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On an example of Nagarajan [☆]

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ABSTRACT

K. R. Nagarajan constructed an example of a formal power series ring of dimension two, over a field of characteristic two, with the action of a cyclic group of order two, such that the ring of invariants is not noetherian. We point out how Nagarajan's example readily extends to each positive prime characteristic, and also to a characteristic zero example: There exists a formal power series ring of dimension two, over a field of characteristic zero, with an action of the infinite cyclic group, such that the ring of invariants is not noetherian. Both the positive characteristic and the characteristic zero examples are sharp in multiple ways.

1. Introduction

Consider a finite group G acting on a noetherian ring R via ring automorphisms. The question whether the invariant ring R^G is noetherian is a classical one, with positive results, in a sense, going back to Hilbert and Noether: If the order of the finite group is

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invertible in R, then R^G is noetherian [6,7,16]; if R is a finitely generated algebra over a noetherian ring A, and the action of G on R is via A-algebra automorphisms, then, again, R^G is noetherian [17].

On the other hand, Nagata gave an example of an artinian local ring R containing a field of characteristic $p > 0$, with an action of a cyclic group G of order p, such that R^G is not noetherian [15, Proposition 0.10], see also [4, §1] and [8, Example 12]; while the ring in this example is of course not an integral domain, in the same paper, Nagata also constructs a pseudo-geometric local integral domain R of dimension one and characteristic $p > 0$, with an action of a cyclic group G of order p, such that R^G is not noetherian [15, Proposition 0.11]. In contrast, if G is a finite group acting on a Dedekind domain R, then R^G is noetherian [15, Proposition 0.3, Remark 0.7].

In light of the above, it is natural to impose stronger hypotheses on R and ask whether R^G is noetherian when G is a finite group, and R is normal [15, Question 0.1], or even regular. These questions were settled in the negative by Nagarajan [12, §4], who constructed a formal power series ring R of dimension two, over a field of characteristic two, with the action of an involution σ such that $R^{\langle\sigma\rangle}$ is not noetherian. Our first goal in this paper is to point out how Nagarajan's example readily extends to each positive prime characteristic p, providing an action of a cyclic group G of order p on a formal power series ring $R := K[\![x, y]\!]$, with K a field of characteristic p, such that the invariant ring R^G is not noetherian. Other variations of Nagarajan's example may be found in [2] and [1].

Our other goal is to note that Nagarajan's construction extends readily to a curious characteristic zero example: there exists a formal power series ring $R := K[\![x, y]\!]$ over a field K of characteristic zero, with an action of the infinite cyclic group G, such that the invariant ring R^G is not noetherian. The positive characteristic and the characteristic zero examples are all sharp: in each case, the dimension of the regular local ring R is the least possible, see Remark 2.2, as is the cardinality and the number of generators of the group G.

While we have focused here on the noetherian property of R^G, related questions on the finite generation of R^G have a rich history: in addition to Nagata's celebrated counterexamples to Hilbert's 14th Problem [13,14], we point the reader towards the papers [11,18,19,3,10,5,9,20], and the references therein.

2. The example

Let $\mathbb{F}$ be a field, and consider the purely transcendental extension field

$$K := \mathbb{F}(a_1, b_1, a_2, b_2, \dots),$$

where the elements a_n, b_n are indeterminates over $\mathbb{F}$. Set $R := K[\![x, y]\!]$, i.e., R is the ring of formal power series in the variables x and y, with coefficients in K. Set

$$f_n := a_n x + b_n y \qquad \text{for } n \geqslant 1.$$

Define an $\mathbb{F}$-algebra endomorphism σ of R as follows:

$$\sigma\colon \begin{cases} x & \longmapsto & x, \\ y & \longmapsto & y, \\ a_n & \longmapsto & a_n + y f_{n+1}, \\ b_n & \longmapsto & b_n - x f_{n+1}. \end{cases}$$

It is readily seen that $\sigma(f_n) = f_n$ for each $n \geqslant 1$, and also that σ is surjective, hence an automorphism of R. With this notation, we prove:

Theorem 2.1. *Let K be a field constructed as above, $R := K[\![x, y]\!]$ a formal power series ring, and $G := \langle \sigma \rangle$ a cyclic group acting on R as described above. If the field K has positive characteristic p, then G is a cyclic group of order p, whereas G is infinite if K has characteristic zero. In either case, the ring of invariants R^G is not noetherian.*

Proof. For each $k \in \mathbb{Z}$, one has

$$\sigma^k(a_n) \;=\; a_n + k y f_{n+1} \quad \text{and} \quad \sigma^k(b_n) \;=\; b_n - k x f_{n+1},$$

so the group $\langle \sigma \rangle$ has order p if K has characteristic $p > 0$, and is infinite cyclic otherwise.

