ON EXTENSION OF GABBER'S WORK

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Let (R, m, K) be a regular local ring of dimension n i.e., m is the maximal ideal of a regular local ring R and $K = R/m$. Let M and N be two finitely generated R-modules such that $\ell(M \otimes_R N) < \infty$. Serre introduced the notion of $\chi(M, N)$ as $\sum(-1)^i \ell(\mathrm{Tor}_i^R(M, N))$ ("ℓ" stands for length) and conjectured that i) $\chi(M, N) \geq 0$ and ii) $\chi(M, N) = 0$ if and only if $\dim M + \dim N < \dim R$. In the mid-nineties Gabber proved part i) of this conjecture. In the course of his proof he introduced the notion of $\chi^{\mathcal{O}_{X'}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'})$ where $X' = \mathbb{P}^n_R$, Y', Z' are closed subvarieties of X' such that Z' is regular and support of $\mathrm{Tor}_i^{O_{X'}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'}) \subset \mathbb{P}^n_K$ for $i \geq 0$. In a highly ingenious way, he then showed that the closed fiber E over s $(= [m])$ of the normal bundle corresponding to the imbedding $Z' \hookrightarrow X'$ is generated by global sections. Finally he observed that if V is a subcone of such an E (i.e., generated by global sections) with $\dim V \leq \mathrm{rank}\, E$, then $\chi^{\mathcal{O}_E}(\mathcal{O}_V, \mathcal{O}_{Z'_s}) \geq 0$. Following Gabber, we define intersection multiplicity on a regular variety X over an excellent discrete valuation ring V with algebraically closed residue field or an algebraically closed field k. Let $\mathcal{F}$ and $\mathcal{G}$ be two coherent $\mathcal{O}_X$-modules. Suppose that $\ell(H^i(X, \mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))) < \infty$ for $i, j \geq 0$. We define

$$\chi^{O_X}(\mathcal{F}, \mathcal{G}) = \sum(-1)^{i+j} \ell(H^i(X, \mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})))$$

(Grothendieck's definition of hypercohomology, EGA III). Our main focus will be n-dimensional projective space over spec(R) and the blow up of spec(R) at the closed point.