Research Statement

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Overview. My research interests lie in Commutative Algebra, with some overlap into Algebraic Geometry. Commutative Algebra is a branch of pure mathematics which is essentially the study of commutative rings. Its roots lie in Algebraic Geometry and Algebraic Number Theory. I study the behavior of maps on divisor class groups and Chow groups determined by hypersurface sections. My theorems represent steps towards understanding the injectivity of these maps on divisor class groups. My current interest involves extending these results to Chow groups. This is a natural progression since the divisor class group makes up one component of the Chow group.

In addition, I am involved in a second research project, concerning powers of ideals. Although the origin of the work stems from basic questions concerning ideals, resulting questions involve increasingly complicated theories, for example integral closure of ideals and valuation theory. This project is discussed briefly in the last section.

1 Background

The development of the divisor class group of a Noetherian normal domain A is due, in large part, to Samuel's work [12], [13] on unique factorization domains (UFD's) in the 1960's. Roughly speaking, the divisor class group of A, denoted by $\mathrm{Cl}(A)$, is a measure of the extent to which A fails to be a UFD. In particular, the divisor class group of A is trivial if and only if A is a UFD. It has long been known that if A is a UFD, then so is a polynomial ring over A, a result due to Gauss. However, the same is not true for the power series ring. In fact, the theory of UFD's includes extensive study of when a similar result holds for power series rings. Samuel conjectured the following: If A is a Noetherian complete local UFD, then the power series ring over A is also a UFD. However, without additional restrictions, this conjecture is false. Counterexamples to this conjecture, as well as subsequent research in the subject of divisor class groups, rely heavily upon methods from algebraic geometry. For instance, using projective schemes, Danilov [3] defined a map from the divisor class group of the power series of A to the divisor class group of A, and, in a series of articles [1], [2], [3], he characterized its injectivity. These results in some ways parallel those of Grothendieck [5], who found conditions under which the homomorphism from the Picard group of the punctured spectrum of A to that of a hypersurface is injective.

Although I am interested in the purely algebraic notions of factoriality and the divisor class group, these concepts also have geometric interpretations. More specifically, in geometry, a domain A arises as the homogeneous coordinate ring of an irreducible variety V. If the dimension of V is n, then to say that A is factorial, roughly speaking, means that for every subvariety $Y \subset V$ of dimension $n-1$, the ideal of functions that vanish on Y is principal. The divisor class group of V consists of isomorphism classes of codimension one cycles on V. It has much importance in algebraic geometry since it is a subtle intrinsic invariant of V that can be used in the classification of algebraic varieties.

2 Divisor Class Groups

Let f be a prime element such that the hypersurface determined by f is normal. Lipman [8] generalized Danilov's map by showing that there is a homomorphism, often referred to as a restriction map, from the divisor class group of A to the divisor class group of the hypersurface determined by f. This map need not be injective. However, Claudia Miller [9] used a generalized notion of the divisor class group to prove that the intersection of the kernels of $\mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A[[T]]/(T^n))$ is trivial. In other words, no non-trivial divisor class can be in each of the kernels.

This motivates the investigation into whether a similar result will hold more generally for a sequence of distinct elements. To be specific, let A be a Noetherian local normal domain and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of prime elements such that each hypersurface $A_n = A/f_nA$ is normal and f_n approaches zero as n goes to infinity. I considered the following two questions:

1. Is the intersection of the kernels of $\mathrm{Cl}(A) \to \mathrm{Cl}(A_n)$ trivial?
2. Are there situations where a positive integer N exists, such that if f_n is in the N-th power of the maximal ideal, then $\mathrm{Cl}(A) \to \mathrm{Cl}(A_n)$ is injective? In other words, if the answer to (1) is yes, must it be true that all but finitely many of the kernels are null? If so, are there effective methods to determine N?

My first result shows that the answer to (1) is affirmative when the ambient ring is *excellent*. This is a very mild hypothesis, satisfied by Noetherian rings which arise in algebraic and arithmetic geometry. Moreover, I relax the condition of normality on the hypersurfaces. Instead, I require only that they are regular in codimension one. As a prerequisite, I defined a map from the divisor class group of A to the divisor class group of the integral closure of a hypersurface, i.e., $\mathrm{Cl}(A) \to \mathrm{Cl}(A_n')$, where A_n' is the integral closure of A_n. As with Miller's result, a positive answer to (1) does not provide a clear connection between a given divisor class of the ambient ring and its image in the divisor class group of a specific hypersurface. However, it does suggest that the pathology of the map lies near the "top" of the maximal ideal. The basic philosophy of my research is that the deeper f lies in powers of the maximal ideal, the better the injective behavior of the restriction map.

Theorem 2.1. *Let A be an excellent, normal, local $\mathbb{Q}$-algebra, such that A is an isolated singularity of dimension at least four. In addition, suppose that A has a small Cohen-Macaulay module. Then there is a positive integer N, depending only on the ring A, such that the following holds: If f lies in the N-th power of the maximal ideal and A/fA is regular in codimension one, then $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ is injective.*

This result has a connection to one of the biggest open problems in Commutative Algebra, namely the Small Cohen-Macaulay Modules Conjecture. The conjecture asserts that if A is a complete Noetherian local ring, then A has a small Cohen-Macaulay module. A consequence of Theorem 2.1 is that if there is an element f lying in the N-th power of the maximal ideal and the hypersurface is regular in codimension one such that the restriction map is not injective, then A could not possess a small Cohen-Macaulay module. Such an example would disprove the long-standing conjecture.

