

D) Arthur's conjectures for unitary representations

JOINT WORK WITH

- Jeff Adams

- Dan Barbasch

References ABV - "The Langlands classification and irr chars for real reductive groups" Birkhäuser 1992

V - "The local Langlands conjecture" in Repn theory of groups + algebras, AMS 199

Ostensible goal: formulate clearly and completely Arthur's conjectures (describing unitary reps which can contribute to the residual spectrum of L^2 -automorphic forms)

Preliminary goals: - recall Langlands' phibs.
- recall what philosophy says about local reps.
- interweave with "strong ratl forms"

What to remember from page 1: spelling of "ostens.

2) A bit about Galois groups and local fields

Recall that a place of \mathbb{Q} (or a global field) is an inclusion of \mathbb{Q} in a locally compact field with dense image:

$$\mathbb{Q} \hookrightarrow \mathbb{Q}_v$$

loc. cpt.

v will generally be a place

The possibilities are $\mathbb{Q}_\infty = \mathbb{R}$ and \mathbb{Q}_p for any prime number p .

Write $\bar{\mathbb{Q}}$ for an algebraic closure,

$$\Gamma = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \quad (\text{pro-finite group})$$

Fix a place v , and pick an alg closure $\bar{\mathbb{Q}}_v$ for \mathbb{Q}_v , put

$$\Gamma_v = \text{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v)$$

= inverse limit of Galois groups of finite Galois extensions

CHOOSE an inclusion.

$$\begin{array}{ccc} \bar{\mathbb{Q}} & \hookrightarrow & \bar{\mathbb{Q}}_v \\ | & & | \\ \mathbb{Q} & \hookrightarrow & \mathbb{Q}_v \end{array}$$

(unique up to action of Γ on $\bar{\mathbb{Q}}$)

GET AN INCLUSION

$$\Gamma_v \hookrightarrow \Gamma$$

(unique up to conj. by Γ)
IMAGE IS CLOSED.

(3) What do the local subgroups $\Gamma_v \hookrightarrow \Gamma$ tell you about Γ ?

Answer - more or less everything ...

The subgroups Γ_v meet a dense set of conjugacy classes in Γ ; so an irr rep φ of Γ is determined by knowing all its restrictions $\varphi_v = \varphi|_{\Gamma_v}$ } the good news

Given a collection of reps φ_v of Γ_v , it isn't easy to say when they're the restrictions of a common φ } the bad news

Model problem: classify reductive groups over \mathbb{Q} .

- ① Fix a group G over \mathbb{Q} (= root datum!)
- ② Fix an inner class of \mathbb{Q} -forms = homomorphism $\delta: \Gamma \rightarrow \text{Out}(G)$
- ③ Form $G^\Gamma = G \rtimes \Gamma$

works for any field char

Prop class

Rational forms of G in inner class $\delta \longleftrightarrow$ homomorphisms

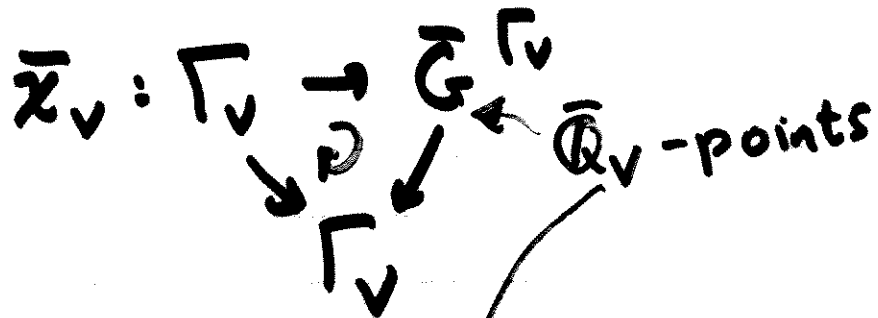
$$\bar{\delta}: \Gamma \rightarrow \bar{G}^\Gamma$$

