p-TORSION ELEMENTS IN LOCAL COHOMOLOGY MODULES II

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1. INTRODUCTION

Gennady Lyubeznik conjectured that if R is a regular ring and $\mathfrak{a}$ is an ideal of R, then the local cohomology modules $H_{\mathfrak{a}}^i(R)$ have only finitely many associated prime ideals, [Ly1, Remark 3.7 (iii)]. While this conjecture remains open in this generality, several results are now available: if the regular ring R contains a field of prime characteristic $p > 0$, Huneke and Sharp showed in [HS] that the set of associated prime ideals of $H_{\mathfrak{a}}^i(R)$ is finite. If R is a regular local ring containing a field of characteristic zero, Lyubeznik showed that $H_{\mathfrak{a}}^i(R)$ has only finitely many associated prime ideals, see [Ly1] and also [Ly2, Ly3]. Recently Lyubeznik has also proved this result for unramified regular local rings of mixed characteristic, [Ly4].

In [Hu] Craig Huneke first raised the following question: for a Noetherian ring R, an ideal $\mathfrak{a} \subset R$, and a finitely generated R-module M, is the number of associated primes ideals of $H_{\mathfrak{a}}^i(M)$ always finite? For some of the work on this problem, we refer the reader to the papers [BL, BRS, He] in addition to those mentioned above. In [Si] we constructed an example of a hypersurface R for which a local cohomology module $H_{\mathfrak{a}}^3(R)$ has p-torsion elements for every prime integer p, and consequently has infinitely many associated prime

ideals. Since this is the only known source of infinitely many associated prime ideals so far, it is worthwhile to investigate whether similar techniques may yield an example of a regular ring R for which a local cohomology module $H_{\mathfrak{a}}^i(R)$ has p-torsion elements for every prime integer p. This leads to some very intriguing questions as we shall see in this paper. Our results thus far support Lyubeznik's conjecture that local cohomology modules of all regular rings have only finitely many associated prime ideals.

Let R be a polynomial ring over the integers and F_i, G_i be elements of R for which

$$F_1 G_1 + F_2 G_2 + \cdots + F_n G_n = 0.$$

Consider the ideal $\mathfrak{a} = (G_1, \ldots, G_n)R$ and the local cohomology module

$$H_{\mathfrak{a}}^n(R) = \varinjlim \frac{R}{(G_1^k, \ldots, G_n^k)R}$$

where the maps in the direct limit system are induced by multiplication by the element $G_1 \cdots G_n$. For a prime integer p and prime power $q = p^e$, let

$$\lambda_q = \frac{(F_1 G_1)^q + \cdots + (F_n G_n)^q}{p}.$$

Note that has λ_q has integer coefficients, i.e., that $\lambda_q \in R$. Consider

$$\eta_q = [\lambda_q + (G_1^q, \ldots, G_n^q)R] \ \in \ H_{\mathfrak{a}}^n(R) = \varinjlim \frac{R}{(G_1^k, \ldots, G_n^k)R}.$$

It is immediately seen that $p\eta_q = 0$ and so if η_q is a nonzero element of $H_{\mathfrak{a}}^n(R)$, then it must be a p-torsion element. Hence if the local cohomology module $H_{\mathfrak{a}}^n(R)$ has only finitely many associated prime ideals then, for all but finitely many prime integers p, the elements η_q as constructed above must be zero i.e., for some $k \in \mathbb{N}$, we have

$$\lambda_q (G_1 \cdots G_n)^k \in (G_1^{q+k}, \ldots, G_n^{q+k})R.$$

This motivates the following conjecture:

Conjecture 1.1. Let R be a polynomial ring over the integers and let $F_i, G_i \in R$ such that

$$F_1 G_1 + \cdots + F_n G_n = 0.$$

Then for every prime integer p and prime power $q = p^e$, there exists $k \in \mathbb{N}$ such that

$$\frac{(F_1 G_1)^q + \cdots + (F_n G_n)^q}{p} (G_1 \cdots G_n)^k \in (G_1^{q+k}, \ldots, G_n^{q+k})R.$$

We shall say that Conjecture 1.1 holds for $G_1, \ldots, G_n \in R$, if it holds for all relations $\sum_{i=0}^{n} F_i G_i = 0$ with $F_1, \ldots, F_n \in R$.

