that (x_n) is bounded in $(\mathbb{R}, d)$, where $d(x_m, x_n) = |x_m - x_n|$.

- (x_n) is not Cauchy in $\mathbb{R}$ since if we pick $\varepsilon = \dfrac{1}{2} > 0$, then for all N, there exists $m, n > N$ (we could choose $m = N+1$, $n = N+2$ for example) such that $d(x_m, x_n) = d(x_{N+1}, x_{N+2}) = 1 > \dfrac{1}{2}$.
- (x_n) is not convergent in $\mathbb{R}$ since it is not Cauchy.

6. If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$, converges. Give illustrative examples.

Solution: Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d) and fix an $\varepsilon > 0$. By definition, there exists N_1, N_2 such that

$$\begin{aligned} d(x_m, x_n) &< \frac{\varepsilon}{2} \quad \text{for all } m, n > N_1. \\ d(y_m, y_n) &< \frac{\varepsilon}{2} \quad \text{for all } m, n > N_2. \end{aligned}$$

Define a sequence (a_n), with $a_n = d(x_n, y_n)$; observe that (a_n) is a sequence in $\mathbb{R}$. Thus to show that (a_n) converges, an alternative way is to show that (a_n) is a Cauchy sequence. First, generalised triangle inequality of d yields the following two inequalities:

$$\begin{aligned} a_m = d(x_m, y_m) &\le d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) \\ &= d(x_m, x_n) + a_n + d(y_m, y_n) \\ \implies a_m - a_n &\le d(x_m, x_n) + d(y_m, y_n). \\ a_n = d(x_n, y_n) &\le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ &= d(x_m, x_n) + a_m + d(y_m, y_n) \\ \implies a_n - a_m &\le d(x_m, x_n) + d(y_m, y_n). \end{aligned}$$

Choose $N = \max\{N_1, N_2\}$. Combining these inequalities yields:

$$\begin{aligned} |a_m - a_n| &\le d(x_m, x_n) + d(y_m, y_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } m, n > N. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows that the sequence (a_n) is Cauchy in $\mathbb{R}$. Consequently, (a_n) must converge.

7. Give an indirect proof of Lemma 1.4-2(b).

Solution: We want to prove that if $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in a metric space (X, d), then $d(x_n, y_n) \longrightarrow d(x, y)$. Choose any $\varepsilon > 0$. By definition, there exists N_1, N_2 such that

$$\begin{aligned} d(x_n, x) &< \frac{\varepsilon}{2} \quad \text{for all } n > N_1. \\ d(y_n, x) &< \frac{\varepsilon}{2} \quad \text{for all } n > N_2. \end{aligned}$$

Generalised triangle inequality of d yields the following two inequalities:

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y) + d(y, y_n) \\ \Longrightarrow \ d(x_n, y_n) - d(x, y) &\leq d(x_n, x) + d(y_n, y). \\ d(x, y) &\leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \\ \Longrightarrow \ d(x, y) - d(x_n, y_n) &\leq d(x_n, x) + d(y_n, y). \end{aligned}$$

Now choose $N = \max\{N_1, N_2\}$. Combining these inequalities yields:

$$\begin{aligned} |d(x_n, y_n) - d(x, y)| &\leq d(x_n, x) + d(y_n, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } n > N. \end{aligned}$$

This proves the statement since $\varepsilon > 0$ was arbitrary.

8. If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$,

$$a d_1(x, y) \leq d_2(x, y) \leq b d_1(x, y), \tag{\dagger}$$

show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same.

Solution: Suppose (x_n) is any Cauchy sequence in (X, d_1). Given $\varepsilon > 0$, there exists an N_1 such that $d(x_m, x_n) < \frac{\varepsilon}{b}$ for all $m, n > N_1$. Using the second inequality in (†),

$$d_2(x_m, x_n) \leq b d_1(x_m, x_n) < \cancel{b}\left(\frac{\varepsilon}{\cancel{b}}\right) = \varepsilon \quad \text{for all } m, n > N_1.$$

Thus, (x_n) is also a Cauchy sequence in (X, d_2).