$\downarrow \cong$ $\downarrow \cong$
 Γ Γ

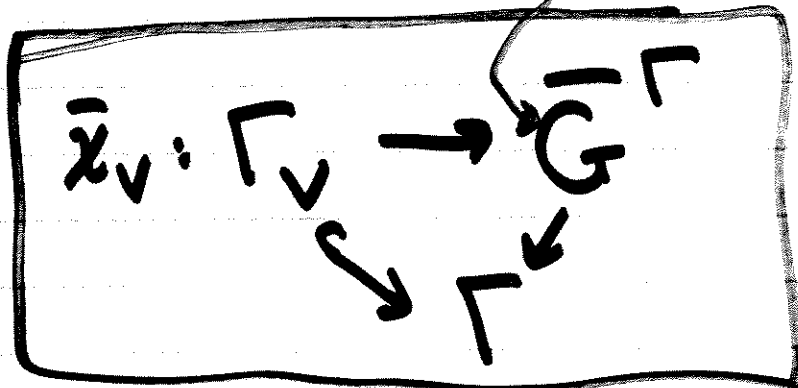
$$\bar{G} = \text{Ad}(G) = G/Z(G)$$

4) Locally... $\gamma: \Gamma \rightarrow \text{Out}(G)$ restricts to $\gamma_v: \Gamma_v \rightarrow \text{Out}(G)$, giving G^{Γ_v}

Prop \mathbb{Q}_v -forms of G in inner class $\gamma_v \leftrightarrow$ homs.



= EXACTLY THE SAME THING AS:



up to $\bar{G}(\bar{\mathbb{Q}}_v)$ conjugacy, we can force $\bar{\chi}_v$ to take values in $\bar{G}(\bar{\mathbb{Q}})$.

To classify reductive gps over \mathbb{Q} , left with two problems:

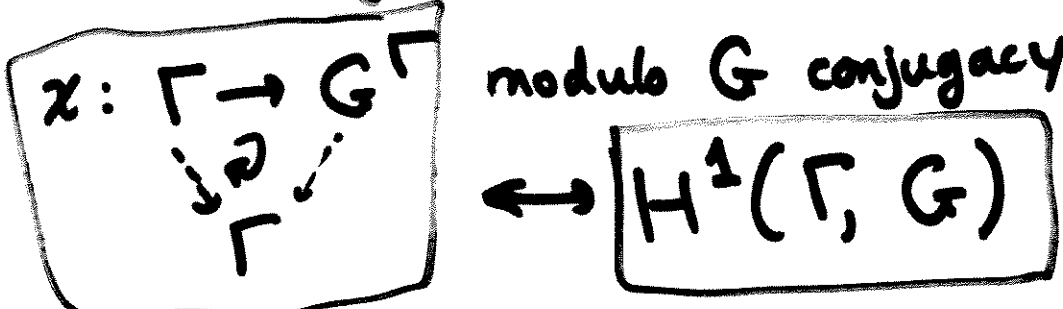
local: for each place v , classify $\bar{\chi}_v$ (the \mathbb{Q}_v -forms)

Collection of one $\bar{\chi}_v$ for each v determines at most one $\bar{\chi}$ [modulo conjugacy].

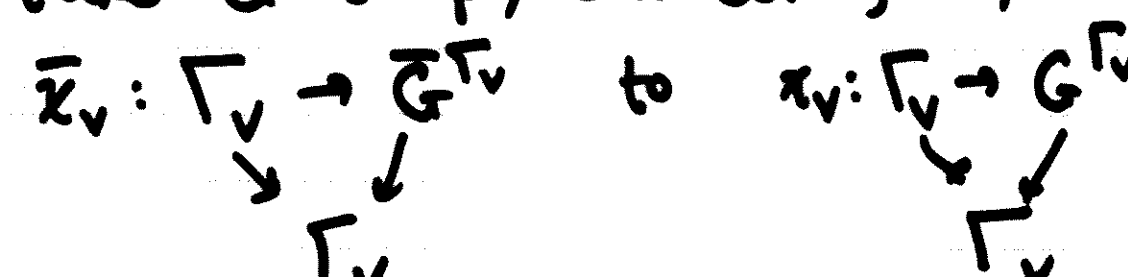
global: which $\{\bar{\chi}_v\}$ can be assembled to $\bar{\chi}$?