Remark 1.2. The above conjecture is easily established if $n = 2$ since in this case we have $F_1 G_1 + F_2 G_2 = 0$ and so

$$\frac{(F_1 G_1)^q + (F_2 G_2)^q}{p} = \frac{(F_1 G_1)^q + (-F_1 G_1)^q}{p} = \begin{cases} (F_1 G_1)^q & \text{if } p = 2, \\ 0 & \text{if } p \neq 2, \end{cases}$$

which is an element of $(G_1^q, G_2^q)R$.

Example 1.3. The hypersurface example. Conjecture 1.1 is false if the condition that R is a polynomial ring over the integers is replaced by the weaker condition that the ring R is a hypersurface over the integers. To see this, let

$$R = \mathbb{Z}[U, V, W, X, Y, Z]/(UX + VY + WZ)$$

and $\mathfrak{a}$ denote the ideal $(x, y, z)R$. (We use lowercase letters here to denote the images of the corresponding variables.) Consider the relation

$$ux + vy + wz = 0$$

where, in the notation of Conjecture 1.1, $F_1 = u$, $F_2 = v$, $F_3 = w$ and $G_1 = x$, $G_2 = y$, $G_3 = z$. By means of a multi-grading argument, it is established in [Si, §4] that

$$\frac{(ux)^p + (vy)^p + (wz)^p}{p}(xyz)^k \notin (x^{p+k}, \ y^{p+k}, \ z^{p+k})R \quad \text{for all } \ k \in \mathbb{N}.$$

Consequently Conjecture 1.1 does not hold here even for the choice of the prime power $q = p$. This is, of course, the example from [Si] of a local cohomology module with infinitely many associated prime ideals: more precisely, for every prime integer p,

$$\left[\frac{(ux)^p + (vy)^p + (wz)^p}{p} + \ (x^p, y^p, z^p)R\right] \ \in \ H^3_{(x,y,z)}(R)$$

is a nonzero p-torsion element.

Conjecture 1.4. We record another formulation of Conjecture 1.1. Let F_i, G_i be elements of a polynomial ring R over the integers such that

$$F_1 G_1 + \cdots + F_n G_n = 0.$$

Let $\mathfrak{a}$ denote the ideal $(G_1, \ldots, G_n)$ of R. For an arbitrary prime integer p and prime power $q = p^e$, we have

$$(F_1 G_1)^q + \cdots + (F_n G_n)^q \equiv 0 \quad \text{mod } p$$

which is a relation on the elements $\overline{G}_1^q, \ldots, \overline{G}_n^q \in R/pR$, where $\overline{}$ denotes the image of an element of R in the ring R/pR. This relation may be viewed as an element $\mu_q \in H^{n-1}_{\mathfrak{a}}(R/pR)$. Conjecture 1.1 is equivalent to the conjecture that this element μ_q is in the image of the natural homomorphism $H^{n-1}_{\mathfrak{a}}(R) \to H^{n-1}_{\mathfrak{a}}(R/pR)$.

To see the equivalence of these conjectures, suppose

$$\mu_q \in \text{Image}\left(H_{\mathfrak{a}}^{n-1}(R) \to H_{\mathfrak{a}}^{n-1}(R/pR)\right).$$

Then the relation $(\overline{F}_1^q, \ldots, \overline{F}_n^q)$ on the elements $\overline{G}_1^q, \ldots, \overline{G}_n^q \in R/pR$ lifts to a relation in $H_{\mathfrak{a}}^{n-1}(R)$, i.e., there exists an integer k and elements $\alpha_i \in R$ such that

$$\alpha_1 G_1^{q+k} + \cdots + \alpha_n G_n^{q+k} = 0 \qquad \text{and}$$

$$\alpha_i \equiv F_i^q (G_1 \cdots G_{i-1} G_{i+1} \cdots G_n)^k \mod p \quad \text{for all} \quad 1 \le i \le n.$$