Now suppose (y_n) is any Cauchy sequence in (X, d_2). Given $\varepsilon > 0$, there exists an $N_2 \in \mathbb{N}$ such that $d(y_m, y_n) < a\varepsilon$ for all $m, n > N_2$. Using the first inequality in (†),

$$d_1(y_m, y_n) \leq \frac{1}{a} d_1(y_m, y_n) < \frac{1}{\cancel{a}}(\cancel{a}\varepsilon) = \varepsilon \quad \text{for all } m, n > N_2.$$

Thus, (y_n) is also a Cauchy sequence in (X, d_1).

9. The Cartesian product $X = X_1 \times X_2$ of two metric spaces (X_1, d_1) and (X_2, d_2) can be made into a metric space (X, d) in many ways. For instance, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we proved previously that the following are metrics for X.

$$\begin{aligned} d_a(x,y) &= d_1(x_1,y_1) + d_2(x_2,y_2). \\ d_b(x,y) &= \sqrt{d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2}. \\ d_c(x,y) &= \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}. \end{aligned}$$

Using Problem 8, show that (X, d_a), (X, d_b) and (X, d_c) all have the same Cauchy sequences.

Solution: We simply need to establish a few related inequalities.

$$\begin{aligned} d_a = d_1 + d_2 \leq 2\max\{d_1, d_2\} = 2d_c &\implies d_a \leq 2d_c. \\ d_c = \max\{d_1, d_2\} \leq d_1 + d_2 = d_a &\implies d_c \leq d_a. \\ d_b^2 = d_1^2 + d_2^2 \leq d_1^2 + d_2^2 + 2d_1d_2 = (d_1+d_2)^2 = d_a^2 &\implies d_b \leq d_a. \\ d_c = \max\{d_1, d_2\} = \max\{\sqrt{d_1^2}, \sqrt{d_2^2}\} \leq \sqrt{d_1^2 + d_2^2} = d_b &\implies d_c \leq d_b. \\ d_a \leq 2d_c \leq 2d_b &\implies d_a \leq 2d_b. \\ d_b \leq d_a \leq 2d_c &\implies d_b \leq 2d_c. \end{aligned}$$

Consequently, we have the following inequality:

$$d_a \leq 2d_c \leq 2d_b \leq 2d_a.$$

10. Using the completeness of $\mathbb{R}$, prove completeness of $\mathbb{C}$.

Solution: Let (z_n) be any Cauchy sequence in $\mathbb{C}$, where $z_n = x_n + iy_n$. For any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n > N$,

$$\begin{aligned} d_{\mathbb{C}}(z_m, z_n) = |z_m - z_n| &= \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} \leq \varepsilon \\ &\implies (x_m - x_n)^2 + (y_m - y_n)^2 \leq \varepsilon^2. \end{aligned}$$

The last inequality implies that for all $m, n > N$,

$$\begin{aligned} (x_m - x_n)^2 \leq \varepsilon^2 &\implies |x_m - x_n| \leq \varepsilon. \\ (y_m - y_n)^2 \leq \varepsilon^2 &\implies |y_m - y_n| \leq \varepsilon. \end{aligned}$$

Thus, both sequences (x_n) and (y_n) are Cauchy in $\mathbb{R}$, which converges to, say, x and y respectively as $n \longrightarrow \infty$ by completeness of $\mathbb{R}$. Define $z = x + iy \in \mathbb{C}$, then convergence of (x_n) and (y_n) implies that $d_{\mathbb{R}}(x_n, x)$ and $d_{\mathbb{R}}(y_n, y)$ both converge to 0 as $n \longrightarrow \infty$. Expanding the definition of $d_{\mathbb{C}}(z_n, z)$ gives

$$\begin{aligned} d_{\mathbb{C}}(z_n, z) = |z_n - z| &= \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &= \sqrt{d_{\mathbb{R}}(x_n, x)^2 + d_{\mathbb{R}}(y_n, y)^2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

This shows that $z \in \mathbb{C}$ is the limit of (z_n). Since (z_n) was an arbitrary Cauchy sequence in $\mathbb{C}$, this proves completeness of $\mathbb{C}$.