3) **VIGRESSION**: hint of how to solve local + global problems above

homs. $\chi: \Gamma \rightarrow G^\Gamma$ modulo G conjugacy $\leftrightarrow H^1(\Gamma, G)$



Roughly: take G simply connected, try to lift $\bar{\chi}_v: \Gamma_v \rightarrow \bar{G}^{\Gamma_v}$ to $\chi_v: \Gamma_v \rightarrow G^{\Gamma_v}$



Obstruction to existence of χ_v is

$$\mathcal{F}(\bar{\chi}_v) \in H^2(\Gamma_v, Z(G))$$

Classes $\mathcal{F}(\bar{\chi}_v)$ classify \mathbb{Q}_v -forms $\bar{\chi}_v$ in p -adic case. (Not obvious.)

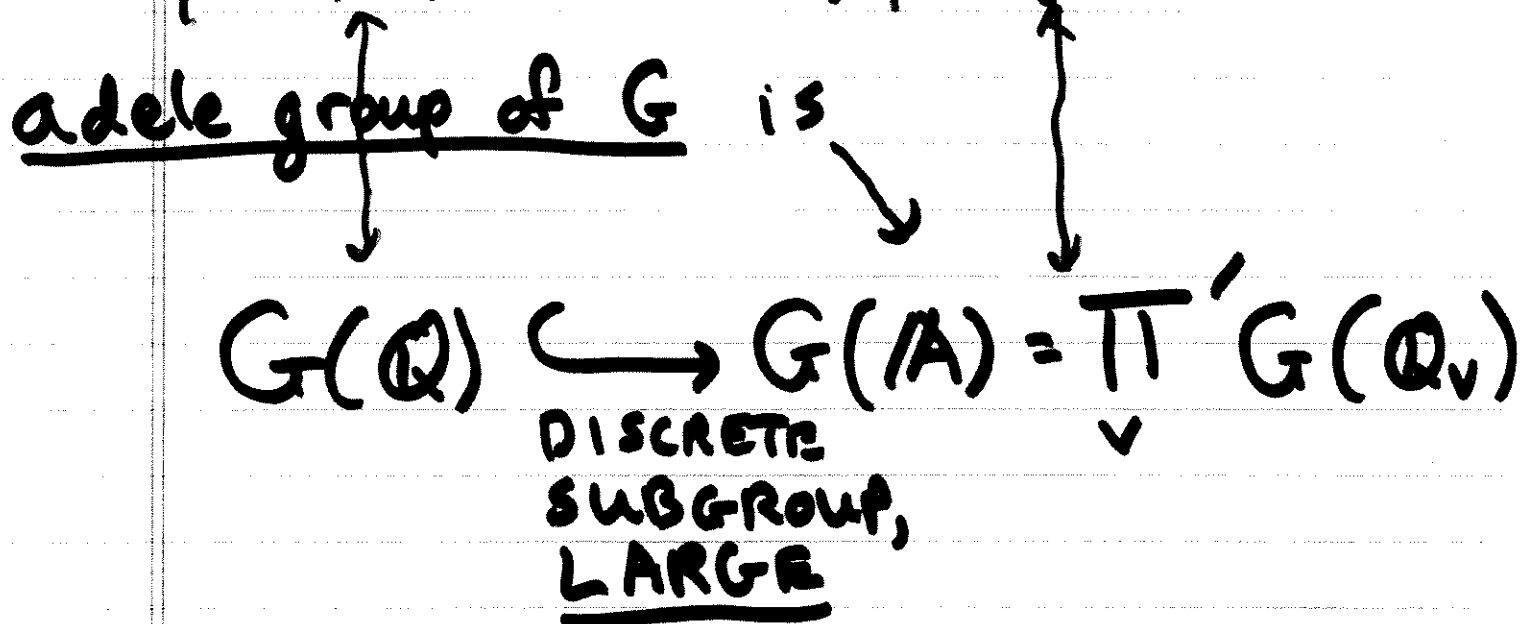
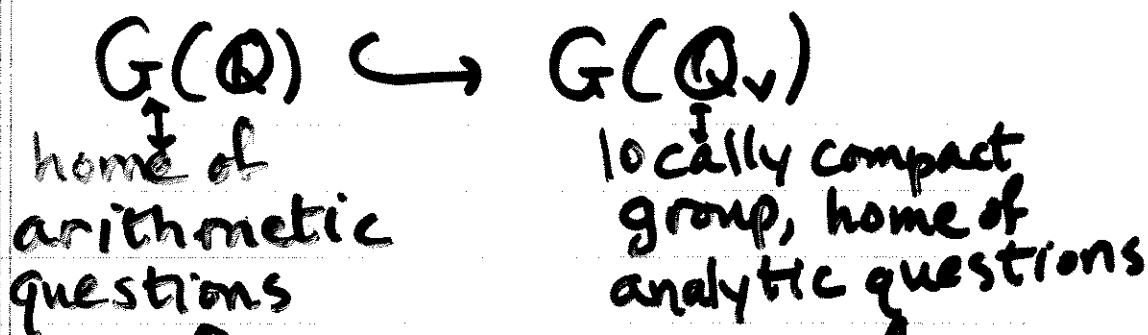
Various $\bar{\chi}_v$ fit together in some $\bar{\chi}$ iff obstructions $\mathcal{F}(\bar{\chi}_v)$ fit together in a global cohom class

$$\mathcal{F}(\bar{\chi}) \in H^2(\Gamma, Z(G))$$

MORAL: correct solutions to local problems tell you how to piece them together to solve global problems.

we return you now to our regularly scheduled program...

G reductive over \mathbb{Q} \forall any place of \mathbb{Q}



$$X = G(\mathbb{Q}) \backslash G(\mathbb{A}) \quad \curvearrowright G(\mathbb{A}) \text{ acts by right mult}$$

AUTOMORPHIC REPRESENTATION

is an irr rep π of $G(\mathbb{A})$ appearing in functions on X

Any nice π of $G(\mathbb{A})$ is

$$\pi = \bigotimes_v \pi_v \quad \pi_v \text{ irr of } G(\mathbb{Q}_v)$$

7) Means: we can look at functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ almost as if they factored:

$$f(g_0, g_2, g_3, g_5 \dots) \\ = f_0(g_0) f_2(g_2) f_3(g_3) f_5(g_5) \dots$$

(Although they don't factor in this way.)

GOALS (LANGLANDS)

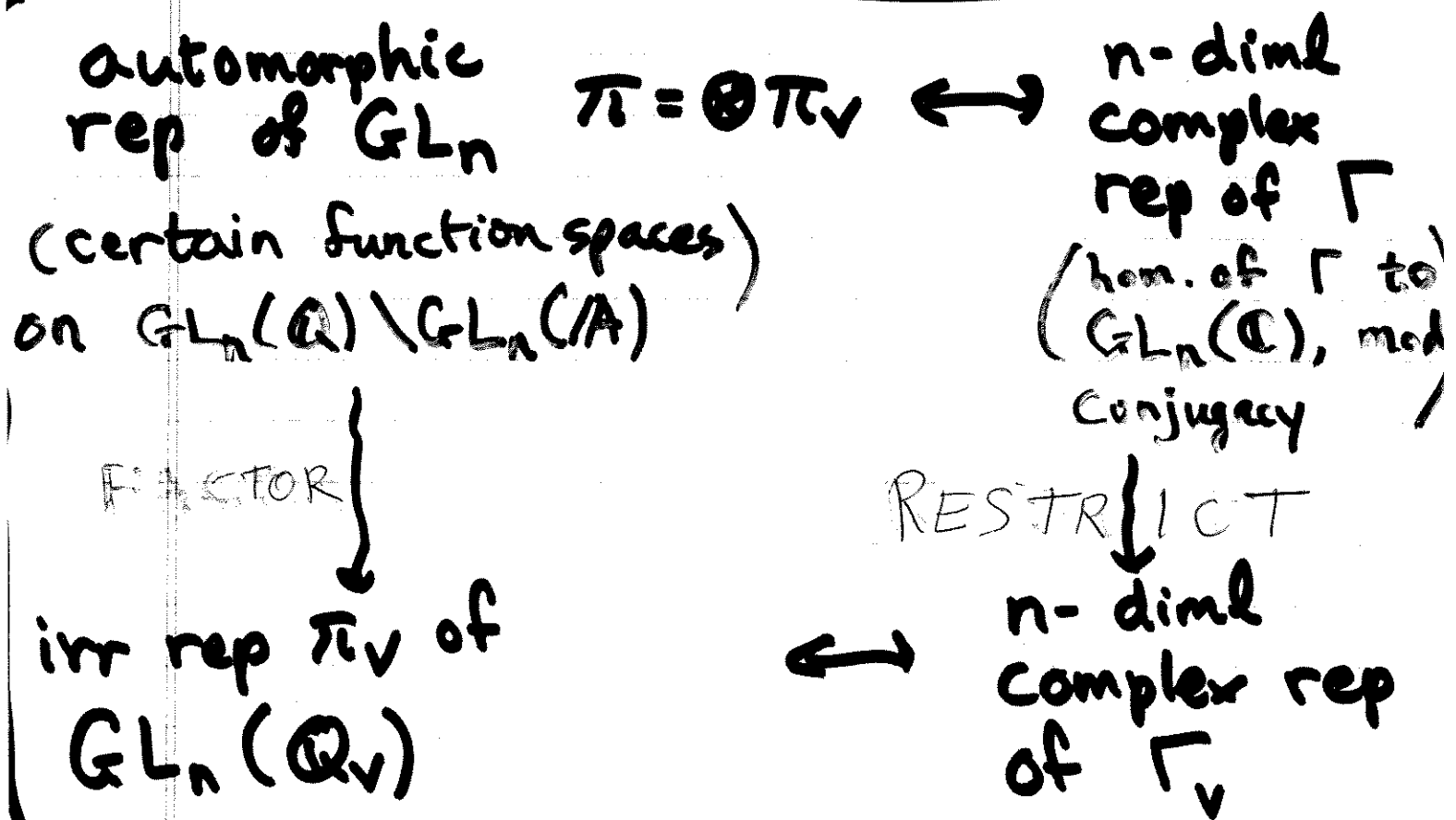
- describe automorphic reps π of G in some way involving $\Gamma = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
- describe all reps π_v of local groups $G(\mathbb{Q}_v)$ using $\Gamma_v = \text{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v)$

