Math. Ann. (2015) 362:551–567
DOI 10.1007/s00208-014-1129-0

Mathematische Annalen

The F-pure threshold of a Calabi–Yau hypersurface

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Received: 16 July 2013 / Revised: 19 February 2014 / Published online: 15 November 2014

Abstract We compute the F-pure threshold of the affine cone over a Calabi–Yau hypersurface, and relate it to the order of vanishing of the Hasse invariant on the versal deformation space of the hypersurface.

Mathematics Subject Classification Primary 13A35; Secondary 13D45 · 14B07 · 14H52

1 Introduction

The F-pure threshold was introduced by Mustaţă, Takagi, and Watanabe [19,23]; it is a positive characteristic invariant, analogous to log canonical thresholds in characteristic zero. We calculate the possible values of the F-pure threshold of the affine cone over a Calabi–Yau hypersurface, and relate the threshold to the order of vanishing of the Hasse invariant, and to a numerical invariant introduced by van der Geer and Katsura in [7].

Theorem 1.1 *Suppose $R = K[x_0, \ldots, x_n]$ is a polynomial ring over a field K of characteristic $p > n + 1$, and f is a homogeneous polynomial defining a smooth*

B. Bhatt was supported by NSF Grants DMS 1160914 and DMS 1128155, and A. K. Singh by NSF Grant DMS 1162585. B. Bhatt and A. K. Singh were supported by NSF Grant 0932078000 while in residence at MSRI.

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Calabi–Yau hypersurface $X = \operatorname{Proj} R/fR$. *Then the* F*-pure threshold of* f *has the form*

$$\operatorname{fpt}(f) = 1 - h/p,$$

where h *is an integer with* $0 \leqslant h \leqslant \dim X$. *If* $p \geqslant n^2 - n - 1$, *then* h *equals the order of vanishing of the Hasse invariant on the versal deformation space of* $X \subset \mathbf{P}^n$.

Hernández has computed F-pure thresholds for binomial hypersurfaces [10] and for diagonal hypersurfaces [11]. The F-pure threshold is computed for a number of examples in [19, Section 4]. Example 4.6 of that paper computes the F-pure threshold in the case of an ordinary elliptic curve, and raises the question for supersingular elliptic curves; this is answered by the above theorem.

The theory of F-pure thresholds is motivated by connections to log canonical thresholds; for simplicity, let f be a homogeneous polynomial with rational coefficients. Using f_p for the corresponding prime characteristic model, one has

$$\operatorname{fpt}(f_p) \leqslant \operatorname{lct}(f) \quad \text{for all } p \gg 0,$$

where $\operatorname{lct}(f)$ denotes the log canonical threshold of f, and

$$\lim_{p \longrightarrow \infty} \operatorname{fpt}(f_p) = \operatorname{lct}(f),$$

see [19, Theorems 3.3, 3.4]; this builds on the work of a number of authors, primarily Hara and Yoshida [9]. It is conjectured that $\operatorname{fpt}(f_p)$ and $\operatorname{lct}(f)$ are equal for infinitely many primes; see [18] for more in this direction.

The F-pure threshold is known to be rational in a number of cases, including for principal ideals in an excellent regular local ring of prime characteristic [15]. Other results on rationality include [2–4,8,22]. For more on F-pure thresholds, we mention [1,12,16,17,20].

2 The F-pure threshold

In [23] the F-pure threshold is defined for a pair $(R, \mathfrak{a})$, where $\mathfrak{a}$ is an ideal in an F-pure ring of prime characteristic. The following special case is adequate for us:

Definition 2.1 Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. For an element f in $\mathfrak{m}$, and integer $q = p^e$, we define

$$\mu_f(q) := \min\left\{k \in \mathbf{N} \mid f^k \in \mathfrak{m}^{[q]}\right\},$$

where $\mathfrak{m}^{[q]}$ denotes the ideal generated by elements r^q for $r \in \mathfrak{m}$. Note that $\mu_f(1) = 1$, and that $1 \leqslant \mu_f(q) \leqslant q$. Moreover, $f^{\mu_f(q)} \in \mathfrak{m}^{[q]}$ implies that $f^{p\mu_f(q)} \in \mathfrak{m}^{[pq]}$, and it follows that $\mu_f(pq) \leqslant p\mu_f(q)$. Thus,

$$\left\{\frac{\mu_f(p^e)}{p^e}\right\}_{e \geqslant 0}$$

is a non-increasing sequence of positive rational numbers; its limit is the *F-pure threshold* of f, denoted $\operatorname{fpt}(f)$.

By definition, $f^{\mu_f(q)-1} \notin \mathfrak{m}^{[q]}$. Taking p-th powers, and using that R is F-pure,

$$f^{p\mu_f(q)-p} \notin \mathfrak{m}^{[pq]}.$$

Combining with the observation above, one has

$$p\mu_f(q) - p + 1 \leqslant \mu_f(pq) \leqslant p\mu_f(q). \tag{2.1}$$

Note that this implies

$$\mu_f(q) = \left\lceil \frac{\mu_f(pq)}{p} \right\rceil \quad \text{for each } q = p^e.$$

The definition is readily adapted to the graded case where R is a polynomial ring with homogeneous maximal ideal $\mathfrak{m}$, and f is a homogeneous polynomial.

Remark 2.2 The numbers $\mu_f(p^e)$ may be interpreted in terms of thickenings of the hypersurface f as follows. Let K be a field of characteristic $p > 0$, and f a homogeneous polynomial of degree d in $R = K[x_0, \dots, x_n]$. Fix integers $q = p^e$ and $t \leqslant q$. The Frobenius iterate $F^e\colon R/fR \longrightarrow R/fR$ lifts to a map $R/fR \longrightarrow R/f^qR$; composing this with the canonical surjection $R/f^qR \longrightarrow R/f^tR$, we obtain a map

$$\widetilde{F_t^e}\colon R/fR \longrightarrow R/f^tR.$$

Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & R(-d) \arrow[r, "f"] \arrow[d, "f^{q-t}F^e"] & R \arrow[r] \arrow[d, "F^e"] & R/fR \arrow[r] \arrow[d, "\widetilde{F_t^e}"] & 0 \\
0 \arrow[r] & R(-dt) \arrow[r, "f^t"] & R \arrow[r] & R/f^tR \arrow[r] & 0,
\end{tikzcd}$$

and the induced diagram of local cohomology modules

$$\begin{tikzcd}
0 \arrow[r] & H^n_{\mathfrak{m}}(R/fR) \arrow[r] \arrow[d, "\widetilde{F_t^e}"] & H^{n+1}_{\mathfrak{m}}(R)(-d) \arrow[r, "f"] \arrow[d, "f^{q-t}F^e"] & H^{n+1}_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & H^n_{\mathfrak{m}}(R/f^tR) \arrow[r] & H^{n+1}_{\mathfrak{m}}(R)(-dt) \arrow[r, "f^t"] & H^{n+1}_{\mathfrak{m}}(R) \arrow[r] & 0.
\end{tikzcd} \tag{2.2}$$

Since the vertical map on the right is injective, it follows that $\widetilde{F_t^e}$ is injective if and only if the middle map is injective, i.e., if and only if the element

$$f^{q-t}F^e\left(\left[\frac{1}{x_0\cdots x_n}\right]\right)=\left[\frac{f^{q-t}}{x_0^q\cdots x_n^q}\right]$$

is nonzero, equivalently, $f^{q-t}\notin\mathfrak{m}^{[q]}$. Hence $\widetilde{F_t^e}\colon H^n_{\mathfrak{m}}(R/fR)\longrightarrow H^n_{\mathfrak{m}}(R/f^tR)$ is injective if and only if $\mu_f(q)>q-t$.

The generating function of the sequence $\{\mu_f(p^e)\}_{e\geqslant 1}$ is a rational function:

Theorem 2.3 *Let $(R,\mathfrak{m})$ be a regular local ring of characteristic $p>0$, and let f be an element of $\mathfrak{m}$. Then the generating function*

$$G_f(z):=\sum_{e\geqslant 0}\mu_f(p^e)z^e$$

is a rational function of z with a simple pole at $z=1/p$; the F-pure threshold of f is

$$\operatorname{fpt}(f)=\lim_{z\longrightarrow 1/p}(1-pz)G_f(z).$$

Proof Since the numbers $\mu_f(p^e)$ are unchanged when R is replaced by its $\mathfrak{m}$-adic completion, there is no loss of generality in assuming that R is a complete regular local ring; the rationality of $\operatorname{fpt}(f)$ now follows from [15, Theorems 3.1, 4.1]. Let $\operatorname{fpt}(f)=a/b$ for integers a and b. By [19, Proposition 1.9], one has

$$\mu_f(p^e)=\lceil p^e\operatorname{fpt}(f)\rceil=\left\lceil\frac{ap^e}{b}\right\rceil\quad\text{for each } q=p^e.$$

Suppose $ap^{e_0}\equiv ap^{e_0+e_1}\mod b$ for integers e_0 and e_1. Then $ap^{e_0}\equiv ap^{e_0+ke_1}\mod b$ for each integer $k\geqslant 0$. Hence there exists an integer c such that

$$\begin{aligned}H(z):=\sum_{k\geqslant 0}\mu_f(p^{e_0+ke_1})z^{e_0+ke_1}&=\sum_{k\geqslant 0}\left\lceil\frac{ap^{e_0+ke_1}}{b}\right\rceil z^{e_0+ke_1}\\&=\sum_{k\geqslant 0}\frac{ap^{e_0+ke_1}+c}{b}\,z^{e_0+ke_1},\end{aligned}$$

is a rational function of z with a simple pole at $z=1/p$. Moreover,

$$\lim_{z\longrightarrow 1/p}(1-pz)H(z)=\frac{a}{be_1}.$$

Partitioning the integers $e\geqslant e_0$ into the congruence classes module e_1, it follows that $G_f(z)$ is the sum of a polynomial in z and e_1 rational functions of the form

$$\sum_{k\geqslant 0}\mu_f(p^{\ell+ke})z^{\ell+ke}.$$

The assertions regarding the pole and the limit now follow. □

The theorem holds as well in the graded setting.

3 Preliminary results

We record some elementary calculations that will be used later. Here, and in the following sections, R will denote a polynomial ring $K[x_0, \dots, x_n]$ over a field K of characteristic $p > 0$, and $\mathfrak{m}$ will denote its homogeneous maximal ideal. By the *Jacobian ideal* of a polynomial f, we mean the ideal generated by the partial derivatives

$$f_{x_i} := \partial f / \partial x_i \quad \text{for } 0 \leqslant i \leqslant n.$$

If f is homogeneous of degree coprime to p, then the Euler identity ensures that f is an element of the Jacobian ideal; this is then the defining ideal of the singular locus of the ring R/fR.

Lemma 3.1 *Let f be a homogeneous polynomial of degree d in $K[x_0, \dots, x_n]$ such that the Jacobian ideal J of f is $\mathfrak{m}$-primary. Then*

$$\mathfrak{m}^{(n+1)(d-2)+1} \subseteq J.$$

Proof Since J is $\mathfrak{m}$-primary, it is a complete intersection ideal. As it is generated by forms of degree $d-1$, the Hilbert–Poincaré series of R/J is

$$P(R/J, t) = \frac{(1 - t^{d-1})^{n+1}}{(1-t)^{n+1}} = (1 + t + t^2 + \cdots + t^{d-2})^{n+1}.$$

It follows that R/J has no nonzero elements of degree greater than $(n+1)(d-2)$. □

Lemma 3.2 *Let $R = K[x_0, \dots, x_n]$ and $\mathfrak{m}^{[q]} = (x_0^q, \dots, x_n^q)$. Then*

$$\mathfrak{m}^{[q]} :_R \mathfrak{m}^{(n+1)(d-2)+1} \subseteq \mathfrak{m}^{[q]} + \mathfrak{m}^{(n+1)(q-d+1)},$$

where $\mathfrak{m}^i = R$ for $i \leqslant 0$.

Proof We prove, more generally, that

$$\mathfrak{m}^{[q]} :_R \mathfrak{m}^k = \begin{cases} \mathfrak{m}^{[q]} + \mathfrak{m}^{nq+q-n-k} & \text{if } 0 \leqslant k \leqslant nq + q - n, \\ R & \text{if } k \geqslant nq + q - n. \end{cases}$$

Suppose r is a homogeneous element of $\mathfrak{m}^{[q]} :_R \mathfrak{m}^k$. Computing the local cohomology module $H^{n+1}_{\mathfrak{m}}(R)$ via a Čech complex on $x_0, \dots, x_n$, the element

$$\left[\frac{r}{x_0^q \cdots x_n^q} \right] \in H^{n+1}_{\mathfrak{m}}(R)$$

is annihilated by $\mathfrak{m}^k$, and hence lies in $\left[H^{n+1}_{\mathfrak{m}}(R) \right]_{\geqslant -n-k}$. If $r \notin \mathfrak{m}^{[q]}$, then

$$\deg r - (n+1)q \geqslant -n - k,$$

i.e., $r \in \mathfrak{m}^{nq+q-n-k}$. The pigeonhole principle implies that $\mathfrak{m}^{nq+q-n}$ is contained in $\mathfrak{m}^{[q]}$, which gives the rest. □

Lemma 3.3 *Let f be a homogeneous polynomial of degree d in $K[x_0, \ldots, x_n]$, such that the Jacobian ideal of f is $\mathfrak{m}$-primary. If $\mu_f(q)$ is not a multiple of p, then*

$$\mu_f(q) \geqslant \frac{(n+1)(q+1) - nd}{d}.$$

Proof Set $k := \mu_f(q)$, i.e., k is the least integer such that

$$f^k \in \mathfrak{m}^{[q]}.$$

Applying the differential operators $\partial/\partial x_i$ to the above, we see that

$$kf^{k-1} f_{x_i} \in \mathfrak{m}^{[q]} \quad \text{for each } i,$$

since $\partial/\partial x_i$ maps elements of $\mathfrak{m}^{[q]}$ to elements of $\mathfrak{m}^{[q]}$. As k is nonzero in K, one has

$$f^{k-1} J \subseteq \mathfrak{m}^{[q]},$$

where J is the Jacobian ideal of f. Lemma 3.1 now implies that

$$f^{k-1} \mathfrak{m}^{(n+1)(d-2)+1} \subseteq \mathfrak{m}^{[q]}.$$

By Lemma 3.2, we then have

$$f^{k-1} \in \mathfrak{m}^{[q]} + \mathfrak{m}^{(n+1)(q-d+1)}.$$

But $f^{k-1} \notin \mathfrak{m}^{[q]}$ by the minimality of k, so $\deg f^{k-1}$ is at least $(n+1)(q-d+1)$, i.e.,

$$d(k-1) \geqslant (n+1)(q-d+1);$$

rearranging the terms, one obtains the desired inequality

$$k \geqslant \frac{(n+1)(q+1) - nd}{d}.$$

□

Lemma 3.4 *Let f be a homogeneous polynomial of degree d in $K[x_0, \ldots, x_n]$, such that the Jacobian ideal of f is $\mathfrak{m}$-primary.*

(1) *If $\frac{\mu_f(q)-1}{q-1} = \frac{n+1}{d}$ for some $q = p^e$, then $\frac{\mu_f(pq)-1}{pq-1} = \frac{n+1}{d}$.*
(2) *Suppose $p \geqslant nd - d - n$. If $\frac{\mu_f(q)}{q} < \frac{n+1}{d}$ for some q, then $\mu_f(pq) = p\mu_f(q)$.*

Proof (1) Since $f^{\mu_f(q)-1}$ has degree $(q-1)(n+1)$ and is not an element of $\mathfrak{m}^{[q]}$, it must generate the socle in $R/\mathfrak{m}^{[q]}$. But then

$$\left(f^{\mu_f(q)-1}\right)^{\frac{pq-1}{q-1}}$$

generates the socle in $R/\mathfrak{m}^{[pq]}$, so

$$\mu_f(pq)-1=\left(\mu_f(q)-1\right)\left(\frac{pq-1}{q-1}\right).$$

For (2), suppose that $\mu_f(pq) < p\mu_f(q)$. Then $\mu_f(pq)$ is not a multiple of p by (2.1). Lemma 3.3 thus implies that

$$(n+1)(pq+1)-nd \leqslant d\mu_f(pq).$$

Combining with $\mu_f(pq) \leqslant p\mu_f(q)-1$ and $d\mu_f(q) \leqslant q(n+1)-1$, we obtain

$$p \leqslant nd-d-n-1,$$

which contradicts the assumption on p. □

We next prove a result on the injectivity of the Frobenius action on negatively graded components of local cohomology modules:

Theorem 3.5 *Let K be a field of characteristic $p > 0$. Let f be a homogeneous polynomial of degree d in $R = K[x_0, \ldots, x_n]$, such that the Jacobian ideal of f is primary to the homogeneous maximal ideal $\mathfrak{m}$ of R.*

If $p \geqslant nd-d-n$, then the Frobenius action below is injective:

$$F\colon \left[H^n_{\mathfrak{m}}(R/fR)\right]_{<0} \longrightarrow \left[H^n_{\mathfrak{m}}(R/fR)\right]_{<0}.$$

Proof Using (2.2) in the case $t=1$ and $e=1$, and restricting to the relevant graded components, we have the diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \left[H^n_{\mathfrak{m}}(R/fR)\right]_{\leqslant -1} & \longrightarrow & \left[H^{n+1}_{\mathfrak{m}}(R)\right]_{\leqslant -d-1} & \longrightarrow & \cdots \\
 & & \big\downarrow F & & \big\downarrow f^{p-1}F & & \\
0 & \longrightarrow & \left[H^n_{\mathfrak{m}}(R/fR)\right]_{\leqslant -p} & \longrightarrow & \left[H^{n+1}_{\mathfrak{m}}(R)\right]_{\leqslant -d-p} & \longrightarrow & \cdots .
\end{array}$$

Thus, it suffices to prove the injectivity of the map

$$f^{p-1}F\colon \left[H^{n+1}_{\mathfrak{m}}(R)\right]_{\leqslant -d-1} \longrightarrow \left[H^{n+1}_{\mathfrak{m}}(R)\right]_{\leqslant -d-p}.$$

A homogeneous element of $[H_{\mathfrak{m}}^{n+1}(R)]_{\leqslant -d-1}$ may be written as

$$\left[\frac{g}{(x_0 \cdots x_n)^{q/p}}\right]$$

for some q, where $g \in R$ is homogeneous of degree at most $(n+1)q/p - d - 1$. If

$$f^{p-1} F\left(\left[\frac{g}{(x_0 \cdots x_n)^{q/p}}\right]\right) = 0,$$

then it follows that $f^{p-1} g^p \in \mathfrak{m}^{[q]}$. Let k be the least integer with

$$f^k g^p \in \mathfrak{m}^{[q]},$$

and note that $0 \leqslant k \leqslant p-1$. If k is nonzero, then applying $\partial/\partial x_i$ we see that

$$f^{k-1} g^p J \subseteq \mathfrak{m}^{[q]}.$$

Lemmas 3.1 and 3.2 show that

$$f^{k-1} g^p \in \mathfrak{m}^{[q]} + \mathfrak{m}^{(n+1)(q-d+1)}.$$

Since $f^{k-1} g^p \notin \mathfrak{m}^{[q]}$, we must have

$$\deg f^{k-1} g^p \geqslant (n+1)(q-d+1).$$

Using $k \leqslant p-1$ and $\deg g^p \leqslant q(n+1) - pd - p$, this gives

$$nd - d - n - 1 \geqslant p,$$

contradicting the assumption on p. It follows that $k = 0$, i.e., that $g^p \in \mathfrak{m}^{[q]}$. But then

$$\left[\frac{g}{(x_0 \cdots x_n)^{q/p}}\right] = 0$$

in $H_{\mathfrak{m}}^{n+1}(R)$, which proves the desired injectivity. □

Remark 3.6 Theorem 3.5 is equivalent to the following geometric statement: if X is a smooth hypersurface of degree d in $\mathbf{P}^n$, then the map

$$H^{n-1}(X, \mathscr{O}_X(j)) \longrightarrow H^{n-1}(X, \mathscr{O}_X(jp)),$$

induced by Frobenius map on X, is injective for $j < 0$ and $p \geqslant nd - d - n$. This statement indeed admits a geometric proof based on the Deligne–Illusie method [6]. One views the de Rham complex $\Omega^*_{X/K}$ as an $\mathscr{O}_{X^{(1)}}$-complex, where $X^{(1)}$ is the Frobenius twist of X over K, and twists it over the latter with $\mathscr{O}_{X^{(1)}}(j)$. For $p > n-1$, the Deligne–Illusie decomposition $\Omega^*_{X/K} \simeq \oplus_i \Omega^i_{X^{(1)}/K}[-i]$, which is available as X clearly lifts to $W_2(K)$, reduces the above injectivity statement to proving that

$$H^{n-1-i}(X, \Omega^i_{X/K}(jp)) = 0$$

for $i > 0$ and $j < 0$. If $p \geqslant nd - d - n$, this vanishing can be proven using standard sequences (details omitted).

4 Calabi–Yau hypersurfaces

We get to the main theorem; see below for the definition of the Hasse invariant.

Theorem 4.1 *Let K be a field of characteristic $p > 0$, and n a positive integer. Let f be a homogeneous polynomial of degree $n + 1$ in $R = K[x_0, \dots, x_n]$, such that the Jacobian ideal of f, i.e., the ideal $(f_{x_0}, \dots, f_{x_n})$, is primary to the homogeneous maximal ideal of R. Then*:

(1) $\mu_f(p) = p - h$, *where h is an integer with* $0 \leqslant h \leqslant n - 1$,
(2) $\mu_f(pq) = p\mu_f(q)$ *for all $q = p^e$ with $q \geqslant n - 1$.*
(3) *If $p \geqslant n - 1$, then* $G_f(z) = \frac{1-hz}{1-pz}$ *and* $\mathrm{fpt}(f) = 1 - \frac{h}{p}$, *where* $0 \leqslant h \leqslant n - 1$.
(4) *Set* $X = \mathrm{Proj}\, R/fR$. *If $p \geqslant n^2 - n - 1$, then the integer h in* (1) *is the order of vanishing of the Hasse invariant on the versal deformation space of $X \subset \mathbf{P}^n$.*

The deformation space in (4) refers to embedded deformations of $X \subset \mathbf{P}^n$; if $n \geqslant 5$, this coincides with the versal deformation space of X as an abstract variety (see Remark 4.7). The following example from [11] shows that all possible values of h from (1) above are indeed attained:

Example 4.2 Consider $f = x_0^{n+1} + \cdots + x_{n+1}^{n+1}$ over a field of prime characteristic p not dividing $n+1$. Let h be an integer with $p \equiv h+1 \mod n+1$ and $0 \leqslant h \leqslant n-1$. Then

$$\mathrm{fpt}(f) = 1 - h/p,$$

for a proof, see [11, Theorem 3.1].

Proof of Theorem 4.1 If $\mu_f(p) = p$, then Lemma 3.4(1) shows that $\mu_f(q) = q$ for all q, and assertions (1–3) follow.

Assume that $\mu_f(p) < p$. Lemma 3.3 gives $\mu_f(p) \geqslant p - n + 1$, which proves (1). As $\mu_f(p) \leqslant p - 1$, it follows that

$$\mu_f(q) \leqslant q - q/p \quad \text{for each } q = p^e.$$

If $\mu_f(pq) < p\mu_f(q)$, then $\mu_f(pq)$ is not a multiple of p by (2.1). Lemma 3.3 implies

$$\mu_f(pq) \geqslant pq - n + 1,$$

and combining with $\mu_f(pq) \leqslant p\mu_f(q) - 1 \leqslant pq - q - 1$, we see that

$$pq - n + 1 \leqslant pq - q - 1,$$

i.e., that $q \leqslant n-2$. This completes the proof of (2), and then (3) follows immediately.
The proof of (4) and the surrounding material occupy the rest of this section. $\square$

The Hasse invariant. We briefly review the construction of the Hasse invariant for suitable families of varieties in positive characteristic p.

Fix a proper flat morphism $\pi\colon \mathscr{X} \longrightarrow S$ of relative dimension N between noetherian $\mathbf{F}_p$-schemes. Assume that the formation of $\mathrm{R}^i\pi_*\mathscr{O}_{\mathscr{X}}$ is compatible with base change, and that $\omega := \omega_{\mathscr{X}/S} := \mathrm{R}^N\pi_*\mathscr{O}_{\mathscr{X}}$ is a line bundle; the key example is a family of degree $(n+1)$ hypersurfaces in $\mathbf{P}^n$. The standard diagram of Frobenius twists of π takes the shape

$$\begin{tikzcd}
\mathscr{X} \arrow[dr, "\mathrm{Frob}_\pi"] \arrow[drr, bend left, "\mathrm{Frob}_{\mathscr{X}}"] \arrow[ddr, bend right, "\pi"'] & & \\
& \mathscr{X}^{(1)} \arrow[r, "\mathrm{Frob}_S"] \arrow[d, "\pi^{(1)}"] & \mathscr{X} \arrow[d, "\pi"] \\
& S \arrow[r, "\mathrm{Frob}_S"] & S,
\end{tikzcd}$$

where the square is Cartesian. Our assumption on π shows that

$$\omega_{\mathscr{X}^{(1)}/S} := \mathrm{R}^N\pi_*\mathscr{O}_{\mathscr{X}^{(1)}} \simeq \mathrm{Frob}_S^*\omega \simeq \omega^p.$$

Using this isomorphism, we define:

Definition 4.3 The *Hasse invariant* H of the family π is the element in

$$\mathrm{Hom}(\omega_{\mathscr{X}^{(1)}/S}, \omega) \simeq \mathrm{Hom}(\mathrm{Frob}_S^*\omega, \omega) \simeq \mathrm{Hom}(\omega^p, \omega) \simeq H^0(S, \omega^{1-p}),$$

defined by pullback along the relative Frobenius map $\mathrm{Frob}_\pi : \mathscr{X} \longrightarrow \mathscr{X}^{(1)}$.

Remark 4.4 The formation of the relative Frobenius map $\mathrm{Frob}_\pi : \mathscr{X} \longrightarrow \mathscr{X}^{(1)}$ is compatible with base change on S. It follows by our assumption on π that the formation of H is also compatible with base change. In particular, given a flat morphism $g\colon S' \longrightarrow S$ and a point $s' \in S'$, the order of vanishing of H at s' coincides with that at $g(s')$. Thus, in proving Theorem 4.1(4), we may assume that K is perfect.

To analyze H, fix a point s in S and an integer $t \geqslant 0$. Write $t[s] \subset S$ for the order t neighbourhood of s, and let $t\mathscr{X}_s \subset \mathscr{X}$ and $t\mathscr{X}_s^{(1)} \subset \mathscr{X}^{(1)}$ be the corresponding neighbourhoods of the fibres of π and $\pi^{(1)}$. The map Frob_π induces maps $t\mathscr{X}_s \longrightarrow t\mathscr{X}_s^{(1)}$, and hence maps

$$\phi_t\colon H^N\big(t\mathscr{X}_s^{(1)},\ \mathscr{O}_{t\mathscr{X}_s^{(1)}}\big) \longrightarrow H^N\big(t\mathscr{X}_s,\ \mathscr{O}_{t\mathscr{X}_s}\big).$$

The order of vanishing of H at s is, by definition, the maximal t such that this map is zero. In favourable situations, one can give a slightly better description of this integer:

Lemma 4.5 *If the map*

$$\psi_t \colon H^N\big(\mathscr{X}_s,\ \mathscr{O}_{X_s}\big) \longrightarrow H^N\big(t\mathscr{X}_s,\ \mathscr{O}_{t\mathscr{X}_s}\big)$$

induced by $\mathrm{Frob}_{\mathscr{X}}$ *is nonzero for some* $t \leqslant p$, *then the minimal such* t *is* $\mathrm{ord}_s H + 1$.

Proof For $t \leqslant p$, by the base change assumption on $\mathrm{R}^N\pi_*\mathscr{O}_{\mathscr{X}}$, one has

$$H^N\big(\mathscr{X}_s,\ \mathscr{O}_{X_s}\big) \otimes_{\kappa(s)} \mathscr{O}_{t[s]} \simeq H^N\big(t\mathscr{X}_s^{(1)},\ \mathscr{O}_{t\mathscr{X}_s^{(1)}}\big),$$

where $\mathscr{O}_{t[s]}$ is viewed as $\kappa(s)$-algebra via the composite

$$\kappa(s) \xrightarrow{\mathrm{Frob}_S} \mathscr{O}_{\mathrm{Frob}_S^{-1}[s]} \xrightarrow{\mathrm{can}} \mathscr{O}_{t[s]},$$

and the isomorphism is induced by the base change $\mathscr{X}^{(1)} \longrightarrow \mathscr{X}$ of $\mathrm{Frob}_S \colon S \longrightarrow S$. Hence, for such t, by adjunction, the map ϕ_t induced by Frob_π is nonzero if and only the map ψ_t induced by $\mathrm{Frob}_{\mathscr{X}}$ is nonzero. But $\mathrm{ord}_s H$ is the maximal integer t with $\phi_t = 0$. □

It is typically hard to calculate H, or even bound its order of vanishing. However, for families of Calabi–Yau hypersurfaces, we have the following remarkable theorem due to Deuring and Igusa; see [13] for $n = 2$, and Ogus [21, Corollary 3] in general:

Theorem 4.6 *Let* $\pi \colon \mathscr{X} \longrightarrow \mathrm{Hyp}_{n+1}$ *be the universal family of Calabi–Yau hypersurfaces in* $\mathbf{P}^n$. *For any point* $[Y] \in \mathrm{Hyp}_{n+1}(K)$ *corresponding to a smooth hypersurface* $Y \subset \mathbf{P}^n$, *we have* $\mathrm{ord}_{[Y]}(H) \leqslant n - 1$ *if* $n \leqslant p$.

Ogus's proof relies on crystalline techniques: he relates $\mathrm{ord}_{[Y]}(H)$ to the relative position of the conjugate and Hodge filtrations on a crystalline cohomology group of Y (following an idea of Katz), and then exploits the natural relation between the Hodge filtration and deformation theory of Y. His result will not be used in proving Theorem 4.1; in fact, our methods yield an alternative proof of Theorem 4.6 avoiding crystalline methods under a mild additional constraint on the prime characteristic p, see Remark 4.9.

Remark 4.7 The universal family $\pi \colon \mathscr{X} \longrightarrow \mathrm{Hyp}_{n+1}$ is, in fact, versal at $[X]$ if $n \geqslant 5$ so $\mathrm{ord}_{[X]}(H)$, i.e., the order of vanishing of H at $[X] \in \mathrm{Hyp}_{n+1}(K)$, is completely intrinsic to X. To see versatility, it suffices to show that the map

$$\mathrm{Hom}(I_X/I_X^2, \mathscr{O}_X) \longrightarrow H^1(X, T_X)$$

obtained from the adjunction sequence

$$0 \longrightarrow I_X/I_X^2 \longrightarrow \Omega^1_{\mathbf{P}^n}|_X \longrightarrow \Omega^1_X \longrightarrow 0$$

is surjective, and that $H^2(X, T_X) = 0$. By the long exact sequence, it suffices to show the vanishing of $H^1(X, T_{\mathbf{P}^n}|_X)$ and $H^2(X, T_X)$. The Euler sequence

$$0 \longrightarrow \mathscr{O}_{\mathbf{P}^n} \longrightarrow \mathscr{O}_{\mathbf{P}^n}(1)^{\oplus n+1} \longrightarrow T_{\mathbf{P}^n} \longrightarrow 0$$

restricted to X immediately shows that $H^i(X, T_{\mathbf{P}^n}|_X) = 0$ for $i = 1, 2$ if $n \geqslant 5$; here we use that $H^i(X, \mathscr{O}_X(j)) = 0$ for $0 < i < n-1$ and all j. The cohomology sequence for the dual of the adjunction sequence then shows that $H^2(X, T_X) = 0$.

The universal family. Fix notation as in Theorem 4.1, with K a perfect field. Let

$$\mathrm{Hyp}_{n+1} := \mathbf{P}\big(H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(n+1))^\vee\big)$$

be the space of hypersurfaces of degree $(n+1)$ in $\mathbf{P}^n$; we follow Grothendieck's conventions regarding projective bundles. Let $\pi\colon \mathscr{X} \longrightarrow \mathrm{Hyp}_{n+1}$ be the universal family, and let $\mathrm{ev}\colon \mathscr{X} \longrightarrow \mathbf{P}^n$ be the evaluation map. Informally, $\mathscr{X}$ parametrizes pairs (x, Y) where $x \in \mathbf{P}^n$ and $Y \in \mathrm{Hyp}_{n+1}$ is a degree $(n+1)$ hypersurface containing x. This description shows that $\mathrm{ev}\colon \mathscr{X} \longrightarrow \mathbf{P}^n$ is a projective bundle, and we can formally write it as $\mathbf{P}(\mathscr{K}^\vee) \longrightarrow \mathbf{P}^n$, where $\mathscr{K} \in \mathrm{Vect}(\mathbf{P}^n)$ is defined as

$$\mathscr{K} := \ker\big(H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(n+1)) \otimes \mathscr{O}_{\mathbf{P}^n} \longrightarrow \mathscr{O}_{\mathbf{P}^n}(n+1)\big),$$

and the map is the evident one. The resulting map

$$\mathscr{X} \longrightarrow \mathbf{P}\big(H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(n+1))^\vee\big)$$

is identified with π. Our chosen hypersurface $X \in \mathbf{P}^n$ gives a point $[X] \in \mathrm{Hyp}_{n+1}(K)$ with $X' := \pi^{-1}([X])$ mapping isomorphically to X via ev. For an integer $t \geqslant 1$, let $tX \subset \mathbf{P}^n$ and $tX' \subset \mathscr{X}$ be the order t neighbourhoods of $X \subset \mathbf{P}^n$ and $X' \subset \mathscr{X}$ respectively.

Lemma 4.8 *The map $H^{n-1}(tX, \mathscr{O}_{tX}) \longrightarrow H^{n-1}(tX', \mathscr{O}_{tX'})$ is injective for all t.*

Proof Let $V = H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(1))$, so $\mathbf{P}^n = \mathbf{P}(V)$ and $\mathrm{Hyp}_{n+1} = \mathbf{P}(\mathrm{Sym}^{n+1}(V)^\vee)$.

For each t, the sheaf $\mathscr{O}_{tX}$ admits a filtration defined by powers of the ideal defining $X \subset tX$, and similarly for tX'. The map $tX' \longrightarrow tX$ is compatible with this filtration as it sends $X' \subset tX'$ to $X \subset tX$. Hence, it suffices to check that the induced map

$$\phi_j\colon H^{n-1}(X, I_X^j/I_X^{j+1}) \longrightarrow H^{n-1}(X', I_{X'}^j/I_{X'}^{j+1})$$

is injective for each $j \geqslant 0$. Fix an isomorphism $\det(V) \simeq K$ and $f \in R_{n+1}$ defining X. These choices determine isomorphisms $I_X \simeq \mathscr{O}_{\mathbf{P}^n}(-n-1) \simeq K_{\mathbf{P}^n}$, and hence an isomorphism

$$H^{n-1}(X, O_X) \simeq H^n(\mathbf{P}^n, I_X) \simeq H^n(\mathbf{P}^n, K_{\mathbf{P}^n}) \simeq K.$$

Tensoring the exact sequence $0 \longrightarrow I_X \longrightarrow \mathscr{O}_{\mathbf{P}^n} \longrightarrow \mathscr{O}_X \longrightarrow 0$ with I_X^j and using Serre duality shows that

$$\begin{aligned} H^{n-1}\big(X, I_X^j/I_X^{j+1}\big) &= \ker\Big(H^n\big(\mathbf{P}^n, I_X^{j+1}\big) \longrightarrow H^n\big(\mathbf{P}^n, I_X^j\big)\Big) \\ &= \operatorname{coker}\Big(H^0\big(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}((j-1)(n+1))\big) \longrightarrow H^0\big(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(j(n+1))\big)\Big)^\vee \\ &= \operatorname{coker}\Big(\mathrm{Sym}^{(j-1)(n+1)}(V) \xrightarrow{f} \mathrm{Sym}^{j(n+1)}(V)\Big)^\vee. \end{aligned}$$

As $X' \subset \mathscr{X}$ is a fibre of π, one has $tX' = \pi^{-1}(t[X])$, where $t[X] \subset \mathrm{Hyp}_{n+1}$ is the order t neighbourhood of $[X] \in \mathrm{Hyp}_{n+1}(K)$. Using flatness of π and the aforementioned isomorphism $H^{n-1}(X, O_X) \simeq K$, we get

$$H^{n-1}(X', I_{X'}^j/I_{X'}^{j+1}) = \big(\mathrm{Sym}^j(\mathrm{Sym}^{n+1}(V)/(f))\big)^\vee.$$

One can check that the pullback ϕ_j above is dual to the map induced by the composition map $\mathrm{Sym}^j(\mathrm{Sym}^{n+1}(V)) \longrightarrow \mathrm{Sym}^{j(n+1)}(V)$ by passage to the appropriate quotients. In particular, the dual map is surjective, so ϕ_j is injective. □

Proof of Theorem 4.1(4) By Remark 4.4, we may assume that the field K is perfect. Fix some $1 \leqslant t \leqslant p$. We get a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d] & tX' \arrow[r, "j"] \arrow[d] & pX' \arrow[r, "a"] \arrow[d] & \mathrm{Frob}^*_{\mathscr{X}} X' \arrow[r, "c"] \arrow[d] & X' \arrow[d] \\ X \arrow[r, "k"] & tX \arrow[r, "\ell"] & pX \arrow[r, "b"] & \mathrm{Frob}^*_{\mathbf{P}^n} X \arrow[r, "d"] & X \end{tikzcd}$$

Here all vertical maps are induced by $\mathrm{ev}\colon \mathscr{X} \longrightarrow \mathbf{P}^n$, the maps c and d are induced by the Frobenius maps on $\mathscr{X}$ and $\mathbf{P}^n$ respectively, and i, j, k, ℓ, a and b are the evident closed immersions; the map b is an isomorphism as $X \subset \mathbf{P}^n$ is a Cartier divisor. In particular, the composite map $X' \longrightarrow X'$ and $X \longrightarrow X$ obtained from each row are the Frobenius maps on X' and X respectively. Passing to cohomology gives a commutative diagram

$$\begin{tikzcd} H^{n-1}(X, \mathscr{O}_X) \arrow[r, equal] \arrow[d, "a_t"] & H^{n-1}(X', \mathscr{O}_{X'}) \arrow[d, "b_t"] \\ H^{n-1}(tX, \mathscr{O}_{tX}) \arrow[r, "c_t"] & H^{n-1}(tX', \mathscr{O}_{tX'}) \end{tikzcd}$$

where a_t and b_t are induced by the Frobenius maps on $\mathbf{P}^n$ and $\mathscr{X}$ respectively, while c_t is injective by Lemma 4.8. To finish the proof, observe that Lemma 4.5 shows that $h+1$ is the minimal value of t for which b_t is injective. Since c_t is injective as well, this is also the minimal value of t for which a_t is injective. It follows by Remark 2.2 and Theorem 3.5 that $\mu_f(p) = p - (h+1) + 1 = p - h$. □

Remark 4.9 We recover Ogus's result, Theorem 4.6, for primes $p \geqslant n^2 - n - 1$; this is immediate from Theorem 4.1(1) and (4).

We conclude by giving an interpretation for $\mathrm{fpt}(f)$ in terms of de Rham cohomology. Write $\mathrm{Fil}^{\mathrm{conj}}_{\bullet}$ and $\mathrm{Fil}^{\bullet}_{H}$ for the increasing conjugate and decreasing Hodge filtrations on $H^{n-1}_{\mathrm{dR}}(X)$ respectively. Then:

Corollary 4.10 *One has* $\mathrm{fpt}(f) = 1 - a/p$, *where a is the largest i such that*

$$\mathrm{Frob}^*_X H^{n-1}_{\mathrm{dR}}(X) \simeq \mathrm{Fil}^{\mathrm{conj}}_0(H^{n-1}_{\mathrm{dR}}(X)) \subset \mathrm{Fil}^i_H(H^{n-1}_{\mathrm{dR}}(X)).$$

Proof This follows from Theorem 4.1 and Ogus's result [21, Theorem 1]. □

Note that the integer a appearing above is the a number defined by van der Geer and Katsura [7] for the special case of a Calabi–Yau family.

5 Quartic hypersurfaces in $\mathbf{P}^2$

Our techniques also yield substantive information for hypersurfaces other than Calabi–Yau hypersurfaces; as an example, we include the case of quartic hypersurfaces in $\mathbf{P}^2$.

When f defines a Calabi–Yau hypersurface X, it is readily seen that the Frobenius action on the $H^{\dim X}(X, \mathscr{O}_X)$ is injective if and only if $\mathrm{fpt}(f) = 1$, i.e., if and only

$$\mathrm{fpt}(f) = \mathrm{lct}(f).$$

For hypersurfaces X of general type, the injectivity of the Frobenius on $H^{\dim X}(X, \mathscr{O}_X)$, or even the ordinarity of X in the sense of Bloch and Kato [5, Definition 7.2]—a stronger condition—does not imply the equality of the F-pure threshold and the log canonical threshold: for example, for each f defining a quartic hypersurface in $\mathbf{P}^2$ over a field of characteristic $p \equiv 3 \mod 4$, we shall see that $\mathrm{fpt}(f) < \mathrm{lct}(f)$; we emphasize that this includes the case of generic hypersurfaces, and that these are ordinary in the sense of Bloch and Kato by a result of Deligne; see [14]. More generally:

Lemma 5.1 *Let f be a homogeneous polynomial of degree d in $K[x_0, \ldots, x_n]$. Then*:

(1) *For each $q = p^e$, one has* $\mu_f(q) \leqslant \left\lceil \frac{nq+q-n}{d} \right\rceil$,
(2) *If $nq + q$ is congruent to any of* $1, 2, \ldots, n \mod d$ *for some q, then* $\mathrm{fpt}(f) < \frac{n+1}{d}$.

For quartics in $\mathbf{P}^2$ one has $n = 2$, $d = 4$. If $p \equiv 3 \mod 4$, then $np + p \equiv 1 \mod d$, so the F-pure threshold is strictly smaller than the log canonical threshold by (2).

Proof The pigeonhole principle implies that $f^k \in \mathfrak{m}^{[q]}$ whenever

$$dk \geqslant (n + 1)(q - 1) + 1,$$

which proves (1). For (2), if $nq + q$ is congruent to any of $1, 2, \ldots, n \mod d$. Then

$$\left\lceil \frac{nq + q - n}{d} \right\rceil \leqslant \frac{nq + q - n + (n - 1)}{d} = \frac{nq + q - 1}{d},$$

and it follows using (1) that $\mu_f(q) < (nq+q)/d$. Thus,

$$\operatorname{fpt}(f) \leqslant \mu_f(q)/q < (n+1)/d.$$

$\square$

Theorem 5.2 *Let K be a field of characteristic $p > 2$. Let f be a homogeneous polynomial of degree 4 in $K[x_0, x_1, x_2]$, such that the Jacobian ideal of f is $\mathfrak{m}$-primary. Then the possible values for $\mu_f(q)$ and the F-pure threshold are*:

$$p \equiv 1 \bmod 4:\quad \mu_f(q) = \begin{cases} \frac{q(3p-3)}{4p} & \textit{for all } q, \quad \operatorname{fpt}(f) = \frac{3p-3}{4p}, \\ \frac{3q+1}{4} & \textit{for all } q, \quad \operatorname{fpt}(f) = \frac{3}{4}, \end{cases}$$

$$p \equiv 3 \bmod 4:\quad \mu_f(q) = \begin{cases} \frac{q(3p-5)}{4p} & \textit{for all } q, \quad \operatorname{fpt}(f) = \frac{3p-5}{4p}, \\ \frac{q(3p-1)}{4p} & \textit{for all } q, \quad \operatorname{fpt}(f) = \frac{3p-1}{4p}. \end{cases}$$

Proof Lemma 3.3 and Lemma 5.1(1) provide the respective inequalities

$$\left\lceil \frac{3p-5}{4} \right\rceil \leqslant \mu_f(p) \leqslant \left\lceil \frac{3p-2}{4} \right\rceil.$$

If $p \equiv 3 \bmod 4$, this reads

$$\frac{3p-5}{4} \leqslant \mu_f(p) \leqslant \frac{3p-1}{4},$$

so there are two possible values for the integer $\mu_f(p)$. The sequence $\{\mu_f(q)/q\}_q$ is constant by Lemma 3.4(2), which completes the proof in this case.

If $p \equiv 1 \bmod 4$, the inequalities read

$$\frac{3p-3}{4} \leqslant \mu_f(p) \leqslant \frac{3p+1}{4}.$$

Again, there are two choices for $\mu_f(p)$. If $\mu_f(p) = (3p-3)/4$, then $\{\mu_f(q)/q\}_q$ is a constant sequence by Lemma 3.4(2). If $\mu_f(p) = (3p+1)/4$, then Lemma 3.4(1) implies that $\mu_f(q) = (3q+1)/4$. $\square$

Remark 5.3 Similarly, for a quintic f in $K[x_0, x_1, x_2]$ with an $\mathfrak{m}$-primary Jacobian ideal, the possibilities for $\operatorname{fpt}(f)$ are easily determined; we do not list the corresponding $\mu_f(q)$ as these are dictated by $\operatorname{fpt}(f)$. We assume below that $p > 5$.

$$\begin{aligned} p &\equiv 1 \bmod 5:\quad \operatorname{fpt}(f):\quad (3p-3)/5p,\ \text{ or } 4/5, \\ p &\equiv 2 \bmod 5:\quad \operatorname{fpt}(f):\quad (3p-6)/5p,\ \text{ or } (3p-1)/5p, \\ p &\equiv 3 \bmod 5:\quad \operatorname{fpt}(f):\quad (3p-4)/5p,\ \text{ or } (3p^2-7)/5p^2,\ \text{ or } (3p^2-2)/5p^2, \\ p &\equiv 4 \bmod 5:\quad \operatorname{fpt}(f):\quad (3p-7)/5p,\ \text{ or } (3p-2)/5p. \end{aligned}$$

The log canonical threshold of a smooth quartic in $\mathbf{P}^2$ is $3/4$; except for the case where it equals $3/4$, the denominator of $\mathrm{fpt}(f)$ in Theorem 5.2 is p. For a quintic as above, if $\mathrm{fpt}(f) \neq \mathrm{lct}(f)$, then the denominator of $\mathrm{fpt}(f)$ is a power of p. More generally, one has:

Proposition 5.4 *Let K be a field of characteristic $p > 0$. Let f be a homogeneous polynomial of degree d in $K[x_0, \ldots, x_n]$ with an $\mathfrak{m}$-primary Jacobian ideal.*

If $p \geqslant nd - d - n$, then either $\mathrm{fpt}(f) = (n+1)/d$, or else the denominator of $\mathrm{fpt}(f)$ is a power of p.

Proof If $\mathrm{fpt}(f) < (n+1)/d$, then there exists an integer q such that

$$\mu_f(q)/q < (n+1)/d.$$

But then $\mathrm{fpt}(f) = \mu_f(q)/q$ by Lemma 3.4(2). □

Acknowledgments We thank Mircea Mustaţă for raising the question for elliptic curves at the AMS-MRC workshop on Commutative Algebra, Snowbird, 2010, the workshop participants, and the American Mathematical Society. We also thank Johan de Jong for a useful conversation.

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