Corollary 2.2. *In the context of Theorem 2.1, if there exists a non-zero element f in the N-th power of the maximal ideal such that $(A/fA)'$ is a UFD, then A is a UFD.*

In the case of characteristic $p > 0$, Griffith and I have attacked questions (1) and (2). One difference in this case is the possibility of p-torsion elements in the kernel. We were able to attain the following result.

Theorem 2.3. *([4]) Let k be a perfect field of positive characteristic p and let S denote an $\mathbb{N}$-graded ring of dimension greater than three such that $S_0 = k$ and S is a normal domain. Assume that Spec(S) $- S_+$ is regular. Let $x_1, \ldots, x_d$ be a system of parameters of S contained in the Noetherian different of the completion of S. If f is a homogeneous prime element in the ideal generated by the x_i^2 such that the hypersurface determined by f is regular in codimension one, then the kernel of the restriction map is at worst a bounded p-group.*

There is a geometric interpretation of this result. If we assume that S is generated over k in degree one and $\mathrm{Proj}(S) = V$ is smooth over k, then we let H be the hypersurface defined by the homogeneous element f. Note that H is smooth in codimension one. There is a commutative diagram in which the column homomorphisms amount to restriction [13], [6].

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Cl}(V) & \longrightarrow & \mathrm{Cl}(S) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Cl}(H) & \longrightarrow & \mathrm{Cl}((S/fS)') & \longrightarrow & 0
\end{array}$$

It follows that there is a natural identification between the kernels of the maps on the divisor class groups of the varieties and the divisor class groups of the rings. Thus, the results of the above theorem apply to $\mathrm{Cl}(V) \to \mathrm{Cl}(H)$ as well.

Remark 2.4. *As a corollary to Theorem 2.3, when it is assumed that the hypersurfaces are normal, rather than just regular in codimension one, then the result is the characteristic p analogue of Theorem 2.1.*

Finally, Griffith and I used the affirmative answer to (1) to construct some important examples where the restriction map is not injective. To be specific, most examples for which the kernel of the restriction of divisor classes is non-trivial come about in two ways: either A is a complete intersection of dimension less than or equal to three, or A is a power series ring, as in Danilov's theory. In particular, our goal was to construct examples of rings that were local isolated singularities of dimension at least four. Such examples are important in light of Grothendieck's results. To be specific, he showed that if A is a complete intersection, as well as an isolated singularity, of dimension greater than or equal to four, then A is a UFD. In this case, injectivity of the restriction map is a moot point. To obtain our examples, we appealed to a combination of the Danilov results together with those in [16].

3 Chow Groups

Currently I am investigating related ideas for Chow groups. More formally, the theme of my research is an analysis of the action of intersection with divisors on Chow groups. These "Lefschetz-type" questions have long interested commutative algebraists.

The *Chow group* of A, denoted $\mathrm{CH}_*(A)$ or $A_*(A)$, is defined to be the group of cycles of A modulo rational equivalence. Two cycles are *rationally equivalent* if their difference lies in the subgroup generated by cycles of the form $[A/(\mathfrak{p}, f)]_i$, where the factor ring $A/\mathfrak{p}$ has dimension i and f is not an element of the prime ideal $\mathfrak{p}$. If A is a Noetherian normal domain of dimension d, then the $(d-1)^{\text{st}}$ component of the Chow group of A is the divisor class group of A.

Let f be a non-zero non-unit of A. Intersection with the divisor (f) gives a map from the Chow group of A to the Chow group of the hypersurface determined by f. It is denoted by $(f) \cap -$. By definition, intersection with (f) maps the class $[A/\mathfrak{p}]_i$ to the class $[A/(\mathfrak{p}, f)]_{i-1}$ if f is not an element of $\mathfrak{p}$ and to zero otherwise.

There are several differences between the maps on divisor classes that I have been considering and those analogous maps on Chow groups. First of all, if a hypersurface is regular in codimension one, then the Chow group of the hypersurface is isomorphic to that of its integral closure. This is not, in general, true of divisor class groups. Moreover, the Chow groups of the hypersurfaces determined by f and by f^n, for n greater than one, are isomorphic. Consequently, some results in the theory of divisor class groups [9] have trivial translations in the realm of Chow groups. To be specific, while one can investigate whether the intersection of the kernels of $\mathrm{CH}_*(A) \to \mathrm{CH}_{*-1}(A'_n)$ is trivial, one can not expect the same connection between the pathology of intersection with (f_n) and the power of the maximal ideal containing f_n. Thus, the question is to find what, if any, relationship exists between the injective behavior of the intersection map and powers of a maximal ideal.

Although my focus thus far has been on injectivity, the study of surjectivity of the intersection map has been investigated by others. The classical "Lefschetz-type" theorems state that $(f) \cap -$ is an isomorphism under certain conditions. Grothendieck, Danilov, and Srinivas [10], for example, have studied these types of problems from the standpoint of algebraic geometry. In [8], Lipman gives a purely algebraic proof of a theorem of Danilov-Samuel. My question is in the spirit of Lipman's approach. Namely, can purely algebraic proofs be given for "Lefschetz-type" results that have been established through methods of algebraic geometry? This query provides a possible avenue for further understanding intersection with divisors on Chow groups.

4 Ideal Filtrations

This project is joint work with Florian Enescu of Georgia State University and Cătălin Ciupercă of North Dakota State University. We begin with the classical situation where J and I are ideals in a Noetherian ring A such that J^p is contained in I^q. Obviously $J^{p'} \subseteq I^{q'}$ for q' less than q and p' greater than p, but what can be said about the supremum of q/p when J^p is contained in I^q?

In the 1950's, Samuel [14] studied the sequence $\{\frac{v_I(J,n)}{n}\}$, where for each n, the numerator is the largest integer such that J^n is contained in $I^{v_I(J,n)}$. Here I and J are two ideals in a commutative Noetherian ring A such that (i) the radical of I contains that of J, (ii) I and J are non-nilpotent, and (iii) the intersection of all the powers of I is zero. Samuel showed that the sequence has a limit $l_I(J)$, and conjectured that this value was a rational number. Rees [11] verified this conjecture using valuation theory.

Our first question concerns extending the above situation to three dimensions. In our study, we include new ideals K and L and such that the product of J^p and K^r is contained in the product of I^q and L^s. For any fixed powers r and s, the supremum of q/p is the same as $l_I(J)$; i.e., the limit remains unchanged by K^r and L^s. However, this is still a two-dimensional consideration. The important question is what happens if r and s are allowed to vary? When we allow r to vary, then we obtain an answer in terms of $l_I(J)$. The proof relies on a result, proved independently, by Katz [7] and Schenzel [15].

Definition 4.1. For each pair (p, r), let $v_I(p, r)$ be the largest integer v such that $J^p \cdot K^r \subseteq I^v$.

Proposition 4.2. *Assume that A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and that the pairs of ideals $\{J, I\}$ and $\{K, I\}$ satisfy conditions (i)-(iii) above. If K is $\mathfrak{m}$-primary, I has analytic spread strictly less than the dimension of A, and $\lim_{p\to\infty} \frac{v_I(J,p)}{p} = \lim_{r\to\infty} \frac{v_I(K,r)}{r} = l$, then $\lim_{p,r\to\infty} \frac{v_I(p,r)}{p+r} = l$.*

A second consideration involves generalizing Samuel's work beyond the power filtrations that he exclusively considered. Many of his ideas extend to cofinal filtrations. However, one important difference is that in this more general case the limits are no longer necessarily rational, as evinced by the example below. In fact, even if $l_G(H)$ is rational, it does not follow that $l_H(G)$ is rational.

Example 4.1. Let $A = k[x]$, for k a field, $G_n = (x^{a_n})$, where $a_n = \lceil n\sqrt{2}\rceil$, and $H_n = (x^n)$. Then $l_H(G) = \sqrt{2}$.

In the case that I and J have the same radical, Samuel defined an equivalence relation $I \sim J$ if and only if the limits $l_I(J)$ and $l_J(I)$ are both one. We generalize this to non-power filtrations. If G and H are cofinal filtrations, then we define $G \sim H$ if and only if $l_G(H)$ and $l_H(G)$ are both one.

Proposition 4.3. *For cofinal filtrations G and H, the following hold:*
(i) $G \sim H$ if and only if $l_G(F) = l_H(F)$ for all filtrations F
(i) $G \sim H$ if and only if $l_F(G) = l_F(H)$ for all filtrations F

With cofinal filtrations, we seek to answer questions such as (a) if $l_F(G)$ is rational, is there a power filtration I such that $l_I(G) = l_F(G)$, and (b) if $l_F(I)$ and $l_I(F)$ are both rational, then does there exist a power filtration J such that F is equivalent to J? In other words, we seek to understand when general filtrations can be approximated by power filtrations.

Finally, our last research direction involves the semigroup S of Samuel's equivalence classes. Since S has the cancellation property, i.e., $[I] + [J] = [I'] + [J]$ implies $[I] = [I']$, one can construct an abelian group, called the *symmetrization* of S. Moreover, from this group we can obtain a $\mathbb{Q}$-vector space V. We define a non-negative real-valued function on S which extends to a norm $|| \cdot ||$ on V. We investigate what information the norm gives us on the elements in the semigroup. In particular, when the ring is regular local of dimension two, then we can investigate the meaning of distance between simple, integrally closed ideals using the work of Zariski-Samuel [17].

Proposition 4.4. *Let R be a two dimensional regular local ring with maximal ideal $\mathfrak{m}$. If I and J are simple, integrally closed, $\mathfrak{m}$-primary ideals, then*

$$||[I] - [J]|| = \max(||[I]||, ||[J]||).$$

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