1.5 Examples. Completeness Proofs.

To prove completeness, we take an arbitrary Cauchy sequence (x_n) in X and show that it converges in X. For different spaces, such proofs may vary in complexity, but they have approximately the same general pattern:

1. Construct an element x (to be used as a limit).
2. Prove that x is an element of the space considered.
3. Prove convergence $x_n \longrightarrow x$ (in the sense of the metric).

1. Let $a, b \in \mathbb{R}$ and $a < b$. Show that the open interval (a, b) is an incomplete subspace of $\mathbb{R}$, whereas the closed interval $[a, b]$ is complete.

Solution: Consider a sequence (x_n) in the metric space $\Big((a, b), |\cdot|\Big)$, where $x_n = a + \frac{1}{n}$. Given any $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$, then for any $m, n > N > \frac{2}{\varepsilon}$,

$$\begin{aligned} d(x_m, x_n) = \left|\frac{1}{m} - \frac{1}{n}\right| &\leq \left|\frac{1}{m}\right| + \left|\frac{1}{n}\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that (x_n) is a Cauchy sequence in (a, b). However, $(x_n) \longrightarrow a \notin (a, b)$ as $n \longrightarrow \infty$. This shows that (a, b) is an incomplete subspace of $\mathbb{R}$. Since $[a, b]$ is a closed (metric) subspace of $\mathbb{R}$ (which is a complete metric space), it follows that the closed interval $[a, b]$ is complete.

2. Let X be the space of all ordered n-tuples $x = (\xi_1, \dots, \xi_n)$ of real numbers and $d(x, y) = \max_j |\xi_j - \eta_j|$, where $y = (\eta_j)$. Show that (X, d) is complete.

Solution: Consider any Cauchy sequence (x_m) in $\mathbb{R}^n$, where $x_m = \Big(\xi_1^{(m)}, \dots, \xi_n^{(m)}\Big)$. Since (x_m) is Cauchy, given any $\varepsilon > 0$, there exists an N such that for all $m, r > N$,

$$d(x_m, x_r) = \max_{j=1,\dots,n} |\xi_j^{(m)} - \xi_j^{(r)}| < \varepsilon$$

In particular, for every fixed $j = 1, \dots, n$,

$$|\xi_j^{(m)} - \xi_j^{(r)}| < \varepsilon \quad \text{for all } m, r > N. \tag{$\dagger$}$$

Hence, for every fixed j, the sequence $(\xi_j^{(1)}, \xi_j^{(2)}, \dots)$ is a Cauchy sequence of real numbers. It converges by completeness of $\mathbb{R}$, say, $\xi_j^{(m)} \longrightarrow \xi_j$ as $m \longrightarrow \infty$.

Using these n limits, we define $x = (\xi_1, \ldots, \xi_n)$. Clearly, $x \in \mathbb{R}^n$. From (†), with $r \longrightarrow \infty$,

$$|\xi_j^{(m)} - \xi_j| < \varepsilon \quad \text{for all } m > N.$$

Since the RHS is independent of j, taking maximum over $j = 1, \ldots, n$ in both sides yields

$$d(x_m, x) = \max_{j=1,\ldots,n} |\xi_j^{(m)} - \xi_j| < \varepsilon \quad \text{for all } m > N.$$

This shows that $x_m \longrightarrow x$. Since (x_m) was an arbitrary Cauchy sequence, $\mathbb{R}^n$ with the metric $d(x, y) = \max |\xi_j - \eta_j|$ is complete.

