REGULAR MORPHISMS DO NOT PRESERVE F-RATIONALITY

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ABSTRACT. For each positive prime integer p we construct a standard graded F-rational ring R, over a field K of characteristic p, such that $R\otimes_K \overline{K}$ is not F-rational. By localizing we obtain a flat local homomorphism $(R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ such that R is F-rational, $S/\mathfrak{m}S$ is regular (in fact, a field), but S is not F-rational. In the process we also obtain standard graded F-rational rings R for which $R\otimes_K R$ is not F-rational.

1. INTRODUCTION

Let $\mathscr{P}$ denote a local property of noetherian rings. The following types of *ascent* have been studied extensively; recall that for K a field, a noetherian K-algebra A is *geometrically regular* over K if $A\otimes_K L$ is regular for each finite extension field L of K.

($\mathrm{ASC_I}$) For a flat local homomorphism $(R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ of excellent local rings, if R is $\mathscr{P}$ and the closed fiber $S/\mathfrak{m}S$ is regular, then S is $\mathscr{P}$.

($\mathrm{ASC_{II}}$) For a flat local homomorphism $(R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ of excellent local rings, if R is $\mathscr{P}$ and the closed fiber $S/\mathfrak{m}S$ is geometrically regular over $R/\mathfrak{m}$, then S is $\mathscr{P}$.

Our main interest here is when $\mathscr{P}$ is F-rationality, a property rooted in Hochster and Huneke's theory of tight closure [HH1]: a local ring $(R,\mathfrak{m})$ of positive prime characteristic is *F-rational* if R is Cohen-Macaulay and each ideal generated by a system of parameters for R is tightly closed. Smith [Sm2] proved that F-rational rings have rational singularities, while Hara [Har] and Mehta-Srinivas [MS] independently proved that rings with rational singularities have F-rational type. Rational singularities of characteristic zero satisfy ($\mathrm{ASC_I}$) by Elkik [El, Théorème 5].

The ascent ($\mathrm{ASC_{II}}$) holds for F-rationality; this, and its variations, are due to Vélez [Ve, Theorem 3.1], Enescu [En1, Theorem 2.27], Hashimoto [Has, Theorem 6.4], and Aberbach-Enescu [AE, Theorem 4.3]. A common thread amongst these is that each affirmative answer requires assumptions along the lines that the fibers are *geometrically* regular. The situation is similar for F-injectivity in this regard; a local ring $(R,\mathfrak{m})$ of positive prime characteristic is *F-injective* if the Frobenius action on local cohomology modules

$$F\colon H^k_{\mathfrak{m}}(R) \longrightarrow H^k_{\mathfrak{m}}(R)$$

is injective for each $k\geqslant 0$. Datta and Murayama [DM, Theorem A] proved that if $(R,\mathfrak{m})$ is F-injective, and $(R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ is a flat local map such that $S/\mathfrak{m}S$ is Cohen-Macaulay and *geometrically* F-injective over $R/\mathfrak{m}$, then S is F-injective; see also [En2, Theorem 4.3] and [Has, Corollary 5.7]. We present examples demonstrating that the geometric assumptions are indeed required, i.e., that F-rationality and F-injectivity do not satisfy ($\mathrm{ASC_I}$):

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Theorem 1.1. *For each positive prime integer p, there exists a flat local ring homomorphism $(R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ of excellent local rings of characteristic p, such that the ring R is F-rational, $S/\mathfrak{m}S$ is regular, but S is not F-rational or even F-injective.*

Enescu had earlier demonstrated that F-injectivity does not satisfy $(\mathrm{ASC_I})$, though the examples [En2, Page 3075] are not normal; the question of whether normal F-injective rings satisfy $(\mathrm{ASC_I})$ has been raised earlier, e.g., [SZ, Question 8.1], and is settled in the negative by Theorem 1.1. There is a more recent notion, *F-anti-nilpotence*, developed in the papers [EH, Ma, MQ]; in view of the implications

$$F\text{-rational} \implies F\text{-anti-nilpotent} \implies F\text{-injective},$$

Theorem 1.1 also shows that F-anti-nilpotence does not satisfy $(\mathrm{ASC_I})$.