Hence we have

$$\left((F_1 G_1)^q + \cdots + (F_n G_n)^q\right)(G_1 \ldots G_n)^k$$

$$= (F_1^q G_2^k \cdots G_n^k - \alpha_1) G_1^{q+k} + \cdots + (F_n^q G_1^k \cdots G_{n-1}^k - \alpha_n) G_n^{q+k}$$

$$\in \left(pG_1^{q+k}, \ldots, pG_n^{q+k}\right) R,$$

and so $\dfrac{(F_1 G_1)^q + \cdots + (F_n G_n)^q}{p}(G_1 \cdots G_n)^k \in \left(G_1^{q+k}, \ldots, G_n^{q+k}\right) R.$

The proof of the converse is similar.

Remark 1.5. We next mention a conjecture due to Mel Hochster. While this was shown to be false in [Si], our entire study of p-torsion elements originates from this conjecture.

Consider the polynomial ring over the integers $R = \mathbb{Z}[u, v, w, x, y, z]$ where $\mathfrak{a}$ is the ideal generated by the size two minors of the matrix

$$M = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix},$$

i.e., $\mathfrak{a} = (\Delta_1, \Delta_2, \Delta_3)R$ where $\Delta_1 = vz - wy$, $\Delta_2 = wx - uz$, and $\Delta_3 = uy - vx$. In the ring R we have $u\Delta_1 + v\Delta_2 + w\Delta_3 = 0$. The relation

$$(u\Delta_1)^q + (v\Delta_2)^q + (w\Delta_3)^q \equiv 0 \mod p$$

may be viewed as an element $\mu_q \in H_{\mathfrak{a}}^2(R/pR)$. Hochster conjectured that for every prime integer there exists a choice of $q = p^e$ such that

$$\mu_q \notin \text{Image}\left(H_{\mathfrak{a}}^2(R) \to H_{\mathfrak{a}}^2(R/pR)\right),$$

and consequently that the image of μ_q in $H_{\mathfrak{a}}^3(R)$ is a nonzero p-torsion element of $H_{\mathfrak{a}}^3(R)$. In [Si] we constructed an equational identity which provided us with an element of $H_{\mathfrak{a}}^2(R)$ that maps to $\mu_q \in H_{\mathfrak{a}}^2(R/pR)$. While we refer the reader to [Si] for the details of the construction, we would like to provide a brief sketch.

Consider the following equational identity:

$$\Delta_1^{2k+1} u^{k+1} \sum_{n=0}^{k} \binom{k}{n} x^n \sum_{i=0}^{n} (-1)^i \binom{k+i}{k} \binom{k+n-i}{k} w^i v^{n-i} \Delta_2^{k-i} \Delta_3^{k-n+i}$$

$$+\Delta_2^{2k+1} v^{k+1} \sum_{n=0}^{k} \binom{k}{n} y^n \sum_{i=0}^{n} (-1)^i \binom{k+i}{k} \binom{k+n-i}{k} u^i w^{n-i} \Delta_3^{k-i} \Delta_1^{k-n+i}$$

$$+\Delta_3^{2k+1} w^{k+1} \sum_{n=0}^{k} \binom{k}{n} z^n \sum_{i=0}^{n} (-1)^i \binom{k+i}{k} \binom{k+n-i}{k} v^i u^{n-i} \Delta_1^{k-i} \Delta_2^{k-n+i}$$

$$= 0.$$

As it is a relation on the elements Δ_i^{2k+1} this identity gives us, for every $k \in \mathbb{N}$, an element $\gamma_k \in H_{\mathfrak{a}}^2(R)$. Using $k = q-1$ and examining the binomial coefficients above mod p, we obtain

$$\left((u\Delta_1)^q + (v\Delta_2)^q + (w\Delta_3)^q \right) (\Delta_1 \Delta_2 \Delta_3)^{q-1} \equiv 0 \mod p.$$