Let $\mathfrak{m}$ denote the maximal ideal of R. We claim that for each α in K, one has

$$\sigma(\alpha) \;\equiv\; \alpha \mod \mathfrak{m}^2 \tag{2.1.1}$$

in R. To see this, suppose $\alpha = g/h$ for nonzero g, h in $\mathbb{F}[a_1, b_1, a_2, b_2, \dots]$. It is immediate from the definition that $\sigma(g) \equiv g \mod \mathfrak{m}^2$. Since g is a unit in R, there exists $g_2 \in \mathfrak{m}^2$ with $\sigma(g) = g(1 - g_2)$. Similarly, there exists $h_2 \in \mathfrak{m}^2$ with $\sigma(h) = h(1 - h_2)$. But then

$$\sigma\left(\frac{g}{h}\right) \;=\; \frac{g(1-g_2)}{h(1-h_2)} \;=\; \frac{g}{h}(1-g_2)(1 + h_2 + h_2^2 + \cdots) \equiv \;\frac{g}{h} \mod \mathfrak{m}^2,$$

which proves the claim.

Given a power series $r \in R$, set $\overline{r}$ to be its constant term, i.e., $\overline{r} \in K$, and $r \equiv \overline{r} \mod \mathfrak{m}$. We next claim that if $r \in R^G$, then (2.1.1) can be strengthened to

$$\sigma(\overline{r}) \;\equiv\; \overline{r} \mod (x^2,\; y^2)R. \tag{2.1.2}$$

Given $r \in R^G$, let α, β, γ be elements of K such that

$$r \;\equiv\; \overline{r} + \alpha x + \beta y + \gamma xy \mod (x^2,\; y^2)R.$$

Since $\sigma(r) = r$, one has

$$\sigma(\overline{r}) + \sigma(\alpha)x + \sigma(\beta)y + \sigma(\gamma)xy \;\equiv\; \overline{r} + \alpha x + \beta y + \gamma xy \mod (x^2,\ y^2)R.$$

By (2.1.1), one has $\sigma(\alpha) \equiv \alpha \mod \mathfrak{m}^2$, and $\sigma(\beta) \equiv \beta \mod \mathfrak{m}^2$, and $\sigma(\gamma) \equiv \gamma \mod \mathfrak{m}^2$, so the above display yields $\sigma(\overline{r}) \equiv \overline{r} \mod (x^2, y^2)R$ as desired.

Lastly, we prove that R^G is not noetherian by showing that

$$f_{n+1} \;\notin\; (f_1, \dots, f_n)R^G \qquad \text{for } n \geqslant 1,$$

which, then, gives a strictly ascending chain of ideals in R^G. Suppose, to the contrary, that there exists an integer n such that

$$f_{n+1} \;=\; \sum_{k=1}^{n} r_k f_k$$

where $r_k \in R^G$ for each k with $1 \leqslant k \leqslant n$. The above may be written as

$$a_{n+1}x + b_{n+1}y \;=\; \sum_{k=1}^{n} r_k(a_k x + b_k y),$$

so comparing the coefficients of x yields

$$a_{n+1} \;=\; \sum_{k=1}^{n} \overline{r}_k\, a_k. \tag{2.1.3}$$

Applying σ to the above equation gives

$$a_{n+1} + y f_{n+2} \;=\; \sum_{k=1}^{n} \sigma(\overline{r}_k)(a_k + y f_{k+1}),$$

i.e.,

$$a_{n+1} + a_{n+2}xy + b_{n+2}y^2 \;=\; \sum_{k=1}^{n} \sigma(\overline{r}_k)(a_k + a_{k+1}xy + b_{k+1}y^2).$$

Since $\sigma(\overline{r}_k) \equiv \overline{r}_k \mod (x^2, y^2)R$ for each k by (2.1.2), one obtains

$$a_{n+1} + a_{n+2}xy \;\equiv\; \sum_{k=1}^{n} \overline{r}_k(a_k + a_{k+1}xy) \mod (x^2,\ y^2)R.$$

In light of (2.1.3), this simplifies to

$$a_{n+2}xy \;\equiv\; \sum_{k=1}^{n} \overline{r}_k\, a_{k+1}xy \mod (x^2,\ y^2)R,$$

from which one obtains

$$a_{n+2} = \sum_{k=1}^{n} \overline{r}_k \, a_{k+1}. \tag{2.1.4}$$

Repeating the argument that (2.1.3) implies (2.1.4) gives

$$a_{n+m+1} = \sum_{k=1}^{n} \overline{r}_k \, a_{k+m} \qquad \text{for each } m \geqslant 1.$$

As $\overline{r}_1, \ldots, \overline{r}_n$ are finitely many elements of the field K, this contradicts the assumption that $a_1, a_2, \ldots$ are infinitely many elements algebraically independent over $\mathbb{F}$. $\square$

Remark 2.2. Consider a discrete valuation ring R, with an action of a group G. We claim that the invariant ring R^G is either a field or a discrete valuation ring; in particular, R^G is noetherian. To see this, let $v\colon R \smallsetminus \{0\} \twoheadrightarrow \mathbb{Z}$ be the discrete valuation, and consider its restriction $\overline{v}\colon R^G \smallsetminus \{0\} \longrightarrow \mathbb{Z}$. If the image of this map is 0, then R^G is a field; otherwise, the image is generated by a positive integer n, which yields the discrete valuation

$$\frac{1}{n}\,\overline{v}\colon R^G \smallsetminus \{0\} \twoheadrightarrow \mathbb{Z}.$$

Data availability

No data was used for the research described in the article.

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