3. Let $M \subset l^\infty$ be the subpace consisting of all sequences $x = (\xi_j)$ with at most finitely many nonzero terms. Find a Cauchy sequence in M which does not converge in M, so that M is not complete.

Solution: Let (x_n) be a sequence in $M \subset l^\infty$, where

$$\xi_j^{(n)} = \begin{cases} \dfrac{1}{j} & \text{if } j \le n, \\ 0 & \text{if } j > n. \end{cases}$$

i.e. $x_n = \left(1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots, \dfrac{1}{n}, 0, 0, \ldots\ldots\right)$. Given any $\varepsilon > 0$, choose N such that $N + 1 > \dfrac{1}{\varepsilon}$, then for any $m > n > N$,

$$d(x_m, x_n) = \sup_{j \in \mathbb{N}} \left|\xi_j^{(m)} - \xi_j^{(n)}\right| = \frac{1}{n+1} \le \frac{1}{N+1} < \varepsilon.$$

This shows that (x_n) is Cauchy in M. However, it is clear that $x_n \longrightarrow x = \left(\dfrac{1}{n}\right)$ as $n \longrightarrow \infty$, but since $x \notin M$, (x_n) does not converge in M.

4. Show that M in Problem 3 is not complete by applying Theorem 1.4-7.

Solution: It is easy to see that M is a subspace of l^∞. The sequence in Problem 3 shows that $x_n \longrightarrow x$ in l^∞ since

$$d(x_n, x) = \frac{1}{n+1} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

However, x doesn't belong to M since it has infinitely many nonzero terms. This shows that M is not a closed subspace of l^∞, and therefore not complete.

5. Show that the set X of all integers with metric d defined by $d(m,n) = |m - n|$ is a complete metric space.

Solution: Observe that for any two distinct integers m, n, $d(m,n) \geq 1$. This implies that the only Cauchy sequences in X are either constant sequences or sequences that are eventually constant. This shows that the set X of all integers with the given metric is complete.

6. Show that the set of all real numbers constitutes an incomplete metric space if we choose $d(x,y) = |\arctan x - \arctan y|$.

Solution: Consider the sequence (x_n), where $x_n = n$. We claim that (x_n) is Cauchy but not convergent in $\mathbb{R}$.

- Since $\arctan n \longrightarrow \dfrac{\pi}{2}$ as $n \longrightarrow \infty$, given any $\varepsilon > 0$, there exists an N such that $\left|\arctan(n) - \dfrac{\pi}{2}\right| < \dfrac{\varepsilon}{2}$ for all $n > N$. Thus, for all $m, n > N$,

$$\begin{aligned} d(x_m, x_n) = |\arctan(m) - \arctan(n)| &\leq \left|\arctan(m) - \frac{\pi}{2}\right| + \left|\frac{\pi}{2} - \arctan(n)\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

- Suppose, for contradiction, that (x_n) converges in $\mathbb{R}$ with the given metric. By definition, there exists an $x \in \mathbb{R}$ such that

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} |\arctan(n) - \arctan(x)| = 0.$$

which then implies that $\arctan(x)$ must equal to $\dfrac{\pi}{2}$, by uniqueness of limits. This contradicts the assumption that $x \in \mathbb{R}$, since $\arctan(x) < \dfrac{\pi}{2}$ for any $x \in \mathbb{R}$.

7. Let X be the set of all positive integers and $d(m,n) = |m^{-1} - n^{-1}|$. Show that (X, d) is not complete.

Solution: Consider a sequence $(x_n) \in X$, where $x_n = n$. With the given metric,

$$d(x_m, x_n) = \left|\frac{1}{m} - \frac{1}{n}\right|$$

and similar argument in Problem 1 shows that (x_n) is a Cauchy sequence. If (x_n) were to converge to some positive integer x, then it must satisfy

$$d(x_n, x) = \left|\frac{1}{n} - \frac{1}{x}\right| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Clearly, $\dfrac{1}{x}$ must be 0, which is a contradiction since no positive integers x gives $\dfrac{1}{x} = 0$.