Consequently $\gamma_{q-1} \mapsto \mu_q$ under the natural homomorphism

$$H_{\mathfrak{a}}^2(R) \to H_{\mathfrak{a}}^2(R/pR).$$

Proposition 1.6. *If Conjecture 1.1 is true for the relations* $\sum_{i=0}^{n} E_i G_i = 0$

and $\sum_{i=0}^{n} F_i G_i = 0$ *where* $E_i, F_i, G_i \in R = \mathbb{Z}[X_1, \ldots, X_m]$, *then it is also*

true for the relation $\sum_{i=0}^{n} (sE_i + tF_i)G_i = 0$ *in* $S = \mathbb{Z}[X_1, \ldots, X_m, s, t]$. *More precisely, if for a prime power* $q = p^e$, *there exists* $k_1, k_2 \in \mathbb{N}$ *such that*

$$\left[\sum_{i=0}^{n} \frac{(E_i G_i)^q}{p} \right] (G_1 \cdots G_n)^{k_1} \in \left(G_1^{q+k_1}, \ldots, G_n^{q+k_1} \right) R \quad and$$

$$\left[\sum_{i=0}^{n} \frac{(F_i G_i)^q}{p} \right] (G_1 \cdots G_n)^{k_2} \in \left(G_1^{q+k_2}, \ldots, G_n^{q+k_2} \right) R, \quad then$$

$$\left[\sum_{i=0}^{n} \frac{(sE_i + tF_i)^q G_i^q}{p} \right] (G_1 \cdots G_n)^{k} \in \left(G_1^{q+k}, \ldots, G_n^{q+k} \right) S$$

for $k = \max\{k_1, k_2\}$.

We leave the proof as an elementary exercise.

2. A SPECIAL CASE OF THE CONJECTURE

In Theorem 2.1 below we prove what is perhaps the first interesting case of Conjecture 1.1.

Theorem 2.1. *Let R be a polynomial ring over the integers and F_i, G_i be elements of R such that F_1, F_2, F_3 form a regular sequence in R and*

$$F_1 G_1 + F_2 G_2 + F_3 G_3 = 0.$$

Let $q = p^e$ be a prime power. Then for $k = q - 1$ we have

$$\frac{(F_1 G_1)^q + (F_2 G_2)^q + (F_3 G_3)^q}{p}(G_1 G_2 G_3)^k \in (G_1^{q+k}, G_2^{q+k}, G_3^{q+k})R.$$

Remark 2.2. In Hochster's conjecture discussed earlier, the relation under consideration is $u\Delta_1 + v\Delta_2 + w\Delta_3 = 0$ and the elements u, v, w certainly form a regular sequence in the polynomial ring $R = \mathbb{Z}[u, v, w, x, y, z]$.

Proof of Theorem 2.1. Since $F_3 G_3 \in (F_1, F_2)R$ and F_1, F_2, F_3 form a regular sequence, there exist $\alpha, \beta \in R$ such that $G_3 = \alpha F_1 + \beta F_2$. In [Si, §2, 3] we showed that in the polynomial ring $\mathbb{Z}[A, B, T]$

$$\frac{(A + B)^q + (-A)^q + (-B)^q}{p}\Big[(A + B)AB\Big]^k$$

is an element of the ideal

$$(A + B)^{q+k}(T, A)^k(T, B)^k + A^{q+k}(T, B)^k(T + B, A + B)^k$$
$$+ B^{q+k}(T, A)^k(T - A, A + B)^k$$

when $k = q - 1$. We shall use this fact with $A = -F_2 G_2$, $B = -F_3 G_3$ and $T = \beta F_2 F_3$. Note that this gives $A + B = F_1 G_1$, $T + B = -\alpha F_1 F_3$, and $T - A = -F_1 G_1 - \alpha F_1 F_3$. Consequently

$$\frac{(F_1 G_1)^q + (F_2 G_2)^q + (F_3 G_3)^q}{p}\Big[F_1 G_1 F_2 G_2 F_3 G_3\Big]^k$$

is an element of the ideal

$$(F_1 G_1)^{q+k}(F_2 F_3, \ F_2 G_2)^k(F_2 F_3, \ F_3 G_3)^k$$
$$+ (F_2 G_2)^{q+k}(F_2 F_3, \ F_3 G_3)^k(F_1 F_3, \ F_1 G_1)^k$$
$$+ (F_3 G_3)^{q+k}(F_2 F_3, \ F_2 G_2)^k(F_1 G_1 + \alpha F_1 F_3, \ F_1 G_1)^k$$
$$\subseteq (F_1 F_2 F_3)^k(G_1^{q+k}, \ G_2^{q+k}, \ G_3^{q+k}).$$