It is worth mentioning that the rings R in Theorem 1.1 are necessarily not Gorenstein, since F-rational Gorenstein rings are F-regular by [HH2, Theorem 4.2], and F-regularity satisfies $(\mathrm{ASC_I})$ by [Ab, Theorem 3.6]. Another subtlety, of course, is that the examples can only exist over imperfect fields, since F-rationality satisfies $(\mathrm{ASC_{II}})$.

Preliminary results are recorded in §2, including an extension of a criterion for F-rationality due to Fedder and Watanabe [FW]. In §3 we construct two families of examples that each imply Theorem 1.1: the first has the advantage that the proofs are more transparent, though the transcendence degree of the imperfect field over $\mathbb{F}_p$ increases with the characteristic p; the second family accomplishes the desired with transcendence degree one, independent of the characteristic $p > 0$, though the calculations are more involved. The examples in §3 are constructed as standard graded rings, with the relevant properties preserved under passing to localizations. In the process, we also obtain standard graded F-rational rings R, with the degree zero component being a field K of positive characteristic, such that the enveloping algebra $R \otimes_K R$ is not F-rational.

2. Preliminaries

Following [Ho, page 125], a local ring of positive prime characteristic is *F-rational* if it is a homomorphic image of a Cohen-Macaulay ring, and each ideal generated by a system of parameters is tightly closed. It follows from this definition that an F-rational local ring is Cohen-Macaulay, [HH2, Theorem 4.2], so the notion coincides with that in §1. Moreover, an F-rational local ring is a normal domain. A localization of an F-rational local ring at a prime ideal is again F-rational; with this in mind, a noetherian ring of positive prime characteristic—which is not necessarily local—is *F-rational* if its localization at each maximal ideal (equivalently, at each prime ideal) is F-rational.

For the case of interest in this paper, let R be an $\mathbb{N}$-graded Cohen-Macaulay normal domain, such that the degree zero component is a field K of characteristic $p > 0$, and R is a finitely generated K-algebra. Then R is F-rational if and only if the ideal generated by some (equivalently, any) homogeneous system of parameters for R is tightly closed; see [HH3, Theorem 4.7] and the preceding remark. An equivalent formulation in terms of local cohomology, following [Sm1, Proposition 3.3], is described next:

Fix a homogeneous system of parameters $x_1, \ldots, x_d$ for R, i.e., a sequence of $d := \dim R$ homogeneous elements that generate an ideal with radical the homogeneous maximal ideal $\mathfrak{m}$ of R. The local cohomology module $H^d_{\mathfrak{m}}(R)$ may then be computed using a Čech complex on $x_1, \ldots, x_d$ as

$$H^d_{\mathfrak{m}}(R) \;=\; \frac{R_{x_1 \cdots x_d}}{\sum_i R_{x_1 \cdots \hat{x}_i \cdots x_d}}.$$

This module admits a natural $\mathbb{Z}$-grading where the cohomology class

$$\eta := \left[\frac{r}{x_1^k \cdots x_d^k}\right] \in H^d_{\mathfrak{m}}(R), \tag{2.0.1}$$

for $r \in R$ a homogeneous element, has

$$\deg \eta := \deg r - k \sum_{i=1}^{d} \deg x_i.$$

The Frobenius endomorphism $F\colon R \longrightarrow R$ induces a map

$$F\colon H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{F(\mathfrak{m})}(R) = H^d_{\mathfrak{m}}(R)$$

that is the *Frobenius action* on $H^d_{\mathfrak{m}}(R)$; this is simply the map

$$\eta = \left[\frac{r}{x_1^k \cdots x_d^k}\right] \longmapsto F(\eta) = \left[\frac{r^p}{x_1^{kp} \cdots x_d^{kp}}\right]. \tag{2.0.2}$$

Since R is Cohen-Macaulay by assumption, R is F-injective precisely when the map (2.0.2) is injective.

The element η as in (2.0.1) belongs to $0^*_{H^d_{\mathfrak{m}}(R)}$, the tight closure of zero in $H^d_{\mathfrak{m}}(R)$, if there exists a nonzero element $c \in R$ such that for all $e \in \mathbb{N}$, one has

$$cF^e(\eta) = 0$$

in $H^d_{\mathfrak{m}}(R)$. This translates as

$$cr^{p^e} \in (x_1^{kp^e}, \ldots, x_d^{kp^e})R$$

for all $e \in \mathbb{N}$. In particular, R is F-rational precisely when

$$0^*_{H^d_{\mathfrak{m}}(R)} = 0.$$

We next review Veronese subrings. Let S be an $\mathbb{N}$-graded ring for which the degree zero component is a field K, and S is a finitely generated K-algebra. Fix a positive integer n. Then the n-th *Veronese subring* of S is the ring

$$S^{(n)} := \bigoplus_{k \in \mathbb{N}} S_{nk}.$$

Set $R := S^{(n)}$. The extension $R \subseteq S$ is pure, so if S is normal ring then so is R; since the extension is moreover finite, if S is Cohen-Macaulay then so is R. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R; then $\mathfrak{m}S$ is primary to the homogeneous maximal ideal $\mathfrak{n}$ of S, and by [GW, Theorem 3.1.1] one has

$$H^d_{\mathfrak{m}}(R) = \bigoplus_{k \in \mathbb{Z}} [H^d_{\mathfrak{n}}(S)]_{nk},$$

where $d := \dim S = \dim R$.

The following is a variation of [FW, Theorem 2.8] and [HH3, Theorem 7.12], and is used in the proof of Theorem 3.2.

Theorem 2.1. *Let S be an $\mathbb{N}$-graded Cohen-Macaulay normal domain, such that the degree zero component is a field K of positive characteristic, and S is a finitely generated K-algebra. Let $\mathfrak{n}$ denote the homogeneous maximal ideal of S, and set $d := \dim S$.*

Suppose each nonzero element of $\mathfrak{n}$ has a power that is a test element, and that there exists an integer $n > 0$ such that the Frobenius action on

$$[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$$

is injective. Then the tight closure of zero in $H^d_{\mathfrak{n}}(S)$ is contained in $[H^d_{\mathfrak{n}}(S)]_{>-n}$.

Proof. The hypotheses ensure that S has a homogeneous system of parameters $x_1,\dots,x_d$ where each x_i is a test element; we compute local cohomology using a Čech complex on such a homogeneous system of parameters. Suppose the assertion of the theorem is false; then there exists a nonzero homogeneous element η in $0^*_{H^d_{\mathfrak{n}}(S)}$ with $\deg\eta \leqslant -n$. After possibly replacing the x_i by powers, we may assume that

$$\eta \;=\; \left[\frac{s}{x_1\cdots x_d}\right],$$

for s a homogeneous element of S. Since each x_i is a test element, one has

$$x_i s^q \;\in\; (x_1^q,\dots,x_d^q)$$

for each $q=p^e$, and hence

$$s^q \;\in\; (x_1^q,\dots,x_d^q) :_R (x_1,\dots,x_d) \;=\; (x_1^q,\dots,x_d^q)+(x_1\cdots x_d)^{q-1},$$

where the equality is because $x_1,\dots,x_d$ is a regular sequence. Since $F^e(\eta)$ is nonzero in view of the injectivity of the Frobenius action on $[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$, one has

$$s^q \;\notin\; (x_1^q,\dots,x_d^q).$$

This implies that $\deg s^q \geqslant \deg(x_1\cdots x_d)^{q-1}$ for each $q=p^e$, which translates as

$$\deg s \;\geqslant\; \frac{q-1}{q}\deg(x_1\cdots x_d).$$

Taking the limit $e\longmapsto\infty$ gives

$$\deg s \;\geqslant\; \deg(x_1\cdots x_d),$$

so $\deg\eta\geqslant 0$. This contradicts $\deg\eta\leqslant -n<0$. □

A ring S is *standard graded* if it is $\mathbb{N}$-graded, with the degree zero component being a field K, such that S is generated as a K-algebra by finitely many elements of S_1.

While Theorem 2.1 requires the injectivity of the Frobenius action on $[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$, additional hypotheses enable one to verify the injectivity of Frobenius on *one* graded component; the following corollary will be used in the proof of Theorem 3.2. Following [GW], the *a-invariant* of a Cohen-Macaulay graded ring S, as in Theorem 2.1, is

$$a(S) := \max\left\{i\in\mathbb{Z} \;\middle|\; [H^d_{\mathfrak{n}}(S)]_i \neq 0\right\}.$$

Corollary 2.2. *Let S be a standard graded Gorenstein normal domain, of characteristic $p>0$, such that the homogeneous maximal ideal $\mathfrak{n}$ is an isolated singular point. Set $d:=\dim S$. Suppose $a(S)<0$, and that there exists an integer n with $-n\leqslant a(S)$ such that the Frobenius action*

$$F\colon [H^d_{\mathfrak{n}}(S)]_{-n} \longrightarrow [H^d_{\mathfrak{n}}(S)]_{-np}$$

is injective. Then the Veronese subring $S^{(n)}$ is F-rational.

Proof. Since S is standard graded and Gorenstein, each nonzero homogeneous element η of $[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$ has a nonzero multiple in $[H^d_{\mathfrak{n}}(S)]_{-n}$. But then $F(\eta)\neq 0$, so Theorem 2.1 implies that the tight closure of zero in $H^d_{\mathfrak{n}}(S)$ is contained in $[H^d_{\mathfrak{n}}(S)]_{>-n}$.

Set $R:=S^{(n)}$. The hypotheses $-n\leqslant a(S)<0$ give

$$H^d_{\mathfrak{m}}(R) \;\subseteq\; [H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$$

where $\mathfrak{m}$ is the homogeneous maximal ideal of R. As the tight closure of zero in $H^d_{\mathfrak{m}}(R)$ is contained in the tight closure of zero in $H^d_{\mathfrak{n}}(S)$, the assertion follows. □

3. THE EXAMPLES

Theorem 3.1. *Fix a prime integer $p > 0$. Let $t_1,\dots,t_p$ be indeterminates over the field $\mathbb{F}_p$ and set $K := \mathbb{F}_p(t_1,\dots,t_p)$. Consider the hypersurface*

$$S := K[x_0,\dots,x_p]/(x_0^p - t_1x_1^p - \cdots - t_px_p^p)$$

with the standard $\mathbb{N}$-grading, and its p-th Veronese subring $R := S^{(p)}$. Then:

1. *The ring R is F-rational.*
2. *The rings $R\otimes_K K^{1/p}$ and $R\otimes_K \overline{K}$ are not F-injective, hence not F-rational.*
3. *The enveloping algebra $R\otimes_K R$ is not F-injective, hence not F-rational.*

Proof. First consider the hypersurface

$$A := \mathbb{F}_p[t_1,\dots,t_p,x_0,\dots,x_p]/(x_0^p - t_1x_1^p - \cdots - t_px_p^p).$$

The Jacobian criterion shows A_{x_i} is regular for each i, so A is normal by Serre's criterion. By inverting an appropriate multiplicative set in A, one obtains the ring S, which therefore is also normal. Since R is a pure subring of the finite extension ring S, it follows that R is normal and Cohen-Macaulay.

Note that S is not F-injective: Set $\mathfrak{n}$ to be the homogeneous maximal ideal of S; computing local cohomology $H^p_{\mathfrak{n}}(S)$ using a Čech complex on the system of parameters $x_1,\dots,x_p$ for S, the cohomology class

$$\left[\frac{x_0}{x_1\cdots x_p}\right] \in H^p_{\mathfrak{n}}(S)$$

maps to zero under the Frobenius action on $H^p_{\mathfrak{n}}(S)$. We claim that the Frobenius action F on $H^p_{\mathfrak{m}}(R)$, with $\mathfrak{m}$ the homogeneous maximal ideal of R, is however injective:

It suffices to prove the injectivity of F on the socle of $H^p_{\mathfrak{m}}(R)$, which is the K-vector space spanned by the cohomology classes

$$\eta_{\boldsymbol{\alpha}} := \left[\frac{x_0^{\alpha_1+\cdots+\alpha_p}}{x_1^{\alpha_1+1}\ \cdots\ x_p^{\alpha_p+1}}\right] \in [H^p_{\mathfrak{m}}(R)]_{-p},$$

where each α_i is a nonnegative integer, $\sum\alpha_i \leqslant p-1$, and $\boldsymbol{\alpha} := (\alpha_1,\dots,\alpha_p)$. Since

$$x_0^p = t_1x_1^p + \cdots + t_px_p^p$$

in the ring S, one has

$$F(\eta_{\boldsymbol{\alpha}}) = \left[\frac{(t_1x_1^p+\cdots+t_px_p^p)^{\sum\alpha_i}}{x_1^{p\alpha_1+p}\ \cdots\ x_p^{p\alpha_p+p}}\right] = \frac{(\sum\alpha_i)!}{\alpha_1!\cdots\alpha_p!}\left[\frac{t_1^{\alpha_1}\cdots t_p^{\alpha_p}}{x_1^p\ \cdots\ x_p^p}\right], \tag{3.1.1}$$

where the latter equality uses the pigeonhole principle. The elements $t_1^{\alpha_1}\cdots t_p^{\alpha_p}$ of K, as $\boldsymbol{\alpha}$ varies subject to the conditions above, are linearly independent over the subfield K^p. It follows that for any nonzero K-linear combination η of the elements $\eta_{\boldsymbol{\alpha}}$, one has $F(\eta)\neq 0$. This proves that the ring R is F-injective.

One may now use Corollary 2.2 to conclude that R is F-rational; alternatively, one can also argue as follows: Equation (3.1.1) shows that the image of $[H^p_{\mathfrak{m}}(R)]_{-p}$ under F lies in the K-span of the cohomology class

$$\mu := \left[\frac{1}{x_1^p\cdots x_p^p}\right],$$

so it suffices to verify that μ does not belong to the tight closure of zero in $H^p_{\mathfrak{m}}(R)$. This holds since no nonzero homogeneous form in R annihilates

$$F^e(\mu) \;=\; \left[\frac{1}{x_1^{p^{e+1}}\cdots\, x_p^{p^{e+1}}}\right]$$

for each $e \geqslant 0$.

For (2), let $\overline{R}$ denote either of $R\otimes_K K^{1/p}$ or $R\otimes_K \overline{K}$. Note that the image of

$$t_2^{1/p}\left[\frac{x_0}{x_1^2x_2\cdots x_p}\right] - t_1^{1/p}\left[\frac{x_0}{x_1x_2^2x_3\cdots x_p}\right] \;\in\; H^p_{\mathfrak{m}}(\overline{R})$$

under the Frobenius action is

$$t_2\left[\frac{t_1x_1^p+\cdots+t_px_p^p}{x_1^{2p}x_2^p\cdots x_p^p}\right] - t_1\left[\frac{t_1x_1^p+\cdots+t_px_p^p}{x_1^px_2^{2p}x_3^p\cdots x_p^p}\right] \;=\; t_2\left[\frac{t_1}{x_1^px_2^p\cdots x_p^p}\right] - t_1\left[\frac{t_2}{x_1^px_2^p\cdots x_p^p}\right]$$

which, of course, is zero.

Lastly, for (3), write the enveloping algebra $S\otimes_K S$ of S as

$$K[x_0,\dots,x_p,\, y_0,\dots,y_p]/(x_0^p - t_1x_1^p - \cdots - t_px_p^p,\; y_0^p - t_1y_1^p - \cdots - t_py_p^p),$$

with the $\mathbb{N}^2$-grading under which $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$ for each i. Then

$$R\otimes_K R \;=\; \bigoplus_{k,l\in\mathbb{N}} [S\otimes_K S]_{(pk,pl)}.$$

Note that $R\otimes_K R$ admits a standard grading; let $\mathfrak{M}$ denote its homogeneous maximal ideal. Then $\mathfrak{M}(S\otimes_K S)$ is primary to the homogeneous maximal ideal $\mathfrak{N} := (x_0,\dots,x_p,\, y_0,\dots,y_p)$ of $S\otimes_K S$, and

$$H^{2p}_{\mathfrak{M}}(R\otimes_K R) \;=\; \bigoplus_{k,l\in\mathbb{N}} [H^{2p}_{\mathfrak{N}}(S\otimes_K S)]_{(pk,pl)}.$$

The nonzero cohomology class

$$\left[\frac{x_0y_1 - x_1y_0}{x_1^2x_2\cdots x_p\, y_1^2y_2\cdots y_p}\right] \;\in\; H^{2p}_{\mathfrak{M}}(R\otimes_K R)$$

is readily seen to be in the kernel of the Frobenius action. □

In the examples provided by the previous theorem, the transcendence degree of K over $\mathbb{F}_p$ increases with p; for the interested reader, the following theorem gets around this, though the proof is perhaps more technical.

Theorem 3.2. *Fix a prime integer $p > 0$. Let t be an indeterminate over the field $\mathbb{F}_p$ and set $K := \mathbb{F}_p(t)$. Consider the hypersurface*

$$S := K[w,x,y,\, z_1,\dots,z_{p-1}]/(w^{p+1} - tx^{p+1} - xy^p - \sum_{i=1}^{p-1} z_i^{p+1})$$

with the standard $\mathbb{N}$-grading, and set $R := S^{(p)}$. Then:

1. *The ring R is F-rational.*
2. *The rings $R\otimes_K K^{1/p}$ and $R\otimes_K \overline{K}$ are not F-injective, hence not F-rational.*
3. *The enveloping algebra $R\otimes_K R$ is not F-injective, hence not F-rational.*

Proof. We begin with the hypersurface

$$A := \mathbb{F}_p[t,w,x,y,z_1,\dots,z_{p-1}]/(w^{p+1} - tx^{p+1} - xy^p - \sum_i z_i^{p+1}).$$

The Jacobian criterion shows that, up to radical, the defining ideal of the singular locus of A contains $(w,x,y,z_1,\dots,z_{p-1})$. The ring S is obtained from A by inverting an appropriate multiplicative set; it follows that S has an isolated singular point at its homogeneous maximal ideal $\mathfrak{n}$. Is particular, S is normal by Serre's criterion.

To prove that R is F-rational, it suffices by Corollary 2.2 to verify that

$$F\colon [H^{p+1}_{\mathfrak{n}}(S)]_{-p} \longrightarrow [H^{p+1}_{\mathfrak{n}}(S)]_{-p^2} \tag{3.2.1}$$

is injective. Using the Čech complex on $x,y,z_1\dots,z_{p-1}$, the vector space $[H^{p+1}_{\mathfrak{n}}(S)]_{-p}$ has a K-basis consisting of cohomology classes

$$\eta_{\alpha,\beta,\boldsymbol{\gamma}} := \left[\frac{w^{1+\alpha+\beta+\sum\gamma_i}}{x^{\alpha+1}\,y^{\beta+1}\prod_i z_i^{\gamma_i+1}}\right]$$

where $\alpha,\beta,\gamma_1,\dots,\gamma_{p-1}$ are nonnegative integers with $\alpha+\beta+\sum\gamma_i \leqslant p-1$. The ring S admits a $(\mathbb{Z}/(p+1))^{p+1}$-grading with

$$\deg z_i = e_i,\quad \deg w = e_p,\quad \deg x = e_{p+1} = \deg y,$$

where $e_1,\dots,e_{p+1}$ denote standard basis vectors modulo $p+1$. Since $\gcd(p,\ p+1)=1$, the action (3.2.1) maps distinct multigraded components to distinct multigraded components, so it suffices to verify the injectivity componentwise. Note that

$$\deg\eta_{\alpha,\beta,\boldsymbol{\gamma}} \;=\; \Big(-\gamma_1-1,\ \dots,\ -\gamma_{p-1}-1,\ 1+\alpha+\beta+\sum_i\gamma_i,\ -\alpha-\beta-2\Big)$$

with respect to the multigrading. Thus, for fixed nonnegative integers k and γ_i with

$$0 \;\leqslant\; k+\sum_i\gamma_i \;\leqslant\; p-1,$$

a homogeneous element of $[H^{p+1}_{\mathfrak{n}}(S)]_{-p}$ with multidegree

$$\Big(-\gamma_1-1,\ \dots,\ -\gamma_{p-1}-1,\ 1+k+\sum_i\gamma_i,\ -k-2\Big)$$

has the form

$$\sum_{\alpha+\beta=k} c_\alpha\eta_{\alpha,\beta,\boldsymbol{\gamma}},$$

where α and β are nonnegative integers with $\alpha+\beta=k$, and $c_\alpha\in K$.

Set $m := k+\sum\gamma_i$, and suppose that the above element

$$\sum_{\alpha+\beta=k} c_\alpha\eta_{\alpha,\beta,\boldsymbol{\gamma}} \;=\; \sum_{\alpha+\beta=k} c_\alpha x^\beta y^\alpha\left[\frac{w^{m+1}}{x^{k+1}\,y^{k+1}\prod_i z_i^{\gamma_i+1}}\right] \tag{3.2.2}$$

belongs to the kernel of the Frobenius action. Then

$$\Big(\sum_{\alpha+\beta=k} c_\alpha^p x^{\beta p}y^{\alpha p}\Big)w^{(m+1)p}$$

belongs to the ideal

$$\Big(x^{(k+1)p},\ y^{(k+1)p},\ z_1^{(\gamma_1+1)p},\ \dots,\ z_{p-1}^{(\gamma_{p-1}+1)p}\Big)S.$$

Since $w^{(m+1)p} \;=\; w^{p-m}\,w^{(p+1)m}$ and $1 \leqslant p-m \leqslant p$, it follows that

$$\Big(\sum_{\alpha+\beta=k} c_\alpha^p x^{\beta p} y^{\alpha p}\Big)\Big(tx^{p+1}+xy^p+\sum_{i=1}^{p-1} z_i^{p+1}\Big)^m \tag{3.2.3}$$

belongs to the monomial ideal

$$\Big(x^{(k+1)p},\ y^{(k+1)p},\ z_1^{(\gamma_1+1)p},\ \dots,\ z_{p-1}^{(\gamma_{p-1}+1)p}\Big) \tag{3.2.4}$$

in the polynomial ring $K[x,y,z_1,\dots,z_{p-1}]$. Bearing in mind that $m = k+\sum\gamma_i$, the terms in the multinomial expansion of (3.2.3) that include the monomial

$$\prod_i z_i^{(p+1)\gamma_i}$$

constitute the polynomial

$$\binom{m}{k,\gamma_1,\dots,\gamma_{p-1}}\Big(\sum_{\alpha+\beta=k} c_\alpha^p x^{\beta p} y^{\alpha p}\Big)(tx^{p+1}+xy^p)^k \prod_i z_i^{(p+1)\gamma_i}$$

which, therefore, also belongs to the monomial ideal (3.2.4). But then

$$\Big(\sum_{\alpha+\beta=k} c_\alpha^p x^{\beta p} y^{\alpha p}\Big)(tx^{p+1}+xy^p)^k \;\in\; \big(x^{(k+1)p},\ y^{(k+1)p}\big)$$

in the polynomial ring $K[x,y]$. This implies that the coefficient of $x^{kp+k}y^{kp}$ in the polynomial above must be zero, i.e., that

$$\sum_{\alpha+\beta=k}\binom{k}{\alpha} c_\alpha^p t^\alpha \;=\; 0.$$

Since $c_\alpha^p \in K^p$ for each α, and $k < [K^p(t):K^p] = p$, this forces each c_α to be zero. But then the element (3.2.2) is zero, so the map (3.2.1) is indeed injective as claimed. This completes the proof of (1).

For (2), let $\mathfrak{m}$ denote the homogeneous maximal ideal of R, and let $\overline{R}$ denote either of $R\otimes_K K^{1/p}$ or $R\otimes_K \overline{K}$. Then

$$\left[\frac{w^2}{x^2y\prod_i z_i}\right] \;-\; t^{1/p}\left[\frac{w^2}{xy^2\prod_i z_i}\right] \;\in\; H_{\mathfrak{m}}^{p+1}(\overline{R})$$

maps, under the Frobenius action on $H_{\mathfrak{m}}^{p+1}(\overline{R})$, to

$$\left[\frac{w^{p-1}tx}{x^py^p\prod_i z_i^p}\right] \;-\; t\left[\frac{w^{p-1}x}{x^py^p\prod_i z_i^p}\right] \;=\; 0,$$

so $\overline{R}$ is not F-injective.

For (3) use w',x',y',z_i' for the second copy of S, and proceed as in the proof of Theorem 3.1. Using $\mathfrak{M}$ for the homogeneous maximal ideal of $R\otimes_K R$, the cohomology class

$$\left[\frac{(ww')^2\,(x'y-xy')}{(xx'yy')^2\prod_i z_i\prod_i z_i'}\right] \;\in\; H_{\mathfrak{M}}^{2p+2}(R\otimes_K R)$$

is readily seen to be nonzero, and in the kernel of the Frobenius action on $H_{\mathfrak{M}}^{2p+2}(R\otimes_K R)$. It follows that the ring $R\otimes_K R$ is not F-injective. $\square$

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REFERENCES

[Ab] I. M. Aberbach, *Extension of weakly and strongly F-regular rings by flat maps*, J. Algebra **241** (2001), 799–807. 2

[AE] I. M. Aberbach and F. Enescu, *Test ideals and base change problems in tight closure theory*, Trans. Amer. Math. Soc. **355**, (2003), 619–636. 1

[DM] R. Datta and T. Murayama, *Permanence properties of F-injectivity*, https://arxiv.org/abs/1906.11399. 1

[El] R. Elkik, *Singularités rationnelles et déformations*, Invent. Math. **47** (1978), 139–147. 1

[En1] F. Enescu, *On the behavior of F-rational rings under flat base change*, J. Algebra **233** (2000), 543–566. 1

[En2] F. Enescu, *Local cohomology and F-stability*, J. Algebra **322** (2009), 3063–3077. 1, 2

[EH] F. Enescu and M. Hochster, *The Frobenius structure of local cohomology*, Algebra Number Theory **2** (2008), 721–754. 2

[FW] R. Fedder and K.-i. Watanabe, *A characterization of F-regularity in terms of F-purity*, in: Commutative algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ. **15**, pp. 227–245, Springer, New York, 1989. 2, 3

[GW] S. Goto and K.-i. Watanabe, *On graded rings I*, J. Math. Soc. Japan **30** (1978), 179–213. 3, 4

[Har] N. Hara, *A characterisation of rational singularities in terms of injectivity of Frobenius maps*, Amer. J. Math. **120** (1998), 981–996. 1

[Has] M. Hashimoto, *Cohen-Macaulay F-injective homomorphisms*, in: Geometric and combinatorial aspects of commutative algebra (Messina, 1999), Lecture Notes in Pure and Appl. Math. **217**, pp. 231–244, Dekker, New York, 2001. 1

[Ho] M. Hochster, *Foundations of tight closure theory*, http://www.math.lsa.umich.edu/~hochster/. 2

[HH1] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116. 1

[HH2] M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62. 2

[HH3] M. Hochster and C. Huneke, *Tight closure of parameter ideals and splitting in module-finite extensions*, J. Algebraic Geom. **3** (1994), 599–670. 2, 3

[LR] S. Loepp and C. Rotthaus, *Some results on tight closure and completion*, J. Algebra **246** (2001), 859–880.

[Ma] L. Ma, *Finiteness properties of local cohomology for F-pure local rings*, Int. Math. Res. Not. **20** (2014), 5489–5509. 2

[MQ] L. Ma and P. H. Quy, *Frobenius actions on local cohomology modules and deformation*, Nagoya Math. J. **232** (2018), 55–75. 2

[MS] V. B. Mehta and V. Srinivas, *A characterization of rational singularities*, Asian J. Math. **1** (1997), 249–271. 1

[SZ] K. Schwede and W. Zhang, *Bertini theorems for F-singularities*, Proc. Lond. Math. Soc. (3) **107** (2013), 851–874. 2

[Sm1] K. E. Smith, *Tight closure of parameter ideals*, Invent. Math. **115** (1994), 41–60. 2

[Sm2] K. E. Smith, *F-rational rings have rational singularities*, Amer. J. Math. **119** (1997), 159–180. 1

[Ve] J. D. Vélez, *Openness of the F-rational locus and smooth base change*, J. Algebra **172** (1995), 425–453. 1

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