8. **(Space $C[a,b]$)** Show that the subspace $Y \subset C[a,b]$ consisting of all $x \in C[a,b]$ such that $x(a) = x(b)$ is complete.

Solution: Consider $Y \subset C[a,b]$ defined by $Y = \{x \in C[a,b] \colon x(a) = x(b)\}$. It suffices to show that Y is closed in $C[a,b]$, so that completeness follows from Theorem 1.4.7. Consider any $f \in \bar{Y}$, the closure of Y. There exists a sequence of functions $(f_n) \in Y$ such that $f_n \longrightarrow f$ in $C[a,b]$. By definition, given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have

$$d(f_n, f) = \max_{t \in [a,b]} |f_n(t) - f(t)| < \varepsilon.$$

In particular, for every $t \in [a,b]$, $|f_n(t) - f(t)| < \varepsilon$ for all $n > N$. This shows that $(f_n(t))$ converges to $f(t)$ uniformly on $[a,b]$. Since the $f_n's$ are continuous function on $[a,b]$ and the convergence is uniform, the limit function f is continuous on $[a,b]$. We are left with showing $f(a) = f(b)$ to conclude that $f \in Y$. Indeed, triangle inequality for real numbers gives:

$$\begin{aligned} |f(a) - f(b)| &\leq |f(a) - f_n(a)| + |f_n(a) - f_n(b)| + |f_n(b) - f(b)| \\ &= |f(a) - f_n(a)| + |f_n(b) - f(b)| \\ &\leq 2 \max_{t \in [a,b]} |f_n(t) - f(t)| \\ &= 2d(f_n, f) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

9. In 1.5-5 we referred to the following theorem of calculus. If a sequence (x_m) of a continuous functions on $[a,b]$ converges on $[a,b]$ and the convergence is uniform on $[a,b]$, then the limit function x is continuous on $[a,b]$. Prove this theorem.

Solution: The proof employs the so called $\varepsilon/3$ proof, which is widely used in proofs concerning uniform continuity. Choose any $t_0 \in [a,b]$ and $\varepsilon > 0$.

- Since (f_n) converges to f uniformly, there exists an $N \in \mathbb{N}$ such that for all $t \in [a,b]$ and for all $n > N$, we have $|f_n(t) - f(t)| < \dfrac{\varepsilon}{3}$.
- Since f_{N+1} is continuous at $t_0 \in [a,b]$, there exists an $\delta > 0$ such that $|f_{N+1}(t) - f_{N+1}(t_0)| < \dfrac{\varepsilon}{3}$ for all $t \in [a,b]$ satisfying $|t - t_0| < \delta$.
- Thus, if $|t - t_0| < \delta$, triangle inequality gives:

  $$|f(t) - f(t_0)| \leq |f(t) - f_{N+1}(t)| + |f_{N+1}(t) - f_{N+1}(t_0)| + |f_{N+1}(t_0) - f(t_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

This shows that f is continuous at t_0.

Since $t_0 \in [a,b]$ was arbitrary, f is continuous on $[a,b]$ or $f \in C[a,b]$.

10. **(Discrete metric)** Show that a discrete metric space is complete.

Solution: Let (X,d) be a discrete metric space, for any two distinct $x, y \in X$, $d(x,y) = 1$. This implies that the only Cauchy sequences in X are either constant sequences or sequences that are eventually constant. This shows that a discrete metric space is complete.

11. **(Space *s*)** Show that in the space s, we have $x_n \longrightarrow x$ if and only if $\xi_j^{(n)} \longrightarrow \xi_j$ for all $j = 1, 2, \ldots$, where $x_n = \left(\xi_j^{(n)}\right)$ and $x = (\xi_j)$.

Solution: The sequence space s consists of the set of all (bounded or unbounded) sequences of complex numbers and the metric d defined by

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}.$$

where $x = (\xi_j)$ and $y = (\eta_j)$.

