On Almost Regular Sequences

by

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Let $(R, \mathfrak{m})$ be a local domain. Let R^+ be the integral closure of R in the algebraic closure of the fraction field of R.

A module M over R is called (balanced) big Cohen-Macaulay module for R if $\mathfrak{m}M \neq M$ and every s.o.p. for R is a regular sequence on M.

Known facts:

(Hochster-Huneke) Let R be an excellent local domain of charac $p > 0$, p prime. Then R^+ is a (balanced) big CM algebra for R (that is, every s.o.p. for R is a regular sequence on R^+). This is false in charac 0 and dimension at least 3.

(Heitmann) Direct summand conjecture holds in dimension 3 for rings of mixed characteristic. More precisely, let R be a complete local domain of dimension 3 and mixed characteristic p. For every $\epsilon > 0$, p^ϵ kills $H^2_{\mathfrak{m}}(R^+)$. In fact, for every s.o.p. x, y, z in R^+ and for every $\epsilon > 0$, if $zu \in (x, y)R^+$, then $p^\epsilon u \in (x, y)R^+$.

Hochster: Let R be a local ring of mixed characteristic and dimension at most 3. Then R has a (balanced) big Cohen-Macaulay algebra. This is also known for the equicharacteristic case. Moreover, this construction of (balanced) big Cohen-Macaulay algebras is weakly functorial for complete local domains.

Let $(R, \mathfrak{m})$ be a d-dimensional formal power series ring over a field or a complete DVR and $R \subset A \subset R^+$, where A is an R-algebra containing arbitrary small powers of p.

Let $v : A^* \to \mathbf{Q}$ be a valuation on A (nonnegative on A) (obtained from a valuation on R^+):

$v(ab) = v(a)+v(b)$ and $v(a+b) \geq min(v(a), v(b))$, for all $a, b \in A^*$.

Let M be an A-module. An element $m \in M$ is called *almost zero* if for all $\epsilon > 0$, there exists $a \in A$ such that $v(a) < \epsilon$ and $am = 0$. The module M is called almost zero (and denoted $M \approx 0$) if all its elements are almost zero.

With this formulation in mind, Heitmann has shown that S is local domain of mixed characteristic p and dimension 3, then $H^i_{\mathfrak{m}}(S^+) \approx 0$ for all $i = 0, 1, 2$.

Roberts has noted that the existence of $S \subset A \subset R^+ = S^+$ such that $H^i_{\mathfrak{m}}(A) \approx 0$ for all $i = 0, ..., d-1$, is relevant for homological conjectures are for S.

Roberts suggested two possible definitions for almost Cohen-Macaulay algebras. One says that A as above with $H^i_{\mathfrak{m}}(A) \approx 0$ for all $i < d$ is called *almost Cohen-Macaulay*. The other definition, most recent in fact, is slightly more general relying on defining almost regular sequences first.

Roberts definition: Let $\underline{x} = x_1, ..., x_d$ be a system of parameters in R. Then $\underline{x}$ is a system of parameters on R^+ if $\underline{x}$ is a system of parameters in a finite extension of R.

Let R and A as above, and let $\underline{x}$ be an s.o.p. on R^+.

Then $\underline{x}$ is an *almost regular sequence* on A if

$$\frac{((x_1, \ldots, x_{i-1}) :_A x_i)}{(x_1, \ldots, x_{i-1})} \approx 0,$$

for all $i = 1, \ldots, d$. Moreover, A is almost Cohen-Macaulay over R if any s.o.p. on R^+ is an almost regular sequence on A.

Proposition 1 *(Roberts) Let $x_1, \ldots, x_d$ be a system of parameters of R. If A is almost Cohen-Macaulay, then $H_i(K_\bullet(x_1, \ldots, x_d; A)) \approx 0$ for $i > 0$. Also, $H^i_{\mathfrak{m}}(A) \approx 0$ for $i < d$.*

He also noted if A is almost Cohen-Macaulay, then the monomial conjecture holds.

Question: Does $H^i_{\mathfrak{m}}(A) \approx 0$ for all $i < d$ imply that A is almost Cohen-Macaulay?

Among the techniques not available anymore are Nakayama's Lemma and Krull's Intersection Theorem.

A satisfactory definition of almost regular sequences and almost Cohen-Macaulay modules should be equivalent to the almost vanishing of local cohomology for $i \neq d$.

Definition 2 *Let A be a ring and M and A-module, $\underline{x} = x_1, ..., x_d$ be elements in A such that $(x_1, ..., x_d)M \neq M$. Such a sequence is called M-quasiregular if for all $k \geq 0$ and for all homogeneous polynomials F of degree k $F \in M \otimes_A A[X_1, ..., X_d]$, if $F(\underline{x}) \in (\underline{x})^{k+1}M$, then the coefficients of F are in $(\underline{x})M$.*

Other terminology: pre-regular (Strooker), appeared first in EGA (Grothendieck) and later in Algebra Ch. 10(Bourbaki).

Well-know fact (Bruns-Herzog for example):

Let $(R, \mathfrak{m})$ local Noetherian of dimension d, $\underline{x} = x_1, \ldots x_d$ a s.o.p. for R and M an R-module. Consider the following assertions

(i) $H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$ and $H^d_{\mathfrak{m}}(M) \neq 0$

(ii) $H_i(\underline{x}; M) = 0$ for all $i = 1, \ldots, d$, and

$H_o(\underline{x}; M) \neq 0$

(iii) $\underline{x}$ is M-quasi-regular,

(iv) $\widehat{M}$ is a balanced big Cohen-Macaulay module.