The required result follows from this statement.   □

3. A Plücker relation

One of the main goals of [Si] was to settle Conjecture 1.1 for the relation

$$u(vz - wy) + v(wx - uz) + w(uy - vx) = 0 \qquad (1)$$

in the polynomial ring $\mathbb{Z}[u, v, w, x, y, z]$. This was accomplished by establishing that for every prime integer p and $q = p^e$ we have, for $k = q - 1$,

$$\frac{1}{p}[u^q(vz - wy)^q + v^q(wx - uz)^q + w^q(uy - vx)^q]$$

$$\times [(vz - wy)(wx - uz)(uy - vx)]^k$$

$$\in \left((vz - wy)^{q+k}, \ (wx - uz)^{q+k}, \ (uy - vx)^{q+k}\right).$$

The syzygy of the matrix $\begin{pmatrix} vz - wy & wx - uz & uy - vx \end{pmatrix}$ is

$$\begin{pmatrix} x & u \\ y & v \\ z & w \end{pmatrix}$$

but, by symmetry, Conjecture 1.1 also holds for the relation

$$x(vz - wy) + y(wx - uz) + z(uy - vx) = 0. \qquad (2)$$

In the light of Proposition 1.6, this gives us the following:

Theorem 3.1. *Conjecture 1.1 holds for*

$$\Delta_1 = vz - wy, \ \Delta_2 = wx - uz, \ \Delta_3 = uy - vx \ \in \ R = \mathbb{Z}[u, v, w, x, y, z],$$

i.e., if $F_1, F_2, F_3 \in R$ satisfy $F_1\Delta_1 + F_2\Delta_2 + F_3\Delta_3 = 0$ then for any prime power $q = p^e$ we have, for $k = q - 1$,

$$\frac{(F_1\Delta_1)^q + (F_2\Delta_2)^q + (F_3\Delta_3)^q}{p}(\Delta_1\Delta_2\Delta_3)^k \in (\Delta_1^{q+k}, \ \Delta_2^{q+k}, \ \Delta_3^{q+k})R.$$

By taking a combination of (1) and (2) above, we get the relation

$$(vz - wy)(ut - xs) + (wx - uz)(vt - ys) + (uy - vx)(wt - zs) = 0$$

in $R = \mathbb{Z}[s, t, u, v, w, x, y, z]$. This is, of course, the Plücker relation

$$-\Delta_{34}\Delta_{12} + \Delta_{24}\Delta_{13} - \Delta_{23}\Delta_{14}$$

where Δ_{ij} is the size two minor formed by picking rows i and j of the matrix

$$\begin{pmatrix} s & u & v & w \\ t & x & y & z \end{pmatrix}.$$

The syzygy of the matrix $\begin{pmatrix} ut - xs & vt - ys & wt - zs \end{pmatrix}$ is

$$\begin{pmatrix} vz - wy & 0 & -wt + zs & vt - ys \\ wx - uz & -wt + zs & 0 & -ut + xs \\ uy - vx & vt - ys & ut - xs & 0 \end{pmatrix}.$$

Since the Koszul relations are easily treated, the following theorem shows that Conjecture 1.1 holds for $ut - xs,\ vt - ys,\ wt - zs$.

Theorem 3.2. *Consider the Plücker relation*

$$(vz - wy)(ut - xs) + (wx - uz)(vt - ys) + (uy - vx)(wt - zs) = 0$$

in the polynomial ring $R = \mathbb{Z}[s, t, u, v, w, x, y, z]$. *Then for* $k = q - 1$, *we have*

$$\frac{1}{p}[(vz - wy)^q(ut - xs)^q + (wx - uz)^q(vt - ys)^q + (uy - vx)^q(wt - zs)^q]$$

$$\times [(ut - xs)(vt - ys)(wt - zs)]^k$$

$$\in \left((ut - xs)^{q+k},\ (vt - ys)^{q+k},\ (wt - zs)^{q+k} \right) R.$$

Towards the proof of this theorem, we first record some identities with binomial coefficients. These identities can be proved using Zeilberger's algorithm (see [PWZ]) and the Maple package EKHAD, but we include proofs for the sake of completeness.