Suppose $x_n \longrightarrow x$ in s, where $x_n = \left(\xi_j^{(n)}\right)$. For every $j \geq 1$, given any $\varepsilon > 0$, there exists an N such that for all $n > N$ we have:

$$\begin{aligned}
\frac{1}{2^j} \frac{\left|\xi_j^{(n)} - \xi_j\right|}{1 + \left|\xi_j^{(n)} - \xi_j\right|} &\leq d(x_n, x) < \frac{1}{2^j} \frac{\varepsilon}{1+\varepsilon} \\
\frac{\left|\xi_j^{(n)} - \xi_j\right|}{1 + \left|\xi_j^{(n)} - \xi_j\right|} &< \frac{\varepsilon}{1+\varepsilon} \\
\left|\xi_j^{(n)} - \xi_j\right|(1+\varepsilon) &< \varepsilon\left[1 + \left|\xi_j^{(n)} - \xi_j\right|\right] \\
\left|\xi_j^{(n)} - \xi_j\right| &< \varepsilon
\end{aligned}$$

This shows that $\xi_j^{(n)} \longrightarrow \xi_j$ as $n \longrightarrow \infty$. Since $j \geq 1$ was arbitrary, the result follows.

Conversely, suppose $\xi_j^{(n)} \longrightarrow \xi_j$ for all $j \geq 1$, where $x_n = \left(\xi_j^{(n)}\right)$ and $x = (\xi_j)$. This implies that for every fixed $j \geq 1$,

$$\frac{1}{2^j}\frac{\left|\xi_j^{(n)} - \xi_j\right|}{1 + \left|\xi_j^{(n)} - \xi_j\right|} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

This shows that $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$.

12. Using Problem 11, show that the sequence space s is complete.

Solution: Consider any Cauchy sequence (x_n) in s, where $x_n = \left(\xi_j^{(n)}\right)$. Since (x_n) is Cauchy, for every $j \geq 1$, given any $\varepsilon > 0$, there exists an N such that for all $m, n > N$ we have

$$\frac{1}{2^j}\frac{\left|\xi_j^{(m)} - \xi_j^{(n)}\right|}{1 + \left|\xi_j^{(m)} - \xi_j^{(n)}\right|} \leq d(x_m, x_n) < \frac{1}{2^j}\frac{\varepsilon}{1+\varepsilon}.$$

In particular, for every $j \geq 1$, $\left|\xi_j^{(m)} - \xi_j^{(n)}\right| < \varepsilon$ for all $m, n > N$. Hence, for every $j \geq 1$, the sequence $\left(\xi_j^{(1)}, \xi_j^{(2)}, \ldots\right)$ is a Cauchy sequence of real numbers. It converges by completeness of $\mathbb{R}$, say, $\xi_j^{(n)} \longrightarrow \xi_j$ as $n \longrightarrow \infty$. Since $j \geq 1$ was arbitrary, this shows that $\xi_j^{(n)} \longrightarrow \xi_j$ as $n \longrightarrow \infty$ for all $j \geq 1$. Identifying $x = (\xi_j)$, we have $x_n \longrightarrow x$ as $n \longrightarrow \infty$ from Problem 11. Since (x_n) was an arbitrary Cauchy sequence in s, this proves completeness of s.

13. Let X be the set of all continuous real-valued functions on $J = [0, 1]$, and let

$$d(x, y) = \int_0^1 |x(t) - y(t)|\, dt.$$

Show that the sequence (x_n) is Cauchy in X, where

$$x_n(t) = \begin{cases} n & \text{if } 0 \leq t \leq \dfrac{1}{n^2}, \\ \dfrac{1}{\sqrt{t}} & \text{if } \dfrac{1}{n^2} \leq t \leq 1. \end{cases}$$

Solution: WLOG, take $m > n$. Sketching out $|x_m(t) - x_n(t)|$, we deduce that

$$d(x_m, x_n) = \int_0^{\frac{1}{m^2}} (m - n)\, dt + \int_{\frac{1}{m^2}}^{\frac{1}{n^2}} \left(\frac{1}{\sqrt{t}} - n\right) dt$$

$$= (m-n)\frac{1}{m^2} + 2\left(\frac{1}{n} - \frac{1}{m}\right) - n\left(\frac{1}{n^2} - \frac{1}{m^2}\right) = \frac{1}{n} - \frac{1}{m}.$$

Similar argument in Problem 1 shows that (x_n) is a Cauchy sequence in $C[0,1]$.

