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Todd Classes of Affine Cones of Grassmannians

Kazuhiko Kurano and Anurag K. Singh

1 Introduction

Let M and N be two modules over a local ring A such that $M \otimes_A N$ has finite length and M has finite projective dimension. In [20], Serre defined the intersection multiplicity of M and N as

$$\chi(M, N) = \sum_{i=0}^{\dim A} (-1)^i \ell\big(\operatorname{Tor}_i^A(M, N)\big), \tag{1.1}$$

where $\ell(-)$ is the length function. If A is a regular local ring, Serre showed that the condition $\ell(M \otimes_A N) < \infty$ implies that $\dim M + \dim N \le \dim A$, and posed the following conjectures.

Vanishing: If $\dim M + \dim N < \dim A$, then $\chi(M, N) = 0$.

Positivity: If $\dim M + \dim N = \dim A$, then $\chi(M, N) > 0$.

Serre settled these conjectures affirmatively for regular local rings A that are equicharacteristic or unramified of mixed characteristic. The positivity conjecture remains open, though Gabber recently proved that $\chi(M, N) \ge 0$ (see [1]). The vanishing conjecture was proved by Roberts in [15] and independently by Gillet and Soulé in [5]. The theorem of Roberts is as follows.

Theorem 1.1 [15]. Let A be a homomorphic image of a regular local ring S such that $\tau_{A/S}([A]) = [\operatorname{Spec} A]_{\dim A}$ (e.g., suppose that A is a regular local ring or a complete intersection). If M and N are finitely generated A-modules of finite projective dimension such that $\ell(M \otimes_A N) < \infty$ and $\dim M + \dim N < \dim A$, then $\chi(M, N) = 0$. □

It draws attention to rings A for which $\tau_{A/S}([A]) = [\operatorname{Spec} A]_{\dim A}$, and such rings were named *Roberts rings* by the first author and studied in [10]. Complete intersections,

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for example, are Roberts rings. Our primary goal in this paper is to determine when affine cones over Grassmann varieties are Roberts rings, and we prove the following result.

Theorem 1.2. For $1 \le d \le n-1$, let $A_d(n)$ denote the affine cone of the Grassmann variety $G_d(n)$ under the Plücker embedding. Then $A_d(n)$ is a Roberts ring if and only if one of the following conditions is satisfied:

(1) $d = 1$;
(2) $d = n - 1$;
(3) $d = 2$ and $n = 4$;
(4) $d = 3$ and $n = 6$. □

Theorem 1.2 gives plenty of examples of Gorenstein factorial rings which are not Roberts rings. The first examples of Gorenstein rings which are not Roberts rings were discovered by the first author in [9] where he computed the Todd classes of certain determinantal rings. A few years later the second author, in collaboration with Miller, [14], found a Gorenstein ring which is not a *numerically Roberts ring* in the sense of [12]. (A local ring is a numerically Roberts ring if and only if the Dutta multiplicity coincides with the Euler characteristic for any bounded free complex with homology of finite length. We remark that a Roberts ring is a numerically Roberts ring but the converse is not true, see [12].) Recently Roberts and Srinivas, using a localization sequence in K-theory, established the existence of large families of Gorenstein rings which are not numerically Roberts rings [19] and, applying their methods, we know that $A_d(n)$ is a Roberts ring if and only if it is a numerically Roberts ring. Therefore, Theorem 1.2 gives many examples of Gorenstein factorial rings which are not numerically Roberts rings.

As a corollary of our results, we show in Section 5 that rings defined by Pfaffian ideals are Roberts rings if and only if they are complete intersections. The first author had earlier established that determinantal rings are Roberts rings if and only if they are complete intersections, [10, Example 6.2]. What is most curious in Theorem 1.2, is that the ring $A_3(6)$ is not a complete intersection, yet it is a Roberts ring.

2 Background

We first review some notation and results from [4, 9, 10] that we use later in our work.

2.1 Roberts rings

Let A be a homomorphic image of a regular local ring S, and $d = \dim A$. The *Chow group* of A is

$$A_*(A) = \bigoplus_{i=0}^{d} A_i(A), \tag{2.1}$$

where $A_i(A)$ is the free abelian group generated by cycles of the form $[A/P]$ for $P \in \operatorname{Spec} A$ with $\dim A/P = i$, considered modulo rational equivalence. Let $G_0(A)$ be the Grothendieck group of finitely generated A-modules. For an abelian group M, we use $M_{\mathbb{Q}}$ to denote the tensor product $M \otimes_{\mathbb{Z}} \mathbb{Q}$. With this notation, consider the *Riemann-Roch map* as in [4, Chapter 18],

$$\tau_{A/S} : G_0(A)_{\mathbb{Q}} \longrightarrow A_*(A)_{\mathbb{Q}}. \tag{2.2}$$