When the range of a summation is not specified, it is assumed to extend over all integers. We set $\begin{pmatrix} k \\ i \end{pmatrix} = 0$ if $i < 0$ or if $k < i$.

Lemma 3.3.
(1)

$$\sum_n (-1)^n \begin{pmatrix} m + s - r \\ m - n \end{pmatrix} \begin{pmatrix} k - n \\ k - r \end{pmatrix} = \begin{cases} (-1)^{r-s} \binom{k-m}{s}, & \text{if } m \leq k - s, \\ (-1)^r \binom{m+s-k-1}{s}, & \text{if } k + 1 \leq m, \\ 0 & \text{else.} \end{cases}$$

(2)

$$\sum_r (-1)^r \begin{pmatrix} 2k - r \\ m - 1 \end{pmatrix} \begin{pmatrix} m \\ r - s \end{pmatrix} = 0 \qquad \text{if} \qquad 1 \leq m \leq 2k - s.$$

Proof. (1) Let

$$F(s, n) = (-1)^n \begin{pmatrix} m + s - r \\ m - n \end{pmatrix} \begin{pmatrix} k - n \\ k - r \end{pmatrix}, \qquad H(s) = \sum_n F(s, n),$$

and $G(s, n) = (-1)(k + 1 - n)F(s, n)$. It is easily verified that

$$\frac{G(s, n+1)}{G(s, n)} = \frac{(-1)(m-n)(r-n)}{(k+1-n)(s-r+n+1)}, \qquad \frac{F(s+1, n)}{F(s, n)} = \frac{m+s+1-r}{n+s+1-r},$$

and these can then be used to obtain the relation

$$(s+1)F(s+1,n) - (m+s-k)F(s,n) = G(s,n+1) - G(s,n).$$

Summing the above equation with respect to n gives

$$(s+1)H(s+1) - (m+s-k)H(s) = 0,$$

and using this recurrence relation for $H(s)$ we get

$$H(s) = \frac{(m-k+s-1)\cdots(m-k)}{s\cdots 1} H(0).$$

The result now follows since $H(0) = (-1)^r$.

(2) Let $F(r) = (-1)^r \binom{2k-r}{m-1}\binom{m}{r-s}$ and $G(r) = (2k+1-r)(r-s)F(r)$. It is a routine verification that

$$G(r) - G(r+1) = m(2k+1-m-s)F(r).$$

Consequently if $1 \leq m \leq 2k - s$ then

$$\sum_r F(r) = \frac{1}{m(2k+1-m-s)} \sum_r (G(r) - G(r+1)) = 0.$$

$\square$

We use the above results to establish the following equational identity:

Lemma 3.4.

$$\sum_{r=0}^{k}\sum_{n=0}^{r} \binom{2k+1-r}{n}\binom{k-n}{r-n}\frac{T^{k-r}}{2k+1-r}(T-Y)^n(Y-Z)^{2k+1-r-n}Y^r Z^r$$

$$= \sum_{s=0}^{k}(-1)^s\binom{k}{s}\frac{T^{k-s}}{2k+1-s}(Y^{2k+1}Z^s - Z^{2k+1}Y^s),$$

Proof. Comparing the coefficients of T^{k-s}, we need to show that

$$\sum_{r=s}^{k}\sum_{n=r-s}^{r}\frac{(-1)^{n-r+s}}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}\binom{n}{r-s}$$

$$\times\ (Y-Z)^{2k+1-r-n}Y^{n+s}Z^r = \frac{(-1)^s}{2k+1-s}\binom{k}{s}(Y^{2k+1}Z^s - Z^{2k+1}Y^s),$$

i.e., that

$$\sum_{r=s}^{k}\sum_{n=r-s}^{r}\frac{(-1)^{n-r}}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}\binom{n}{r-s}$$

$$\times\ (Y-Z)^{2k+1-r-n}Y^n Z^{r-s} = \frac{1}{2k+1-s}\binom{k}{s}(Y^{2k+1-s} - Z^{2k+1-s}).$$

The coefficient of $Y^m Z^{2k+1-s-m}$ in the expression on the left hand side is

$$
\sum_{r,n} \frac{(-1)^{n+m+1}}{2k+1-r} \binom{2k+1-r-n}{m-n} \binom{2k+1-r}{n} \binom{k-n}{r-n} \binom{n}{r-s}
$$

$$
= \sum_r \frac{(-1)^{m+1}}{2k+1-r} \binom{m}{r-s} \binom{2k+1-r}{m} \sum_n (-1)^n \binom{m+s-r}{m-n} \binom{k-n}{r-n}.
$$

which we denote by $\gamma_{m,s}$. Note that $\gamma_{m,s} = 0$ unless $m \leq k-s$ or $m \geq k+1$ since $\displaystyle\sum_n (-1)^n \binom{m+s-r}{m-n} \binom{k-n}{r-n} = 0$ for such m by Lemma 3.3 (1).

We next consider the case $m \leq k-s$. It follows from Lemma 3.3 (1) that

$$
\gamma_{m,s} = \sum_r \frac{(-1)^{m+1+r-s}}{2k+1-r} \binom{m}{r-s} \binom{2k+1-r}{m} \binom{k-m}{s}.
$$

If $m = 0$, then the only term that contributes to this sum is when $r = s$, and we get $\gamma_{0,s} = \dfrac{-1}{2k+1-s} \dbinom{k}{s}$. We may now assume $1 \leq m \leq k-s$ and then

$$
\gamma_{m,s} = \frac{(-1)^{m+1-s}}{m} \binom{k-m}{s} \sum_r (-1)^r \binom{m}{r-s} \binom{2k-r}{m-1} = 0
$$

by Lemma 3.3 (2).

For $m \geq k+1$, Lemma 3.3 (1) gives

$$
\gamma_{m,s} = \sum_r \frac{(-1)^{m+1+r}}{2k+1-r} \binom{m}{r-s} \binom{2k+1-r}{m} \binom{m+s-k-1}{s}
$$

$$
= \frac{(-1)^{m+1}}{m} \binom{m+s-k-1}{s} \sum_r (-1)^r \binom{m}{r-s} \binom{2k-r}{m-1}.
$$

By Lemma 3.3 (2), this is zero if $m \leq 2k-s$. If $m = 2k+1-s$ the only term that contributes to this sum is when $r = s$, and $\gamma_{2k+1-s,s} = \dfrac{1}{2k+1-s} \dbinom{k}{s}$.

$\square$

This lemma enables us to establish the following crucial identity:

Lemma 3.5.

$$\sum_{r=0}^{k}\sum_{n=0}^{r}\frac{k+1}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}T^{k-r}$$

$$\times\Bigg[X^{2k+1}(T-Y)^n(Y-Z)^{2k+1-r-n}Y^rZ^r$$

$$+Y^{2k+1}(T-Z)^n(Z-X)^{2k+1-r-n}Z^rX^r$$

$$+Z^{2k+1}(T-X)^n(X-Y)^{2k+1-r-n}X^rY^r\Bigg]=0$$

Proof. Using Lemma 3.4, the left hand side in the equation equals

$$\sum_{s=0}^{k}(-1)^s\binom{k}{s}\frac{k+1}{2k+1-s}T^{k-s}\times\Bigg[X^{2k+1}(Y^{2k+1}Z^s-Z^{2k+1}Y^s)$$

$$+\ Y^{2k+1}(Z^{2k+1}X^s-X^{2k+1}Z^s)\ +\ Z^{2k+1}(X^{2k+1}Y^s-Y^{2k+1}X^s)\Bigg]\ =\ 0.$$

$\square$

Proof of Theorem 3.2. Since $k=q-1$, we have $(k+1)/(2k+1-r)=q/(2q-1-r)$. For $0\le r\le q-2$, if $q/(2q-1-r)=a/b$ for relatively prime integers a and b, then p divides a since $q+1\le 2q-1-r\le 2q-1$. Consequently there exists an integer $d\in\mathbb{Z}$ such that d is relatively prime to p and

$$\frac{dq}{2q-1-r}\in\mathbb{Z}\quad\text{for all}\quad 0\le r\le q-1.$$

After replacing d by a suitable multiple, if necessary, we may assume that $d\equiv 1\mod p$. Using Lemma 3.5 with

$$T=\frac{t}{s},\qquad X=\frac{t}{s}-\frac{x}{u},\qquad Y=\frac{t}{s}-\frac{y}{v},\qquad Z=\frac{t}{s}-\frac{z}{w},$$

we get

$$\sum_{r=0}^{k}\sum_{n=0}^{r}\frac{k+1}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}\left(\frac{t}{s}\right)^{k-r}$$

$$\times\Bigg[\left(\frac{ut-xs}{su}\right)^{2k+1}\left(\frac{y}{v}\right)^n\left(\frac{vz-wy}{vw}\right)^{2k+1-r-n}\left(\frac{vt-ys}{sv}\right)^r\left(\frac{wt-zs}{sw}\right)^r$$

$$\left(\frac{vt-ys}{sv}\right)^{2k+1}\left(\frac{z}{w}\right)^n\left(\frac{wx-uz}{wu}\right)^{2k+1-r-n}\left(\frac{wt-zs}{sw}\right)^r\left(\frac{ut-xs}{su}\right)^r$$

$$\left(\frac{wt-zs}{sw}\right)^{2k+1}\left(\frac{x}{u}\right)^n\left(\frac{uy-vx}{uv}\right)^{2k+1-r-n}\left(\frac{ut-xs}{su}\right)^r\left(\frac{vt-ys}{sv}\right)^r\Bigg]$$

$$=0.$$

Multiplying by $ds^{4k+1}(uvw)^{2k+1}$ clears denominators, and gives us the equational identity (with integer coefficients):

$$\sum_{r=0}^{k}\sum_{n=0}^{r}\frac{d(k+1)}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}(st)^{k-r}$$

$$\times\Bigg[(ut-xs)^{2k+1}(wy)^{n}(vz-wy)^{2k+1-r-n}(vt-ys)^{r}(wt-zs)^{r}$$

$$+\;(vt-ys)^{2k+1}(uz)^{n}(wx-uz)^{2k+1-r-n}(wt-zs)^{r}(ut-xs)^{r}$$

$$+\;(wt-zs)^{2k+1}(vx)^{n}(uy-vx)^{2k+1-r-n}(ut-xs)^{r}(vt-ys)^{r}\Bigg]=0.$$

Note that for $0\le r\le k=q-1$ and $0\le n\le r$,

$$\frac{d(k+1)}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}\equiv\begin{cases}1\mod p,&\text{if }r=k\text{ and }n=0,\\0\mod p,&\text{else.}\end{cases}$$

Consequently we get

$$(ut-xs)^{2k+1}(vz-wy)^{k+1}(vt-ys)^{k}(wt-zs)^{k}$$

$$+\;(vt-ys)^{2k+1}(wx-uz)^{k+1}(wt-zs)^{k}(ut-xs)^{k}$$

$$+\;(wt-zs)^{2k+1}(uy-vx)^{k+1}(ut-xs)^{k}(vt-ys)^{k}$$

$$\in\left(p(ut-xs)^{2k+1},\;p(vt-ys)^{2k+1},\;p(wt-zs)^{2k+1}\right)R.$$

Finally, this shows that

$$\frac{1}{p}[(vz-wy)^{q}(ut-xs)^{q}+(wx-uz)^{q}(vt-ys)^{q}+(uy-vx)^{q}(wt-zs)^{q}]$$

$$\times[(ut-xs)(vt-ys)(wt-zs)]^{k}$$

$$\in\left((ut-xs)^{q+k},\;(vt-ys)^{q+k},\;(wt-zs)^{q+k}\right)R.$$

$\square$

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