14. Show that the Cauchy sequence in Problem 13 does not converge.

Solution: For every $x \in C[0,1]$,

$$\begin{aligned} d(x_n, x) &= \int_0^1 |x_n(t) - x(t)|\,dt \\ &= \int_0^{\frac{1}{n^2}} |n - x(t)|\,dt + \int_{\frac{1}{n^2}}^1 \left|\frac{1}{\sqrt{t}} - x(t)\right|\,dt \end{aligned}$$

Since the integrands are nonnegative, so is each integral on the right. Hence, $d(x_n, x) \longrightarrow 0$ would imply that each integral approaches zero and, since x is continuous, we should have $x(t) = \frac{1}{\sqrt{t}}$ if $t \in (0,1]$. But this is impossible for a continuous function, otherwise we would have discontinuity at $t = 0$. Hence, (x_n) does not converge, that is, does not have a limit in $C[0,1]$.

15. Let X be the metric space of all real sequences $x = (\xi_j)$ each of which has only finitely many nonzero terms, and $d(x,y) = \sum |\xi_j - \eta_j|$, where $y = (\eta_j)$. Note that this is a finite sum but the number of terms depends on x and y. Show that (x_n) with $x_n = \left(\xi_j^{(n)}\right)$,

$$\xi_j^{(n)} = \begin{cases} \frac{1}{j^2} & \text{for } j = 1, \ldots, n, \\ 0 & \text{for } j > n. \end{cases}$$

is Cauchy but does not converge.

Solution: Since $\sum_{j=1}^{\infty} \frac{1}{j^2}$ is convergent and it is a sum of positive terms, given any $\varepsilon > 0$, there exists an N_1 such that $\sum_{j=n}^{\infty} \frac{1}{j^2} < \varepsilon$ for all $n > N_1$. Choose $N = N_1$, then for all $m > n > N$,

$$d(x_m, x_n) = \sum_{j=n+1}^{m} \frac{1}{j^2} \le \sum_{j=n+1}^{\infty} \frac{1}{j^2} \le \sum_{j=N+1}^{\infty} \frac{1}{j^2} < \varepsilon.$$

This shows that (x_n) is a Cauchy sequence. For every $x = (\xi_j) \in X$, there exists an $N = N_x$ such that $\xi_j = 0$ for all $j > N$. Then for all $n > N$,

$$d(x_n, x) = |1 - \xi_1| + \left|\frac{1}{4} - \xi_2\right| + \ldots + \left|\frac{1}{N^2} - \xi_N\right| + \frac{1}{(N+1)^2} + \ldots + \frac{1}{n^2}.$$

We can clearly see that, even if $\xi_j = \dfrac{1}{j^2}$ for all $j \leq N$, $d(x_n, x)$ does not converge to 0 as $n \longrightarrow \infty$.

1.6 Completion of Metric Spaces.

1. Show that if a subspace Y of a metric space consists of finitely many points, then Y is complete.

   Solution:

2. What is the completion of (X, d), where X is the set of all rational numbers $\mathbb{Q}$ and $d(x, y) = |x - y|$?

   Solution:

3. What is the completion of a discrete metric space X?

   Solution:

4. If X_1 and X_2 are isometric and X_1 is complete, show that X_2 is complete.

   Solution:

5. **(Homeomorphism)** A *homeomorphism* is a continuous bijective mapping $T\colon X \longrightarrow Y$ whose inverse is continuous; the metric spaces X and Y are then said to be *homeomorphic.*

   (a) Show that if X and Y are isometric, they are homeomorphic.