This is an isomorphism of $\mathbb{Q}$-vector spaces and it is known that under mild hypotheses (e.g., if A is complete, or is essentially of finite type over a field or over $\mathbb{Z}$) it does not depend on the choice of the regular local ring S, see [10, 18]. When $\tau_{A/S}$ does not depend on the choice of S, we denote it simply by τ_A. Let $[A]$ denote the class of the ring A in $G_0(A)_{\mathbb{Q}}$. Then A is said to be a *Roberts ring* if

$$\tau_{A/S}\big([A]\big) \in A_d(A)_{\mathbb{Q}} \tag{2.3}$$

for some choice of S. In other words, if we write

$$\tau_{A/S}\big([A]\big) = \tau_d + \tau_{d-1} + \cdots + \tau_0 \quad \text{for } \tau_i \in A_i(A)_{\mathbb{Q}}, \tag{2.4}$$

then A is a Roberts ring if and only if $\tau_{d-1} = \cdots = \tau_0 = 0$ for some choice of S. We summarize some properties of Roberts rings; see [10] for the proofs.

Theorem 2.1. Consider a ring $(A, \mathfrak{m})$ which is a homomorphic image of a regular local ring.

(1) If $(A, \mathfrak{m})$ is a Roberts ring, so are the local rings $A_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$, and the $\mathfrak{m}$-adic completion $\widehat{A}$;
(2) let $x \in \mathfrak{m}$ be a nonzerodivisor. If A is a Roberts ring, so is A/xA;
(3) if A is a subring of a regular local ring T such that T is a module-finite extension of A, then A is a Roberts ring;
(4) if A is a normal domain with a Noether normalization $S \subseteq A$ such that the extension A/S is generically Galois, then A is a Roberts ring. □

We next record some facts about $\tau_{A/S}$.

Theorem 2.2. Let A be a local ring of dimension d which is a homomorphic image of a regular local ring S. Let

$$\tau_{A/S}([A]) = \tau_d + \tau_{d-1} + \cdots + \tau_0 \quad \text{for } \tau_i \in A_i(A)_{\mathbb{Q}}; \tag{2.5}$$

(1) $\tau_d \neq 0$;
(2) if A is a complete intersection, then $\tau_i = 0$ for all $i < d$;
(3) if A is a Cohen-Macaulay ring with canonical module ω_A, then

$$\tau_{A/S}([\omega_A]) = \tau_d - \tau_{d-1} + \tau_{d-2} - \cdots + (-1)^d \tau_0; \tag{2.6}$$

(4) if A is a Gorenstein ring, then $\tau_{d-i} = 0$ for odd integers i;
(5) if A is a normal Roberts ring, then it is $\mathbb{Q}$-Gorenstein. A normal domain A of dimension two is a Roberts ring if and only if A is a $\mathbb{Q}$-Gorenstein ring.

□

2.2 Affine cones of smooth projective varieties

Let $R = \oplus_{n \geq 0} R_n$ be a graded ring over a field $R_0 = K$ which is generated, as a K-algebra, by finitely many elements of degree one. Let $\mathfrak{m}$ be the unique homogeneous maximal ideal of R. Assume that $X = \operatorname{Proj} R$ is a smooth projective variety of dimension t. Let $A_*(X) = \bigoplus_{i=0}^t A_i(X)$ and $\mathrm{CH}(X) = \bigoplus_{i=0}^t \mathrm{CH}^i(X)$ denote the *Chow group* and the *Chow ring* of X, respectively, where

$$\mathrm{CH}^i(X) = A_{t-i}(X) \quad \text{for all } 0 \leq i \leq t. \tag{2.7}$$

Set $\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{CH}(X) \otimes \mathbb{Q}$. Let $h = c_1(\mathcal{O}_X(1)) \cap [X] \in \mathrm{CH}^1(X)_{\mathbb{Q}}$ be the first Chern class of the invertible sheaf $\mathcal{O}_X(1)$. One of the main results of [9] is the following theorem.

Theorem 2.3. There is an exact sequence of graded modules

$$\mathrm{CH}(X)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}(X)_{\mathbb{Q}} \xrightarrow{\xi} A_*(R_\mathfrak{m})_{\mathbb{Q}} \longrightarrow 0, \tag{2.8}$$

where ξ is a map satisfying $\xi([\operatorname{Proj} R/\mathfrak{p}]) = [R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}]$ for each homogeneous prime ideal $\mathfrak{p}$ of R. Under this map, $\xi(\mathrm{td}(\Omega_X^\vee)) = \tau_{R_\mathfrak{m}}([R_\mathfrak{m}])$ where $\mathrm{td}(\Omega_X^\vee)$ is the Todd class of the tangent sheaf $\Omega_X^\vee$, and $\tau_{R_\mathfrak{m}} : G_0(R_\mathfrak{m})_{\mathbb{Q}} \to A_*(R_\mathfrak{m})_{\mathbb{Q}}$ is the Riemann-Roch isomorphism. In particular,

$$\begin{aligned} &A_0(R_\mathfrak{m})_{\mathbb{Q}} \cong 0, \\ &A_i(R_\mathfrak{m})_{\mathbb{Q}} \cong \frac{\mathrm{CH}^{t+1-i}(X)_{\mathbb{Q}}}{h\mathrm{CH}^{t-i}(X)_{\mathbb{Q}}} \quad \text{for all } 1 \leq i \leq t, \\ &A_{t+1}(R_\mathfrak{m})_{\mathbb{Q}} \cong \mathrm{CH}^0(X)_{\mathbb{Q}} \cong \mathbb{Q}. \end{aligned} \tag{2.9}$$

□

2.3 Dutta multiplicity

Let A be a complete local ring of dimension d over a perfect field of prime characteristic p, and let $G_\bullet$ be a bounded complex of free modules with homology of finite length. We denote by $F^n(-)$ the nth iteration of the Frobenius functor. The *Dutta multiplicity* of $G_\bullet$ is the limit

$$\chi_\infty(G_\bullet) = \lim_{n\to\infty} \frac{\chi(F^n(G_\bullet))}{p^{nd}}, \tag{2.10}$$

studied by Dutta in [3]. The Dutta multiplicity behaves, in many ways, better than the usual multiplicity, and Roberts used the Dutta multiplicity in an essential way in his proof of the new intersection theorem in mixed-characteristic [16]. While we do not pursue it here, the Dutta multiplicity can be defined in a characteristic-free way, as was accomplished by the first author in [8]. One of the motivating reasons for the study of Roberts rings is that over these rings, the Dutta multiplicity of a complex coincides with its Euler characteristic.

By the support of a complex $G_\bullet$, denoted by $\mathrm{Supp}(G_\bullet)$, we mean the union of the supports of its homology modules. We summarize some results from [2, 4, 13, 15, 17] which illustrate the behavior of $\chi_\infty(G_\bullet)$.

Theorem 2.4. Let $G_\bullet$ and $F_\bullet$ be bounded complexes of finitely generated free modules over a complete local ring A of dimension d and characteristic $p > 0$. Assume furthermore that A has a perfect residue class field.

(1) If $\mathrm{Supp}(G_\bullet) = \{\mathfrak{m}\}$, then $\chi_\infty(G_\bullet{}^\vee) = (-1)^d \chi_\infty(G_\bullet)$;
(2) if $\mathrm{Supp}(G_\bullet) = \{\mathfrak{m}\}$, then $\chi_\infty(F^n(G_\bullet)) = p^{nd}\chi_\infty(G_\bullet)$ for all $n \in \mathbb{N}$;
(3) if $\dim \mathrm{Supp}(G_\bullet) + \dim \mathrm{Supp}(F_\bullet) < d$ and $\mathrm{Supp}(G_\bullet) \cap \mathrm{Supp}(F_\bullet) = \{\mathfrak{m}\}$, then $\chi_\infty(G_\bullet \otimes_A F_\bullet) = 0$;
(4) if $\dim \mathrm{Supp}(G_\bullet) + \dim \mathrm{Supp}(F_\bullet) \leq d$ and $\mathrm{Supp}(G_\bullet) \cap \mathrm{Supp}(F_\bullet) = \{\mathfrak{m}\}$, then $\chi_\infty(G_\bullet \otimes_A F_\bullet) = (-1)^{d-\dim \mathrm{Supp}(G_\bullet)} \chi_\infty(G_\bullet^\vee \otimes_A F_\bullet)$;
(5) if $G_\bullet$ has length d and $\mathrm{Supp}(G_\bullet) = \{\mathfrak{m}\}$ (in particular, $G_\bullet$ is not exact), then $\chi_\infty(G_\bullet) > 0$. □

The assertions (1), (2), (4), and (5) of Theorem 2.4 are not true in general if the Dutta multiplicity χ_∞ is replaced by the usual Euler characteristic χ. However, if the ring A is a Roberts ring, all assertions of Theorem 2.4 are true for the Euler characteristic χ since, in this case, $\chi_\infty(G_\bullet) = \chi(G_\bullet)$ for a bounded free complex $G_\bullet$ with $\mathrm{Supp}(G_\bullet) = \{\mathfrak{m}\}$.

Remark 2.5. Assume that A is a d-dimensional local ring (not necessary of positive characteristic) which is a homomorphic image of a regular local ring. In this generality, the Dutta multiplicity of complexes with support in $\{\mathfrak{m}\}$ is defined in [8].

The statements (1), (3), (4) in Theorem 2.4 hold true in this case. Furthermore, if we replace F with the pth Adams operation ψ^p [6, 13], statement (2) is also valid for any positive integer p. If A contains a field, then (5) is true, see [13]. However, this is an open problem in the case of mixed-characteristic. The positivity of the Dutta multiplicity is deeply connected to the positivity conjecture of Serre, [11, Theorem 1.2].

Remark 2.6. The concept of a *numerically Roberts ring* is defined in [12]. It is proved there that a local ring A is a numerically Roberts ring if and only if, over the ring A, the Dutta multiplicity always coincides with the Euler characteristic. Consequently, a Roberts ring is a numerically Roberts ring, but there are many examples of numerically Roberts rings which are not Roberts rings.

However, using a method established in [19], the affine cone $A_d(n)$ of a Grassmann variety is a Roberts ring if and only if it is a numerically Roberts ring. A key point here is that for a Grassmann variety $G = G_d(n)$, we have $CH(G)_{\mathbb{Q}} \cong CH_{num}(G)_{\mathbb{Q}}$.

3 Vector bundles

We review definitions and basic facts on Chern characters, [4, Section 3.2], that we use later.

Let E be a vector bundle on a scheme X. We use $c_t(E)$ to denote its *Chern polynomial*,

$$c_t(E) = 1 + c_1(E)t + c_2(E)t^2 + c_3(E)t^3 + \cdots . \tag{3.1}$$

For an exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, the *Whitney sum* formula gives

$$c_t(E) = c_t(E')c_t(E''). \tag{3.2}$$

If the vector bundle E has rank r, then $c_i(E) = 0$ for all $i > r$. If its Chern polynomial is factored formally as

$$c_t(E) = \prod_{i=1}^{r} (1 + \alpha_i t), \tag{3.3}$$

the α_i's are called the *Chern roots* of E, and the Chern classes of E are elementary symmetric functions of $\alpha_1, \dots, \alpha_r$. The *Chern character* of E is

$$ch(E) = \sum_{i=1}^{r} \exp(\alpha_i) = \sum_{i=1}^{r} \sum_{n \geq 0} \frac{\alpha_i^n}{n!}. \tag{3.4}$$

The first few terms, as found in [4, Example 3.2.3], are

$$\begin{aligned}\operatorname{ch}(E) &= r + c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \frac{1}{6}\left(c_1^3 - 3c_1c_2 + 3c_3\right) \\ &\quad + \frac{1}{24}\left(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4\right) + \cdots ,\end{aligned} \tag{3.5}$$

where $c_i = c_i(E)$.

The Chern character of a tensor product of vector bundles is

$$\operatorname{ch}\left(E \otimes E'\right) = \operatorname{ch}(E)\operatorname{ch}\left(E'\right), \tag{3.6}$$

and for an exact sequence $0 \to E' \to E \to E'' \to 0$, we have

$$\operatorname{ch}(E) = \operatorname{ch}\left(E'\right) + \operatorname{ch}\left(E''\right). \tag{3.7}$$

The Chern classes of the dual bundle $E^\vee$ are given by

$$c_i\left(E^\vee\right) = (-1)^i c_i(E). \tag{3.8}$$

The *Todd class* $\operatorname{td}(E)$ of a vector bundle E with Chern roots $\alpha_1, \ldots, \alpha_r$ is

$$\operatorname{td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - \exp\left(-\alpha_i\right)}, \tag{3.9}$$

and the first few terms of the expansion are

$$\begin{aligned}\operatorname{td}(E) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}\left(c_1^2 + c_2\right) + \frac{1}{24}\left(c_1c_2\right) \\ &\quad + \frac{1}{720}\left(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4\right) + \cdots ,\end{aligned} \tag{3.10}$$

see [4, Example 3.2.4].

4 Grassmannians

Let $X = (x_{ij})$ be an $n \times d$ matrix of indeterminates over a field K, and consider the ring R generated, as a K-algebra, by all the $d \times d$ minors of the matrix X. Then R is the homogeneous coordinate ring of the Grassmann variety $G_d(n)$ of d-dimensional subspaces in an n-dimensional vector space, that is, $G_d(n) = \operatorname{Proj} R$. The relations between the minors are quadratic, and are the well-known Plücker relations, see [7, Section 6, Chapter VII].

Setting $G = G_d(n)$ for the notational convenience, we have the universal exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O}_G^n \longrightarrow Q \longrightarrow 0, \tag{4.1}$$

where Q (resp., S) is the *universal* rank $(n-d)$ *quotient bundle* (resp., *universal* rank d *subbundle*) on G, see [4, Section 14.6]. We briefly explain the construction of Q and S. Consider the $K[x_{ij}]$-module T which is the submodule of the free module $K[x_{ij}]^n$ generated by the columns of X. Let N denote the set of elements of T which have entries in R. Then N is a graded submodule of R^n, and S is the locally free sheaf corresponding to N. Similarly, the locally free sheaf Q corresponds to the graded R-module R^n/N, see [18, Section 10.2].

By [4, Appendix B.5.8], we have

$$\Omega_G^\vee = \mathrm{Hom}(S, Q) = S^\vee \otimes Q. \tag{4.2}$$

From the universal exact sequence we also get

$$\wedge^n \mathcal{O}_G^n \cong \mathcal{O}_G \cong \wedge^d S \otimes \wedge^{n-d} Q, \tag{4.3}$$

and so $\wedge^{n-d} Q \cong (\wedge^d S)^\vee \cong \wedge^d (S^\vee)$. Here, $\wedge^{n-d} Q$ is the very ample invertible sheaf corresponding to the Plücker embedding $G = G_d(n) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$. By [4, Remark 3.2.3(c)], $c_1(\wedge^{n-d} Q) = c_1(Q)$ and setting $h = c_1(Q)$, Theorem 2.3 gives us the exact sequence

$$\mathrm{CH}(G)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}(G)_{\mathbb{Q}} \xrightarrow{\xi} A_*(A)_{\mathbb{Q}} \longrightarrow 0 \tag{4.4}$$

such that

$$\xi\big(\operatorname{td}(\Omega_G^\vee)\big) = \tau_A\big([A]\big) \in A_*(A)_{\mathbb{Q}}, \tag{4.5}$$

where $A = R_m$. Let $t = d(n-d)$, which is the dimension of the projective variety G. Then $\dim A = t + 1$, and suppose that

$$\tau_A\big([A]\big) = \tau_{t+1} + \tau_t + \tau_{t-1} + \cdots + \tau_0, \tag{4.6}$$

where $\tau_i \in A_i(A)_{\mathbb{Q}}$ for each i. Here, since A is essentially of finite type over a field, the Riemann-Roch map $\tau_{A/S}$ is independent of the choice of a regular local ring S. Comparing terms with the expansion of $\mathrm{td}(\Omega_G^\vee)$, we see that

$$\begin{aligned}
\tau_{t+1} &= 1, \\
\tau_t &= \frac{1}{2} c_1(\Omega_G^\vee) \bmod h\mathrm{CH}^0(G)_{\mathbb{Q}}, \\
\tau_{t-1} &= \frac{1}{12}\Big(c_1(\Omega_G^\vee)^2 + c_2(\Omega_G^\vee)\Big) \bmod h\mathrm{CH}^1(G)_{\mathbb{Q}}, \\
\tau_{t-2} &= \frac{1}{24} c_1(\Omega_G^\vee)c_2(\Omega_G^\vee) \bmod h\mathrm{CH}^2(G)_{\mathbb{Q}}, \\
\tau_{t-3} &= \frac{1}{720}\Big(-c_1(\Omega_G^\vee)^4 + 4c_1(\Omega_G^\vee)^2 c_2(\Omega_G^\vee) + 3c_2(\Omega_G^\vee)^2 \\
&\qquad + c_1(\Omega_G^\vee)c_3(\Omega_G^\vee) - c_4(\Omega_G^\vee)\Big) \bmod h\mathrm{CH}^3(G)_{\mathbb{Q}},
\end{aligned} \tag{4.7}$$

and so forth. Recall that A is a Roberts ring if and only if $\tau_i = 0$ for all $i \le t$, and we will prove Theorem 1.2 essentially by establishing the vanishing or nonvanishing of τ_i's.

Proof of Theorem 1.2. If $d = 1$ or $d = n-1$, the affine cone $A_d(n)$ is a regular local ring, and therefore is a Roberts ring. Consequently, we may assume that $2 \le d \le n-2$. In the case $d = 2$ and $n = 4$, it is easily seen that there is exactly one Plücker relation, and so $A_2(4)$ is a hypersurface, hence a Roberts ring.

In general, the Whitney sum formula, applied to the universal exact sequence $0 \to S \to \mathcal{O}_G^n \to Q \to 0$, gives $c_t(S)c_t(Q) = c_t(\mathcal{O}_G^n) = (c_t(\mathcal{O}_G))^n = 1$, which says that

$$\begin{aligned}
&\big(1 + c_1(S)t + c_2(S)t^2 + c_3(S)t^3 + c_4(S)t^4 + \cdots\big) \\
&\quad \times \big(1 + c_1(Q)t + c_2(Q)t^2 + c_3(Q)t^3 + c_4(Q)t^4 + \cdots\big) = 1.
\end{aligned} \tag{4.8}$$

Comparing the coefficients, we obtain

$$\begin{aligned}
&c_2(S) + c_1(S)c_1(Q) + c_2(Q) = 0, \\
&c_4(S) + c_3(S)c_1(Q) + c_2(S)c_2(Q) + c_1(S)c_3(Q) + c_4(Q) = 0.
\end{aligned} \tag{4.9}$$

By Lemma 4.1(1) below, the graded component of $\mathrm{CH}(G)_{\mathbb{Q}}/h\,\mathrm{CH}(G)_{\mathbb{Q}}$ in degree one is

$$\mathrm{CH}^1(G)_{\mathbb{Q}}/h\mathrm{CH}^0(G)_{\mathbb{Q}} = 0. \tag{4.10}$$

In particular, $c_1(E) \equiv 0 \bmod h\,\mathrm{CH}(G)_{\mathbb{Q}}$ for any vector bundle E on G. Hence we have $c_2(S) \equiv -c_2(Q)$ and $c_4(S) \equiv c_2(Q)^2 - c_4(Q)$, which will be used later. Furthermore, the expansion of $\mathrm{ch}(E)$ is simplified as

$$\begin{aligned} \mathrm{ch}(E) &\equiv \operatorname{rank} E + \frac{1}{2}\big(-2c_2(E)\big) + \frac{1}{6}\big(3c_3(E)\big) + \frac{1}{24}\big(2c_2(E)^2 - 4c_4(E)\big) + \cdots \\ &\equiv \operatorname{rank} E - c_2(E) + \frac{1}{2}c_3(E) + \frac{1}{12}\big(c_2(E)^2 - 2c_4(E)\big) + \cdots . \end{aligned} \tag{4.11}$$

Recall that $\mathrm{ch}(\Omega_G^\vee) = \mathrm{ch}(S^\vee)\,\mathrm{ch}(Q)$ since $\Omega_G^\vee = S^\vee \otimes Q$. This equation gives us

$$\begin{aligned} &d(n-d) - c_2(\Omega_G^\vee) + \frac{1}{2}c_3(\Omega_G^\vee) + \frac{1}{12}\Big(c_2(\Omega_G^\vee)^2 - 2c_4(\Omega_G^\vee)\Big) + \cdots \\ &\quad\equiv \left[d - c_2(S^\vee) + \frac{1}{2}c_3(S^\vee) + \frac{1}{12}\Big(c_2(S^\vee)^2 - 2c_4(S^\vee)\Big) + \cdots\right] \\ &\quad\quad \times \left[n - d - c_2(Q) + \frac{1}{2}c_3(Q) + \frac{1}{12}\Big(c_2(Q)^2 - 2c_4(Q)\Big) + \cdots\right]. \end{aligned} \tag{4.12}$$

Comparing the components of degree two in (4.12), we see that

$$c_2(\Omega_G^\vee) \equiv dc_2(Q) + (n-d)c_2(S^\vee) \equiv dc_2(Q) + (n-d)c_2(S). \tag{4.13}$$

Since $c_2(S) \equiv -c_2(Q)$, we have $c_2(\Omega_G^\vee) \equiv (2d-n)c_2(Q)$. Consequently,

$$\begin{aligned} \tau_{t-1} &= \frac{1}{12}\Big(c_1(\Omega_G^\vee)^2 + c_2(\Omega_G^\vee)\Big) \bmod h\mathrm{CH}^1(G)_{\mathbb{Q}} \\ &= \frac{1}{12}(2d-n)c_2(Q) \bmod h\mathrm{CH}^1(G)_{\mathbb{Q}}. \end{aligned} \tag{4.14}$$

We need the following lemma to complete the proof of Theorem 1.2.

Lemma 4.1. Let G denote the Grassmann manifold $G_d(n)$ where $2 \le d \le n-2$. Then

(1) $\mathrm{CH}^1(G)_{\mathbb{Q}} = h\mathrm{CH}^0(G)_{\mathbb{Q}}$;
(2) $c_2(Q) \notin h\mathrm{CH}^1(G)_{\mathbb{Q}}$;
(3) if $d \ge 4$ and $n = 2d$, then $c_2(Q)^2 \notin h\mathrm{CH}^3(G)_{\mathbb{Q}}$;
(4) if $d = 3$ and $n = 6$, then $h\mathrm{CH}^i(G)_{\mathbb{Q}} = \mathrm{CH}^{i+1}(G)_{\mathbb{Q}}$ for $3 \le i \le 8$. □

We first complete the proof of Theorem 1.2 using this lemma. Recall that we may assume that $2 \le d \le n-2$. Since

$$\tau_{t-1} = \frac{1}{12}(2d-n)c_2(Q) \bmod h\mathrm{CH}^1(G)_{\mathbb{Q}}. \tag{4.15}$$

Lemma 4.1(2) implies that τ_{t-1} is nonzero if $n \ne 2d$. Consequently, $A_d(n)$ is not a Roberts ring in this case.

We next assume that $n = 2d$. Since $c_2(\Omega_G^\vee) \equiv (2d-n)c_2(Q)$, we have $c_2(\Omega_G^\vee) \equiv 0$. Comparing the components of degree four in (4.12), we get

$$\begin{aligned} -\frac{1}{6}c_4(\Omega_G^\vee) &\equiv \frac{d}{12}\left(c_2(Q)^2 - 2c_4(Q)\right) + c_2(S^\vee)c_2(Q) + \frac{d}{12}\left(c_2(S^\vee)^2 - 2c_4(S^\vee)\right) \\ &\equiv c_2(S)c_2(Q) + \frac{d}{12}\left(c_2(Q)^2 - 2c_4(Q) + c_2(S)^2 - 2c_4(S)\right). \end{aligned} \tag{4.16}$$

Since $c_2(S) \equiv -c_2(Q)$ and $c_4(S) \equiv c_2(Q)^2 - c_4(Q)$, we have

$$-\frac{1}{6}c_4(\Omega_G^\vee) \equiv -c_2(Q)^2. \tag{4.17}$$

Consequently, $c_4(\Omega_G^\vee) \equiv 6c_2(Q)^2$ and so

$$\tau_{t-3} = -\frac{1}{120}c_2(Q)^2 \bmod hCH^3(G)_{\mathbb{Q}}. \tag{4.18}$$

If $n = 2d$ and $d \geq 4$, then τ_{t-3} is nonzero by Lemma 4.1(3). Hence $A_d(n)$ is not a Roberts ring in this case.

Suppose that $n = 6$ and $d = 3$. Then $A_3(6)$ is a Gorenstein ring of dimension 10 and so $\tau_i = 0$ for odd integers i. Since $n = 2d$, we have $\tau_8 = \tau_{t-1} = 0$. The equality $\tau_{9-i} = 0$ for $i \geq 3$ follows from Lemma 4.1(4). Hence $A_3(6)$ is a Roberts ring. ■

We now record the proof of Lemma 4.1.

Proof of Lemma 4.1. We use the notation and results of [4, Sections 14.5–14.7] for Schubert cycles. The Chow ring $CH(G)_{\mathbb{Q}}$ has a basis over $\mathbb{Q}$ represented by the set of partitions

$$\lambda = (\lambda_1, \ldots, \lambda_d), \quad \text{where } n - d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0. \tag{4.19}$$

We denote the cycle corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ by $\{\lambda\}$ or $\{\lambda_1, \ldots, \lambda_d\}$. Set $|\lambda| = \sum \lambda_i$. Then $CH^l(G)_{\mathbb{Q}}$ has a basis which consists of the set of cycles $\{\lambda\}$ such that $|\lambda| = l$. For $1 \leq m \leq n - d$, the classes $c_m(Q)$ are called the *special Schubert classes* and $\sigma_m = c_m(Q)$ coincides with the cycle $\{m, 0, \ldots, 0\}$. The multiplication by σ_m is determined by *Pieri's formula*:

$$\{\lambda\} \times \sigma_m = \sum \{\mu\}, \tag{4.20}$$

where the sum runs over μ with

$$n - d \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_d \geq \lambda_d, \quad |\mu| = |\lambda| + m. \tag{4.21}$$

(1) The group $CH^1(G)_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space of dimension one, whose generator, in terms of a Young diagram, is $\ydiagram{1}$. Since $h \in CH^1(G)_{\mathbb{Q}}$ corresponds to a very ample divisor, h does not vanish. Therefore, $CH^1(G)_{\mathbb{Q}} = hCH^0(G)_{\mathbb{Q}}$ is satisfied.

(2) The group $CH^2(G)_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space of dimension two spanned by $\ydiagram{2}$ and $\ydiagram{1,1}$. The image $hCH^1(G)_{\mathbb{Q}}$ is the $\mathbb{Q}$-span of

$$\ydiagram{1} \times \ydiagram{1} = \ydiagram{1,1} + \ydiagram{2}, \quad \text{and so} \quad c_2(Q) = \ydiagram{2} \notin hCH^1(G)_{\mathbb{Q}}. \tag{4.22}$$

(3) Since $CH^3(G)_{\mathbb{Q}}$ is spanned by $\ydiagram{3}$, $\ydiagram{2,1}$, and $\ydiagram{1,1,1}$, it follows that $hCH^3(G)_{\mathbb{Q}}$ is spanned by $\ydiagram{3} \times \ydiagram{1} = \ydiagram{3,1} + \ydiagram{4}$,

$$\ydiagram{2,1} \times \ydiagram{1} = \ydiagram{3,1} + \ydiagram{2,2} + \ydiagram{2,1,1}, \qquad \ydiagram{1,1,1} \times \ydiagram{1} = \ydiagram{2,1,1} + \ydiagram{1,1,1,1}. \tag{4.23}$$

Then it is easy to see that

$$c_2(Q)^2 = \ydiagram{2} \times \ydiagram{2} = \ydiagram{4} + \ydiagram{3,1} + \ydiagram{2,2} \tag{4.24}$$

is not an element of $hCH^3(G)_{\mathbb{Q}}$.

(4) If $d = 3$ and $n = 6$, then $hCH^3(G)_{\mathbb{Q}}$ is spanned by

$$\ydiagram{3} \times \ydiagram{1} = \ydiagram{3,1}, \qquad \ydiagram{2,1} \times \ydiagram{1} = \ydiagram{3,1} + \ydiagram{2,2} + \ydiagram{2,1,1}, \qquad \ydiagram{1,1,1} \times \ydiagram{1} = \ydiagram{2,1,1} \tag{4.25}$$

which, in this case, generate $CH^4(G)_{\mathbb{Q}}$. This shows that $hCH^3(G)_{\mathbb{Q}} = CH^4(G)_{\mathbb{Q}}$, and the remaining cases may be computed similarly. ∎

Remark 4.2. The ring $A_3(6)$ is not a complete intersection. It is a ring of dimension 10, and is the homomorphic image of a regular local ring of dimension 20 (which is the number of 3×3 minors of a 6×3 matrix) modulo an ideal generated minimally by 35 Plücker relations. The number of minimal generators may be checked using [7, Section 6, Chapter VII] and eliminating redundant relations, or by a computer algebra package such as Macaulay 2.

5 Pfaffian ideals

We determine next when the rings $S/\operatorname{Pf}_m(Y)$ defined by Pfaffian ideals are Roberts rings.

Let $Z = (z_{ij})$ be a $2m \times 2m$ antisymmetric matrix, that is, $z_{ij} = -z_{ji}$ for $1 \le i < j \le 2m$ and $z_{ii} = 0$ for $1 \le i \le 2m$. We call

$$\operatorname{Pf}(Z) = \sum_{\sigma} \operatorname{sgn}(\sigma) z_{\sigma(1)\sigma(2)}\, z_{\sigma(3)\sigma(4)} \cdots z_{\sigma(2m-1)\sigma(2m)} \tag{5.1}$$

the *Pfaffian* of Z, where the sum is taken over permutations of $\{1, 2, \ldots, 2m\}$ which satisfy $\sigma(1) < \sigma(3) < \cdots < \sigma(2m-1)$ and

$$\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \ldots, \sigma(2m-1) < \sigma(2m). \tag{5.2}$$

It is easy to see that $\operatorname{Pf}(Z)^2 = \det(Z)$.

Let m and n be positive integers such that $2m \le n$, and let $Y = (y_{ij})$ be the $n \times n$ antisymmetric matrix with variables y_{ij} for $1 \le i < j \le n$. For a set of integers such that $1 \le s_1 < \cdots < s_{2m} \le n$, we denote by $\operatorname{Pf}(s_1, \ldots, s_{2m})$ the Pfaffian of the $2m \times 2m$ antisymmetric matrix $(y_{s_i s_j})$. Let K be a field and S be the localization of the polynomial ring $K[y_{ij} | 1 \le i < j \le n]$ at its homogeneous maximal ideal. We denote by $\operatorname{Pf}_m(Y)$ the ideal of S generated by all the elements $\operatorname{Pf}(s_1, \ldots, s_{2m})$ for $1 \le s_1 < \cdots < s_{2m} \le n$. Set $B_m(n) = S/\operatorname{Pf}_m(Y)$. It is well known that $B_m(n)$ is a factorial Gorenstein ring and that

$$\dim B_m(n) = \dim S - (n - 2m + 1)(n - 2m + 2)/2. \tag{5.3}$$

With this notation we have the following theorem.

Theorem 5.1. The following conditions are equivalent:

(1) $B_m(n)$ is a Roberts ring;
(2) $B_m(n)$ is a complete intersection;
(3) $n = 2m$ or $m = 1$. □

Proof. The minimal number of generators of the ideal $\operatorname{Pf}_m(Y)$ is $\binom{n}{2m}$, and its height is $(n - 2m + 1)(n - 2m + 2)/2 = \binom{n-2m+2}{2}$. Using these facts, the equivalence of (2) and (3) is easily verified.

In the case $m = 2$, the ideal $\operatorname{Pf}_2(Y)$ is generated by the elements

$$y_{ij}y_{kl} - y_{ik}y_{jl} + y_{il}y_{jk}, \quad \text{for } 1 \le i < j < k < l \le n. \tag{5.4}$$

These are precisely the Plücker relations for the Grassmann variety $G_2(n)$, and so $B_2(n)$ coincides with $A_2(n)$. It then follows from Theorem 1.2 that $B_2(n)$ is a Roberts ring if and only if $n = 4$.

Next, assume that $m \geq 3$. If $n = 2m$, then $B_m(n)$ is a complete intersection and, therefore, a Roberts ring. If $n > 2m$, then a suitable localization of $B_m(n)$ gives a Pfaffian ring $B_{m-1}(n-2)$ over a different base field. By induction on m, we may assume that $B_{m-1}(n-2)$ is not a Roberts ring and it follows from Theorem 2.1(1) that $B_m(n)$ is not a Roberts ring. This completes the proof of the theorem. ∎

Remark 5.2. The ring $A_d(n)$ is a Roberts ring if and only if it is a numerically Roberts ring. Consequently, $B_2(n)$ is a Roberts ring if and only if it is a numerically Roberts ring. However, the authors do not know whether or not the rings $B_m(n)$ are numerically Roberts rings in the case $m \geq 3$.

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Kazuhiko Kurano: Department of Mathematics, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan
E-mail address: kurano@comp.metro-u.ac.jp

Anurag K. Singh: Department of Mathematics, University of Utah, 155 South 1400 East, JWB 233, Salt Lake City, UT 84112-0090, USA
E-mail address: singh@math.utah.edu