Then (i) is equivalent to (ii), (iii) is equivalent to (iv), while (i) implies (iii).

Counterexample to (iii) implies (ii) (communicated by Roberts):

Let A be a DVR and let x be a nonzerodivisor on A. Denote K the fraction field of A.

One has that $xK/A = K/A$.

Let $M = K/A \oplus A$.

One can check that x is quasiregular on M, nut not a nonzerodivisor on M (or an almost nonzerodivisor).

Important point: M is not separated.

It is also know that a sequence of elements $x_1, ..., x_d$ is M-quasiregular if it is M-regular. The converse it is also true if all modules

$$M/(x_1, ..., x_i)M,$$

$1 \leq i \leq d-1$, are separated.

In fact, x is quasi-regular if and only if x is regular on $\hat{M}$.

We will call a polynomial $f \in A[X_1, ..., x_d]$ almost zero if its coefficients are almost zero or equivalently if for any $\epsilon > 0$ there is $a \in A$ with $v(a) < \epsilon$ such that $af = 0$.

Definition 3 *Let $\underline{x} = x_1, ..., x_d$ be sequence of elements in A. Let M be an A-module such that $(\underline{x})M \neq M$. Then $\underline{x}$ is an almost quasi-regular sequence if for every $\epsilon > 0$ and every polynomial $f \in M[X_1, ..., X_d]$ homogeneous of degree k, such that $f(\underline{x}) \in (\underline{x})^{k+1}M$ there exists $a \in A$ with $v(a) < \epsilon$ and $af \in (\underline{x})M[X_1, ..., X_d]$.*

The condition is equivalent to the condition that the homomorphism

$$T : A/(\underline{x})A[X_1, ..., X_d] \otimes M \to \mathrm{gr}_{\underline{x}} M$$

induced by $X_i \to \mathrm{class}(x_i) \in \frac{(\underline{x})}{(\underline{x})^2}$

is an almost isomorphism which is equivalent to $Ker(T) \approx 0$.

It is also equivalent to the condition that for all $t > 0$, the natural maps given by multiplication by $(x_1 \cdots x_d)^t$

$$M/(x_1, ..., x_d)M \to M/(x_1^{t+1}, ..., x_d^{t+1})M$$

are almost injective.

Proposition 4 *Let $R \subset S$ be a module finite extension and $\underline{x}$ be a s.o.p. for S. Then if there exists $S \subset A \subset R^+$ such that $\underline{x}$ is almost A-quasiregular, then $(x_1 \cdots x_d)^t \notin (x_1^{t+1}, ..., x_d^{t+1})S$, for all t.*

Proof.

if $(x_1 \cdots x_d)^t \in (x_1^{t+1}, ..., x_d^{t+1})S$ then $(x_1 \cdots x_d)^t \in (x_1^{t+1}, ..., x_d^{t+1})A$ so by one of the equivalent formulations of almost quasi-regularity it follows that for all $\epsilon > 0$, there exists $a \in A$ with $v(a) < \epsilon$ such that

$$a \in (x_1, ..., x_d)A.$$

However, if $a = a_1x_1 + \cdots + a_dx_d$, then $v(a) \geq \min(v(x_i)) > 0$, which is impossible. $\square$

Proposition 5 *Let A and M as above. Let $x_1, ..., x_n$ be an almost regular sequence on M. Then it is an almost quasi-regular sequence on M.*

Relation to Koszul cohomology:

Proposition 6 *Let A, M as above and $\underline{x}$ sequence of elements of A such that $\underline{x}M \neq M$.*

Consider the following assertions:

(i) $\underline{x}$ is almost regular on M

(ii) $H_i(\underline{x}; M) \approx 0$ for all $i \geq 1$;

(iii) $H_1(\underline{x}; M) \approx 0$;

(iv) $\underline{x}$ is almost quasi-regular on M.

Then (i) implies (ii), (ii) implies (iii), (iii) implies (iv).

Under what conditions is the converse true?

The condition that the quotient modules

$$M/(x_1,..,x_i)M$$

of M are separated needs to replaced by a condition like the following:

M is said to satisfy Condition (S) relative to I:

Let $m \in M$. If for all $k > 0$ and all $\epsilon > 0$ there exists $a \in A$, $v(a) < \epsilon$ such that $am \in I^k M$, then there exists $b \in A$ with $v(b) < \epsilon$ and $bm = 0$.

It implies that $\cap_{k=1}^{\infty} I^k M \approx 0$, but not equivalent to it.

We are concerned with the cases $I = (\underline{x})$ or $I = m$.

Hochster: R^+ is separated.

Question: Does R^+ satisfy Condition (S)?

Proposition 7 *Let $\underline{x}$, A, M as above.*

Then $\underline{x}$ is almost M-quasiregular if and only if $\underline{x}$ is almost $\widehat{M}$-quasiregular. Moreover, $\underline{x}$ almost $\widehat{M}$-regular implies $\underline{x}$ is almost $\widehat{M}$-quasiregular.

Question: Does $\widehat{M}$ satisfy Condition (S)?

Let $H = \min\{i : H^i_{(\underline{x})}(M) \neq 0\}$ and $h = \max\{i : H_i(\underline{x}, M) \neq 0\}$.

It is known that whenever A is Noetherian of dimension d and M is arbitrary then $H + h = d$.

This can be extended to our situation where the word almost is inserted appropriately.

Goal:

find algebras A that are almost CM (resp. almost quasiCM), or that admit almost CM (resp. almost quasiCM) modules.

Roberts:

a possibility is to investigate the *Fontaine* rings $E(R^+)$.