   Solution:

   (b) Illustrate with an example that a complete and an incomplete metric space may be homeomorphic.

   Solution:

6. Show that $C[0, 1]$ and $C[a, b]$ are isometric.

   Solution: Consider the mapping T defined by

$$T\colon C[0,1] \longrightarrow C[a,b]\colon f \mapsto g(s) = f\left(\frac{s-a}{b-1}\right).$$

(a) T is an isometry. Indeed, for any $f_1, f_2 \in C[0,1]$ we have

$$\begin{aligned} d(Tf_1, Tf_2) &= \max_{t\in[a,b]} |Tf_1(t) - Tf_2(t)| \\ &= \max_{t\in[a,b]} \left| f_1\left(\frac{t-a}{b-a}\right) - f_2\left(\frac{t-a}{b-1}\right)\right| \\ &= \max_{s\in[0,1]} |f_1(s) - f_2(s)| \\ &= d(f_1, f_2). \end{aligned}$$

(b) T is injective. Indeed, suppose $Tf_1 = Tf_2$, then $0 = d(Tf_1, Tf_2) = d(f_1, f_2)$ since T is an isometry. This implies that $d(f_1, f_2) = 0 \implies f_1 = f_2$.

(c) T is surjective by construction. Indeed, for any $g \in C[a,b]$, define f such that $g(s) = f\left(\frac{s-a}{b-a}\right)$. Note that $f \in C[0,1]$ since $\frac{s-a}{b-a} \in [0,1]$ for all $s \in [a,b]$, and g is continuous on $[a,b]$.

7. If (X,d) is complete, show that $(X,\tilde{d})$, where $\tilde{d} = \dfrac{d}{1+d}$, is complete.

Solution:

8. Show that in Problem 7, completeness of $(X,\tilde{d})$ implies completeness of (X,d).

Solution:

9. If (x_n) and (x_n') in (X,d) are such that $\lim\limits_{n\to\infty} d(x_n, x_n') = 0$ holds and $x_n \longrightarrow l$, show that (x_n') converges and has the limit l.

Solution:

10. If (x_n) and (x_n') are convergent sequences in a metric space (X,d) and have the same limit l, show that they satisfy $\lim\limits_{n\to\infty} d(x_n, x_n') = 0$.

Solution:

11. Show that $\lim\limits_{n\to\infty} d(x_n, x_n') = 0$ defines an equivalence relation on the set of all Cauchy sequences of elements of X.

Solution:

12. If (x_n) is Cauchy in (X, d) and (x_n') in X satisfies $\lim_{n\to\infty} d(x_n, x_n') = 0$, show that (x_n') is Cauchy in X.

Solution:

13. **(Pseudometric)** A *finite pseudometric* on a set X is a function $d\colon X \times X \longrightarrow \mathbb{R}$ satisfying (M1), (M3), (M4) and

$$d(x, x) = 0. \tag{M2*}$$

What is the difference between a metric and a pseudometric? Show that $d(x, y) = |\xi_1 - \eta_1|$ defines a pseudometric on the set of all ordered pairs of real numbers, where $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$. (We mention that some authors use the term *semimetric* instead of *pseudometric*.)

Solution:

14. Does

$$d(x, y) = \int_a^b |x(t) - y(t)|\, dt$$

define a metric or pseudometric on X if X is

(a) the set of all real-valued continuous function on $[a, b]$,

Solution:

(b) the set of all real-valued Riemann integrable functions on $[a, b]$?

Solution:

15. If (X, d) is a pseudometric space, we call a set

$$B_r(x_0) = \{x \in X \colon d(x, x_0) < r\}$$

an *open ball* in X with center $x_0 \in X$ and *radius* $r > 0$. What are open balls of radius 1 in Problem 13?

Solution: