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An asymptotic vanishing theorem for the cohomology of thickenings

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Abstract

Let X be a closed equidimensional local complete intersection subscheme of a smooth projective scheme Y over a field, and let X_t denote the t-th thickening of X in Y. Fix an ample line bundle $\mathcal{O}_Y(1)$ on Y. We prove the following asymptotic formulation of the Kodaira vanishing theorem: there exists an integer c, such that for all integers $t \geq 1$, the cohomology group $H^k(X_t, \mathcal{O}_{X_t}(j))$ vanishes for $k < \dim X$ and $j < -ct$. Note that there are no restrictions on the characteristic of the field, or on the singular locus of X. We also construct examples illustrating that a linear bound is indeed the best possible, and that the constant c is unbounded, even in a fixed dimension.

1 Introduction

Let Y be a projective scheme over a field, and let X be a closed subscheme defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$. For integers $t \geq 1$, let X_t denote the t-th *thickening* of X in Y, i.e., the closed subscheme of Y defined by $\mathcal{I}^t$. In [2], we proved the following version of the Kodaira vanishing theorem for thickenings of local complete intersection (lci) subvarieties of projective space $\mathbb{P}^n$:

Theorem 1.1 [2, Theorem 1.4] *Let X be a closed lci subvariety of $\mathbb{P}^n$ over a field of characteristic zero. Then, for each $t \geq 1$ and $k <$ codim(Sing X), one has*

$$H^k(X_t,\ \mathcal{O}_{X_t}(j)) = 0 \quad \textit{for } j < 0.$$

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When X is smooth and $t = 1$, this is precisely what is obtained from the Kodaira vanishing theorem. There are well-known counterexamples in the case of positive characteristic [9,12]; the condition on the singular locus is needed as well in view of the examples from [1]. Nonetheless, as we prove here, there is an *asymptotic* version of the above vanishing theorem that holds in good generality:

Theorem 1.2 *Let Y be a smooth projective scheme over a field, equipped with an ample line bundle $\mathcal{O}_Y(1)$. Let X be a closed equidimensional lci subscheme of Y. Then there exists an integer $c \geq 0$, such that for each $t \geq 1$ and $k < \dim X$, one has*

$$H^k(X_t,\ \mathcal{O}_{X_t}(j)) = 0 \qquad \textit{for all } j < -ct,$$

where, for a closed subscheme $Z \subset Y$ and integer j, we write $\mathcal{O}_Z(j) := \mathcal{O}_Y(1)^{\otimes j}|_Z$.

Unlike Theorem 1.1 that relies on Hodge-theoretic input (via Kodaira vanishing), the proof of Theorem 1.2 only uses Serre vanishing; this is why we do not need any assumption on the characteristic of the field in Theorem 1.2.

In the case where $Y = \mathbb{P}^n$, with $\mathcal{O}_Y(1)$ the standard ample line bundle, Theorem 1.2 answers [6, Questions 7.1 and 7.2] in the lci case; see Corollaries 3.3 and 3.4. The linear bound in Theorem 1.2 is best possible in view of Example 4.1 where, for each integer $c \geq 2$, we construct an lci scheme X of dimension 1 such that, for each $t \geq 1$, the cohomology group $H^0(X_t,\ \mathcal{O}_{X_t}(j))$ vanishes for $j \leq -ct$, and is nonzero for $j = -ct + 1$. Theorem 1.2 may fail—even in characteristic zero—when X is not lci, see Example 4.2, or when X is lci but not equidimensional, see Example 4.3.

2 Preliminaries

Let X be a projective scheme over a field $\mathbb{F}$. Set $d := \dim X$. We use $D_{coh}(X)$ to denote the derived category of complexes

$$\cdots \longrightarrow P^{i-1} \longrightarrow P^i \longrightarrow P^{i+1} \longrightarrow \cdots$$

of $\mathcal{O}_X$-modules with coherent cohomology, and $D^b_{coh}(X)$ for the full triangulated subcategory of bounded complexes, i.e., those with only finitely many nonzero cohomology groups. We use $D^{\leq a}_{coh}(X)$ (resp. $D^{\geq a}_{coh}(X)$) for complexes whose cohomology vanishes for $i > a$ (resp. $i < a$). It is straightforward that each complex in $D^{\leq a}_{coh}(X)$ (resp. $D^{\geq a}_{coh}(X)$) is quasi-isomorphic to a complex $P^\bullet$ such that $P^i = 0$ for $i > a$ (resp. $i < a$). In particular, each complex in $D^b_{coh}(X)$ is quasi-isomorphic to a complex $P^\bullet$ such that $P^i \neq 0$ only for finitely many integers i.

We use $D^{\leq a}(\mathbb{F})$ to denote the derived category of complexes of $\mathbb{F}$-vector spaces whose cohomology vanishes for $i > a$, with $D^{\geq a}(\mathbb{F})$ defined analogously.

Since the global section functor $R\Gamma(X, -)$ sends a coherent sheaf E on X to a complex in $D^{\leq d}(\mathbb{F})$, and since each element P in $D^b_{coh}(X) \cap D^{\leq a}_{coh}(X)$ is represented by a complex $P^\bullet$ such that $P^i \neq 0$ only for finitely many i and $P^i = 0$ for $i > a$, it follows by applying the hypercohomology spectral sequence to $P^\bullet$ that the complex

$R\Gamma(X, P^\bullet)$ lies in $D^{\leq a+d}(\mathbb{F})$; while we do not need it here, this is true even without the boundedness assumption.

A key technical ingredient is the derived m-th divided power functor

$$\Gamma^m : D^{\leq 0}_{coh}(X) \longrightarrow D^{\leq 0}_{coh}(X)$$

constructed in [8], see also [10, Chapter 25] or [11]. We summarize the properties of Γ^m that we use in this paper. For a locally free sheaf E of finite rank, Γ^m is the usual m-th divided power of E. In particular, one has in this case,

$$\Gamma^m(E) = \mathrm{Sym}^m(E^\vee)^\vee,$$

where $(-)^\vee = \mathcal{H}om(-, \mathcal{O}_X)$. By [10, 25.2.4.1], the functor Γ^m preserves $D^{\leq a}_{coh}(X)$ for all integers $a \leq 0$. Just as divided powers are not an additive functor, neither is Γ^m; the functor Γ^m does not preserve shifts or exact triangles in general. However, Γ is compatible with direct sums in the following sense: if $P = \bigoplus P^i$ is a (finite) direct sum, then

$$\Gamma^m(P) \cong \bigoplus_{a_i \geq 0,\ \sum a_i = m} \bigotimes_i \Gamma^{a_i}(P^i).$$

More generally, by [8, 5.4] or [10, 25.2], $\Gamma^* := \bigoplus_m \Gamma^m$ extends to a monoidal functor on the filtered derived category, which is compatible with the formation of the associated graded object in the above sense. In particular, if $P^\bullet$ is a complex with a finite filtration whose associated graded object is $\bigoplus P^i$, then $\Gamma^m(P^\bullet)$ has a finite filtration with the associated graded object given by

$$\bigoplus_{a_i \geq 0,\ \sum a_i = m} \bigotimes_i \Gamma^{a_i}(P^i).$$

In our applications, an ample line bundle $\mathcal{O}_X(1)$ on X is usually fixed at the outset. Thus, for $E \in D_{coh}(X)$ and any integer n, we write $E(n) := E \otimes_{\mathcal{O}_X} (\mathcal{O}_X(1))^{\otimes n}$ as expected.

3 Proof of the main theorem, and some consequences

To prove Theorem 1.2, we shall need a result which, very roughly speaking, is a variant of Serre vanishing where tensor powers of a sufficiently ample line bundle are replaced by divided powers of a sufficiently ample vector bundle. To make the proof flow better, it is convenient to formulate a more general statement involving complexes as follows:

Proposition 3.1 *Let X be a projective scheme over a field $\mathbb{F}$, equipped with an ample line bundle $\mathcal{O}_X(1)$. Fix a coherent sheaf F and $E \in D^b_{coh}(X) \cap D^{\leq 0}_{coh}(X)$. Then, for $c \gg 0$, one has*

$$R\Gamma\big(X,\ \Gamma^m(E(c)\big) \otimes F(l)) \in D^{\leq 0}(\mathbb{F})$$

for all integers $l \geq 0$ *and* $m > 0$.

The idea of the proof is to choose a representative of E where each term is a direct sum of twists of the structure sheaf $\mathcal{O}_X$, and then use Serre vanishing. However, to avoid working with unbounded complexes, we only choose an "approximate representative" for E, i.e., one that does not change cohomology in a certain range of degrees. The key point is Lemma 3.2, which ensures that applying derived divided powers to a shift of a "positive" complex can only increase "positivity."

Proof Fix a coherent sheaf F on X as in the statement of the proposition. By Serre vanishing, there exists an integer $j_0 > 0$ such that $H^i(X, F(j)) = 0$ for all $i > 0$ and $j \geq j_0$. Stated differently, $R\Gamma(X, F(j)) \in D^{\leq 0}(\mathbb{F})$ for $j \geq j_0$.

For the purpose of the proof, we may replace E by any complex quasi-isomorphic to E. By constructing a resolution of E whose terms consist of finite direct sums of twists of $\mathcal{O}_X$, we may hence assume that E is bounded above by zero, and that each E^i is a finite direct sum of twists of $\mathcal{O}_X$. Set $d := \dim X$. For an integer r with $r > d$, set $P^\bullet$ to be

$$0 \longrightarrow E^{-r} \longrightarrow E^{-(r-1)} \longrightarrow \cdots \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow 0.$$

Then each P^i is a finite direct sum of twists of $\mathcal{O}_X$, and the cokernel $Q^\bullet$ of the injective map $P^\bullet \longrightarrow E^\bullet$ lies in $D^b_{coh}(X) \cap D^{\leq -r}_{coh}(X)$.

For each integer c, we view

$$\varphi \colon P^\bullet(c) \hookrightarrow E^\bullet(c)$$

as a one-step decreasing filtration of $E^\bullet(c)$, normalized so that $\mathrm{gr}^1(E^\bullet(c)) = P^\bullet(c)$ and $\mathrm{gr}^0(E^\bullet(c)) = Q^\bullet(c)$. By the compatibility of Γ^m with filtrations, as discussed in §2, we obtain an induced filtration on $\Gamma^m(E^\bullet(c))$ with the associated graded pieces given by

$$\mathrm{gr}^a(\Gamma^m(E^\bullet(c)) \;=\; \Gamma^a(P^\bullet(c)) \otimes \Gamma^b(Q^\bullet(c)), \qquad \text{with } a + b = m,$$

where negative divided powers are understood to be 0. Thus, the graded pieces vanish unless $0 \leq a \leq m$, and $a = 0$ gives the "top" graded piece (i.e., the quotient) while $a = m$ gives the "bottom" graded piece (i.e., a subobject). In particular, the map

$$\Gamma^m(\varphi) \colon \Gamma^m(P^\bullet(c)) \longrightarrow \Gamma^m(E^\bullet(c))$$

identifies with the inclusion

$$\mathrm{gr}^m(\Gamma^m(E^\bullet(c))) \xleftarrow{\simeq} \mathrm{Fil}^m(\Gamma^m(E^\bullet(c))) \longrightarrow \Gamma^m(E^\bullet(c)),$$

and hence its cokernel (which we regard as a representative for its cone in the derived category) carries a filtration whose graded pieces have the form

$$\Gamma^a(P^\bullet(c)) \otimes \Gamma^b(Q^\bullet(c)), \qquad \text{with } a + b = m \text{ and } b > 0.$$

Since Γ^a preserves $D^{\leq i}_{coh}(X)$ for $i \leq 0$, we have $\Gamma^a(P^\bullet) \in D^{\leq 0}_{coh}(X)$ and $\Gamma^b(Q^\bullet) \in D^{\leq -d}_{coh}(X)$ provided $b > 0$, and hence their tensor product lies in $D^{\leq -d}_{coh}(X)$. Since tensoring with $F(j)$ preserves $D^{\leq -d}_{coh}(X)$, we see that the cone of

$$\Gamma^m(P^\bullet(c)) \otimes F(j) \longrightarrow \Gamma^m(E(c)) \otimes F(j)$$

also lies in $D^{\leq -d}_{coh}(X)$ for all $m \geq 0$ and $c, j \in \mathbb{Z}$.

Since $R\Gamma(X, -)$ takes $D^{\leq -d}_{coh}(X)$ to $D^{\leq 0}(\mathbb{F})$, the cone of

$$R\Gamma(X,\ \Gamma^m(P^\bullet(c)) \otimes F(j)) \longrightarrow R\Gamma(X,\ \Gamma^m(E(c)) \otimes F(j))$$

lies in $D^{\leq 0}(\mathbb{F})$ for all $m \geq 0$ and $c, j \in \mathbb{Z}$. It is thus sufficient to prove the proposition when E is replaced by $P^\bullet$; indeed, for the remainder of the proof, we take E to be $P^\bullet$.

By construction, $P^i = 0$ for $i > 0$ and $i < -r$. Consider the filtration on $P^\bullet(c)$ with the i-th filtered piece given by

$$0 \longrightarrow P^{-i}(c) \longrightarrow \cdots \longrightarrow P^0(c) \longrightarrow 0.$$

By the compatibility of Γ^m with filtrations, we get that $\Gamma^m(P^\bullet(c))$ has a filtration with associated graded object

$$\bigoplus_{a_i \geq 0,\ \sum a_i = m} \Gamma^{a_0}(P^0(c)) \otimes \Gamma^{a_1}(P^{-1}(c)[1]) \otimes \cdots \otimes \Gamma^{a_r}(P^{-r}(c)[r])$$

for each $m \geq 0$ and $c \in \mathbb{Z}$. Tensoring with $F(j)$, we see that for each $c, j \in \mathbb{Z}$ and $m \geq 0$, the complex $\Gamma^m(P^\bullet(c)) \otimes F(j)$ has a finite filtration with associated graded object

$$\bigoplus_{a_i \geq 0,\ \sum a_i = m} \Gamma^{a_0}(P^0(c)) \otimes \Gamma^{a_1}(P^{-1}(c)[1]) \otimes \cdots \otimes \Gamma^{a_r}(P^{-r}(c)[r]) \otimes F(j).$$

It is thus enough to show: for $m > 0$, $j \geq 0$, and $c \gg 0$, applying $R\Gamma(X, -)$ to each of the terms in the direct sum above produces an object in $D^{\leq 0}(\mathbb{F})$. Fix such a term corresponding to an index of the form $m = \sum_i a_i$ with $a_i \geq 0$.

As each P^{-i} is a finite direct sum of twists of the structure sheaf, and only finitely many terms P^{-i} are nonzero, we know that for $c \gg 0$, each $P^{-i}(c)$ is a direct sum of line bundles of the form $\mathcal{O}_X(j)$ for $j \geq j_0$, where j_0 was the integer chosen at the start of the proof. By Lemma 3.2 below, there are now two possibilities for the term $\Gamma^{a_i}(P^{-i}(c)[i])$ appearing above: if $a_i = 0$, we simply get $\mathcal{O}_X$, while for $a_i > 0$, we get a complex which is a direct sum of complexes of the form $\mathcal{O}_X(j) \otimes_{\mathbb{F}} V$ with $V \in D^{\leq 0}(\mathbb{F})$. Since $m = \sum_i a_i$ is positive, we must have $a_i > 0$ for at least one i. Thus, the complex displayed above is a direct sum of complexes of the form $F(j) \otimes_{\mathbb{F}} V$ for some $j \geq j_0$ and $V \in D^{\leq 0}(\mathbb{F})$. By our choice of j_0, we know that

$$R\Gamma(X,\ F(j) \otimes_{\mathbb{F}} V) \ \in\ D^{\leq 0}(\mathbb{F})$$

if $j \geq j_0$ and $V \in D^{\leq 0}(\mathbb{F})$, which completes the proof. □

Lemma 3.2 *Let X be a projective scheme over a field $\mathbb{F}$, equipped with an ample line bundle $\mathcal{O}_X(1)$. Let $b, j_1, \ldots, j_s$ be integers, where $b \geq 0$, and set*

$$E := \bigoplus_{i=1}^{s} \mathcal{O}_X(j_i)[b],$$

which is a shift of a direct sum of twists of $\mathcal{O}_X$. Then, for each integer $a \geq 0$, one has

$$\Gamma^a(E) \ = \bigoplus_{a_i \geq 0,\ \sum a_i = a} \mathcal{O}_X(a_1 j_1 + \cdots + a_s j_s) \otimes_{\mathbb{F}} \Gamma^{a_1}(\mathbb{F}[b]) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \Gamma^{a_s}(\mathbb{F}[b]),$$

where each $\Gamma^{a_i}(\mathbb{F}[b])$ is a complex of $\mathbb{F}$-vector spaces lying in $D^{\leq 0}(\mathbb{F})$.

Proof As $\Gamma^*(-)$ preserves $D^{\leq 0}(\mathbb{F})$, the containment in $D^{\leq 0}(\mathbb{F})$ asserted at the end is automatic. The rest follows from the behavior of Γ^a under direct sums, and the fact that

$$\Gamma^a(\mathcal{O}_X(j)[b]) \ \simeq\ \mathcal{O}_X(aj) \otimes_{\mathbb{F}} \Gamma^a(\mathbb{F}[b])$$

for integers a, b, j with $a, b \geq 0$. □

Proof of Theorem 1.2 Set $d := \dim X$, and let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal sheaf of the lci subscheme $X \hookrightarrow Y$, so $\mathcal{I}/\mathcal{I}^2$ is the conormal bundle of this closed immersion. Since X is lci and equidimensional, its dualizing complex has the form $\omega_X[d]$ for a line bundle ω_X, so Serre duality says

$$H^i(X,\ \mathcal{O}_X(j)) \ \cong\ H^{d-i}(X,\ \omega_X(-j))^\vee.$$

By Serre vanishing, there exists an integer $c_0 \geq 1$ such that

$$H^{d-i}(X,\ \omega_X(-j)) = 0 \qquad \text{for all } -j \geq c_0 \text{ and } i < d.$$

Equivalently, we have

$$R\Gamma(X,\ \mathcal{O}_X(j)) \ \in\ D^{\geq d}(\mathbb{F}) \qquad \text{for } j \leq -c_0.$$

We shall reduce the rest of the proof to the following assertion:

There exists an integer $c_1 \geq 0$ such that, for each integer $s \geq 1$, one has

$$R\Gamma\big(X,\ \operatorname{Sym}^s(\mathcal{I}/\mathcal{I}^2)(j)\big) \ \in\ D^{\geq d}(\mathbb{F}) \qquad \text{for } j < -c_1 s. \tag{3.1}$$

We claim that (3.1) implies the theorem. Indeed, given an integer $t \geq 1$ as in the theorem, summing the conclusion of (3.1) for $s = 1, \ldots, t-1$ implies that

$$R\Gamma\big(X_t,\ \mathcal{I}/\mathcal{I}^t\big) \in\ D^{\geq d}(\mathbb{F})$$

for $j < -c_1(t-1) = -c_1 t + c_1$, and hence also for $j < -c_1 t$. Taking $c = \max(c_0, c_1)$ gives the theorem.

It remains to prove (3.1). Let $\mathcal{N} := (\mathcal{I}/\mathcal{I}^2)^\vee$ denote the normal bundle. Using Serre duality, it suffices to show that there exists $c_1 \geq 0$, such that for each $s \geq 1$, one has

$$R\Gamma\big(X,\ \Gamma^s(\mathcal{N})(j) \otimes \omega_X\big) \ \in\ D^{\leq 0}(\mathbb{F}) \qquad \text{for } j > c_1 s.$$

But this follows from Proposition 3.1, since

$$\Gamma^s(\mathcal{N})(as+b) \ =\ \Gamma^s(\mathcal{N}(a))(b)$$

for all integers a, b. □

We record implications of Theorem 1.2 for local cohomology modules. By a *standard graded ring* over a field $\mathbb{F}$, we mean an $\mathbb{N}$-graded ring R with $R_0 = \mathbb{F}$ that is generated, as an $\mathbb{F}$-algebra, by finitely many elements of R_1. Let R be a standard graded polynomial ring over a field, and let I be a homogeneous ideal. For $t \geq 1$, set $X_t := \operatorname{Proj} R/I^t$. Let j be an arbitrary integer. Using $\mathfrak{m}$ to denote the homogeneous maximal ideal of R, one has an exact sequence relating local cohomology and sheaf cohomology:

$$\begin{aligned} 0 \longrightarrow H^0_{\mathfrak{m}}(R/I^t)_j \longrightarrow (R/I^t)_j \longrightarrow H^0(X_t,\ \mathcal{O}_{X_t}(j)) \\ \longrightarrow H^1_{\mathfrak{m}}(R/I^t)_j \longrightarrow 0. \end{aligned} \tag{3.2}$$

Moreover, for each $k \geq 1$, one has

$$H^k(X_t,\ \mathcal{O}_{X_t}(j)) \ =\ H^{k+1}_{\mathfrak{m}}(R/I^t)_j.$$

The asymptotic behavior of lengths of local cohomology modules has been studied extensively, see [4] and the references therein. For R an analytically unramified local ring and I an arbitrary ideal, the limit

$$\lim_{t\longrightarrow\infty} \ell(H^0_{\mathfrak{m}}(R/I^t))/t^{\dim R}$$

exists by [4, Corollary 6.3]. In [5, Theorem 1.2] the authors give an example where this limit is irrational, for I defining a smooth complex projective curve. In the context of local cohomology, Theorem 1.2 yields the following:

Corollary 3.3 *Let R be a standard graded polynomial ring over a field, and $\mathfrak{m}$ the homogeneous maximal ideal of R. Suppose I is a homogeneous ideal such that R/I is equidimensional and* Proj R/I *is lci. Then*

$$\limsup_{t\longrightarrow\infty} \frac{\ell\big(H^k_{\mathfrak{m}}(R/I^t)\big)}{t^{\dim R}} < \infty$$

for each $k < \dim R/I$.

Proof The case $k = 0$ is covered by [4, Corollary 6.3], so assume $k \geq 1$. By Theorem 1.2 applied to $Y = \mathbb{P}^n$, with $\mathcal{O}_Y(1)$ being the standard ample line bundle, there exists an integer $c \geq 0$, such that for each $t \geq 1$ and $k < \dim R/I$, one has

$$H^k_{\mathfrak{m}}(R/I^t)_j = 0 \qquad \text{for } j < -ct.$$

The result now follows from [6, Theorem 5.3]. □

Corollary 3.4 *Let R be a standard graded polynomial ring over a field, with homogeneous maximal ideal $\mathfrak{m}$. Suppose I is a homogeneous radical ideal such that R/I is equidimensional and $\ell\big(H^k_{\mathfrak{m}}(R/I^t)\big) < \infty$ for each $k < \dim R/I$ and $t \geq 1$. Then, for each $k < \dim R/I$,*

$$\limsup_{t\longrightarrow\infty} \frac{\ell\big(H^k_{\mathfrak{m}}(R/I^t)\big)}{t^{\dim R}} < \infty.$$

Proof For a radical ideal $\mathfrak{a}$ in a regular local ring A, a theorem of Cowsik and Nori implies that $A/\mathfrak{a}^t$ is Cohen–Macaulay for each $t \geq 1$ if and only if $A/\mathfrak{a}$ is a complete intersection ring, [3, page 219]. The finiteness of the length of each local cohomology module $H^k_{\mathfrak{m}}(R/I^t)$, for $k < \dim R/I$, implies that $(R/I^t)_{\mathfrak{p}}$ is Cohen–Macaulay for each $t \geq 1$ and $\mathfrak{p} \in \operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$. It follows that $(R/I)_{\mathfrak{p}}$ is a complete intersection ring for each $\mathfrak{p} \neq \mathfrak{m}$, and hence that Proj R/I is lci. The desired result is now immediate from Corollary 3.3.

Remark 3.5 In the recent paper [7], the authors prove the following result: let R be a standard graded ring over a field of characteristic zero; let $\mathfrak{m}$ denote the homogeneous maximal ideal of R. Suppose I is a homogeneous ideal such that R/I is Cohen–Macaulay and of dimension at least 2, and I is locally a complete intersection on

Spec $R \smallsetminus \{\mathfrak{m}\}$. Fix an integer k with $k < \dim R/I$. Then, for $t \geq 1$, the lowest degree in which the local cohomology module $H^k_{\mathfrak{m}}(R/I^t)$ is nonzero is bounded below by a linear function of t.

The hypotheses in [7] are somewhat different from those in Theorem 1.2 of the present paper, where there is no assumption on the characteristic, nor do we require the ring R/I to be Cohen–Macaulay.

4 Examples

The following example, which is a variation of [2, Example 5.7], shows that the bound in Theorem 1.2 cannot be better than linear; the example also shows that the constant c in the theorem may be unbounded, even when $\dim X$ is fixed.

Example 4.1 Consider the polynomial ring $R := \mathbb{F}[x, y, u, v, w]$, where $\mathbb{F}$ is a field of arbitrary characteristic. Fix an integer $c \geq 2$, and set

$$I := (uy - vx,\ vy - wx) + (u, v, w)^c.$$

The ring R/I has dimension 2, and the elements x, y form a system of parameters. Since

$$(R/I)_x = \mathbb{F}[x, x^{-1}, y, u]/(u^c) \quad \text{and} \quad (R/I)_y = \mathbb{F}[x, y, y^{-1}, w]/(w^c),$$

one sees that $X := \operatorname{Proj} R/I$ is lci. We prove that for all integers $t \geq 1$, the asymptotic vanishing in this example takes the form $H^0(X_t,\ \mathcal{O}_{X_t}(j)) = 0$ for $j \leq -ct$, whereas

$$H^0(X_t,\ \mathcal{O}_{X_t}(-ct + 1)) \neq 0.$$

The argument is via local cohomology; the sequence (3.2) shows that for $j < 0$, one has

$$H^0(X_t,\ \mathcal{O}_{X_t}(j)) \ = \ H^1_{\mathfrak{m}}(R/I^t)_j.$$

We analyze $H^1_{\mathfrak{m}}(R/I^t)$ using the Čech complex

$$0 \longrightarrow R/I^t \longrightarrow (R/I^t)_x \oplus (R/I^t)_y \longrightarrow (R/I^t)_{xy} \longrightarrow 0,$$

and claim that

$$\left[\left(\frac{u}{x^2}\right)^{ct-1},\ \left(\frac{w}{y^2}\right)^{ct-1} \right] \ \in \ (R/I^t)_x \oplus (R/I^t)_y \tag{4.1}$$

determines a nonzero element of $H^1_{\mathfrak{m}}(R/I^t)_{-ct+1}$. To verify that the displayed element is indeed a Čech cocycle, it suffices to verify that

$$(uy^2)^{ct-1} - (wx^2)^{ct-1} \ \in \ I^t.$$

Since the ideal I contains $uy^2 - wx^2$ as well as $(uy^2)^c$, it suffices to check that

$$(uy^2)^{ct-1} - (wx^2)^{ct-1} \in \left(uy^2 - wx^2,\ (uy^2)^c\right)^t$$

in the polynomial ring $\mathbb{F}[x, y, u, v, w]$, and hence in its subring $\mathbb{F}[uy^2,\ wx^2]$. Setting $a := uy^2$ and $b := wx^2$ for notational simplicity, it suffices to check that

$$a^{ct-1} - b^{ct-1} \in \left(a - b,\ a^c\right)^t$$

in the polynomial ring $\mathbb{F}[a, b]$. Replacing b by $a - b$, we need to show

$$a^{ct-1} - (a - b)^{ct-1} \in \left(b,\ a^c\right)^t,$$

which is evident by considering the binomial expansion of $(a-b)^{ct-1}$. This completes the argument that (4.1) is indeed a Čech cocycle.

To verify that $(u/x^2)^{ct-1}$ is nonzero in $(R/I^t)_x$, note that its image under the surjection

$$(R/I^t)_x \longrightarrow\!\!\!\!\rightarrow \left(\frac{R}{(uy - vx,\ vy - wx) + (u, v, w)^{ct}}\right)_x = \mathbb{F}[x, x^{-1}, y, u]/(u^{ct})$$

is nonzero. As it has negative degree, the element (4.1) cannot be in the image of

$$R/I^t \longrightarrow (R/I^t)_x \oplus (R/I^t)_y,$$

which completes the argument that

$$H^0(X_t,\ \mathcal{O}_{X_t}(-ct + 1)) = H^1_{\mathfrak{m}}(R/I^t)_{-ct+1} \neq 0.$$

Next, we examine the intersection of $(R/I^t)_x$ and $(R/I^t)_y$ in $(R/I^t)_{xy}$. For this, consider the $\mathbb{Z}^3$-grading with

$$\begin{aligned} &\deg u = (2, 0, -1), &\quad &\deg x = (1, 0, 0),\\ &\deg v = (1, 1, -1), &\quad &\deg y = (0, 1, 0),\\ &\deg w = (0, 2, -1). & & \end{aligned}$$

Each homogeneous element of $(R/I^t)_x$ has degree (i, j, k) with $j \geq 0$ and $k > -ct$, whereas, in $(R/I^t)_y$, each homogeneous element has degree (i, j, k) with $i \geq 0$ and $k > -ct$. Thus, a homogeneous element in the intersection must have degree (i, j, k) satisfying $i \geq 0$, $j \geq 0$, and $k > -ct$. But the $\mathbb{Z}^3$-grading specializes to the standard $\mathbb{N}$-grading on R under the map

$$\mathbb{Z}^3 \longrightarrow \mathbb{Z} \qquad \text{with} \qquad (i, j, k) \longmapsto i + j + k,$$

implying that each homogeneous element in the kernel of

$$(R/I^t)_x \oplus (R/I^t)_y \longrightarrow (R/I^t)_{xy}$$

has degree greater than $-ct$. It follows that

$$H^0(X_t,\ \mathcal{O}_{X_t}(j)) \;=\; H^1_{\mathfrak{m}}(R/I^t)_j \;=\; 0 \qquad \text{for } j \le -ct.$$

Theorem 1.2 may fail if X is not lci:

Example 4.2 Let Z denote the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$, over a field $\mathbb{F}$ of characteristic zero, and set $X \subset \mathbb{P}^6$ to be the projective cone over Z. Then X has dimension 4, and is Cohen–Macaulay though not lci. If $t \ge 2$, we claim that

$$H^3(X_t,\ \mathcal{O}_{X_t}(j)) \neq 0 \qquad \text{for each } j < 0.$$

By [2, Example 5.1], if $t \ge 2$, then $H^2(Z_t,\ \mathcal{O}_{Z_t}) \neq 0$, i.e., $H^3_{\mathfrak{m}_R}(R/I^t)_0 \neq 0$, where R/I is the homogeneous coordinate ring for $Z \subset \mathbb{P}^5$. But then $X \subset \mathbb{P}^6$ has homogeneous coordinate ring S/IS, where $S := R[y]$ with y being a new indeterminate, so

$$H^4_{\mathfrak{m}_S}(S/I^tS) \;\cong\; H^3_{\mathfrak{m}_R}(R/I^t) \otimes_{\mathbb{F}} H^1_{(y)}(\mathbb{F}[y])$$

has a nonzero graded component in each negative degree, which proves the claim.

Lastly, Theorem 1.2 may fail if X is lci but not equidimensional:

Example 4.3 Consider the polynomial ring $R := \mathbb{F}[x, y, z]$, where $\mathbb{F}$ is a field of arbitrary characteristic, and set $I := (xy,\ xz)$. Then R/I has dimension 2, and $X :=$ Proj R/I is smooth, hence lci. Fix $t \ge 1$. The exact sequence

$$0 \longrightarrow R/I^t \longrightarrow R/(x^t) \oplus R/(y,z)^t \longrightarrow R/(x^t + (y,z)^t) \longrightarrow 0$$

induces an isomorphism

$$H^1_{\mathfrak{m}}(R/I^t)_j \;\cong\; H^1_{\mathfrak{m}}(R/(y,z)^t)_j \qquad \text{for } j < 0$$

which shows that $H^1_{\mathfrak{m}}(R/I^t)$ has a nonzero graded component in each negative degree, so

$$H^0(X_t,\ \mathcal{O}_{X_t}(j)) \;\neq\; 0 \qquad \text{for each } j < 0.$$

References

1. Arapura, D., Jaffe, D.B.: On Kodaira vanishing for singular varieties. Proc. Am. Math. Soc. **105**, 911–916 (1989)
2. Bhatt, B., Blickle, M., Lyubeznik, G., Singh, A.K., Zhang, W.: Stabilization of the cohomology of thickenings. Am. J. Math. **141**, 531–561 (2019)
3. Cowsik, R.C., Nori, M.V.: On the fibres of blowing up. J. Indian Math. Soc. **40**, 217–222 (1976)
4. Cutkosky, S.D.: Asymptotic multiplicities of graded families of ideals and linear series. Adv. Math. **264**, 55–113 (2014)
5. Cutkosky, S.D., Hà, H.T., Srinivasan, H., Theodorescu, E.: Asymptotic behavior of the length of local cohomology. Can. J. Math. **57**, 1178–1192 (2005)
6. Dao, H., Montaño, J.: Length of local cohomology of powers of ideals. Trans. Am. Math. Soc. **371**, 3483–3503 (2019)
7. Dao, H., Montaño, J.: On asymptotic vanishing behavior of local cohomology. Math. Z. **295**, 73–86 (2020)
8. Illusie, L.: Complexe Cotangent et Déformations. I. Lecture Notes in Mathematics, vol. 239. Springer, Berlin (1971)
9. Lauritzen, N., Rao, A.P.: Elementary counterexamples to Kodaira vanishing in prime characteristic. Proc. Indian Acad. Sci. Math. Sci. **107**, 21–25 (1997)
10. Lurie, J.: Spectral algebraic geometrie. https://www.math.ias.edu/~lurie/
11. Quillen, D.: Homology of Commutative Rings. Mimeographed Notes. MIT Press, New York (1968)
12. Raynaud, M.: Contre-exemple au "vanishing theorem" en caractéristique $p > 0$. C. P. Ramanujam—A Tribute, Tata Institute of Fundamental Research. Studies in Mathematics, vol. 8, pp. 273–278. Springer, Berlin (1978)

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