SUBJECT TO

- if π automorphic corresponds to a " Γ -thing" φ , then $\pi = \otimes \pi_v$, where π_v corresponds to " Γ_v -thing" $\varphi_v = \varphi$ "restricted to" $\Gamma_v \hookrightarrow \Gamma$

⑧ When you're a hammer, the whole world looks like a nail...

One possible " Γ -thing" that can be restricted to subgroups is a representation. Here's the most naive version of the Langlands conjecture.



Arrows \leftarrow are actually supposed to exist (known for \mathbb{Q}_v , not for \mathbb{Q}). But they're very far from ONTO

1) Want (locally) - ~~some~~

1) Extension of naive idea from GL_n to other groups

$$GL_2 \rightsquigarrow G_2$$

$G / \mathbb{Q} \rightsquigarrow$ root datum

$\gamma: \Gamma \rightarrow \text{Out}(G)$

$$(X^*, R, X_*, R^\vee)$$

\uparrow roots \uparrow coroots

DUAL GROUP: ${}^\vee G =$ complex gp,
root datum (X_*, R^\vee, X^*, R)

${}^\vee \gamma: \Gamma \rightarrow \text{Out}({}^\vee G)$

L-group of G : ${}^\vee G^\Gamma \stackrel{\text{def}}{=} {}^\vee G \rtimes \Gamma$

G version of GL_n naive: replace "n-dim reps of Γ " by SPLITTINGS

$$\varphi: \Gamma \rightarrow \begin{matrix} {}^\vee G^\Gamma \\ \downarrow \cong \\ \Gamma \end{matrix} \quad / \quad G \text{ conjugation}$$

10) CORRECTIONS/emendations

① Proof that an irr rep of $G(A)$ is of the form $\bigotimes_{\nu} \pi_{\nu}$ is ~~due to~~ ^{existed} by D. FLATH (Corvallis volumes)

② My group G^{Γ} is a little different from Jeff Adams': Γ action on G related to rational structure, so not an algebraic action; Jeff's related to Cartan involution. For Jeff, passage $\gamma \rightarrow \nu\gamma$ is a little subtle, involving $-w_0$; for me it's just inverse transpose

Jeff's $\Gamma(\mathbb{C}/\mathbb{R}) \rightarrow \text{Out}(G)$

my $\Gamma(\mathbb{C}/\mathbb{R}) \rightarrow \text{Out}(G)$

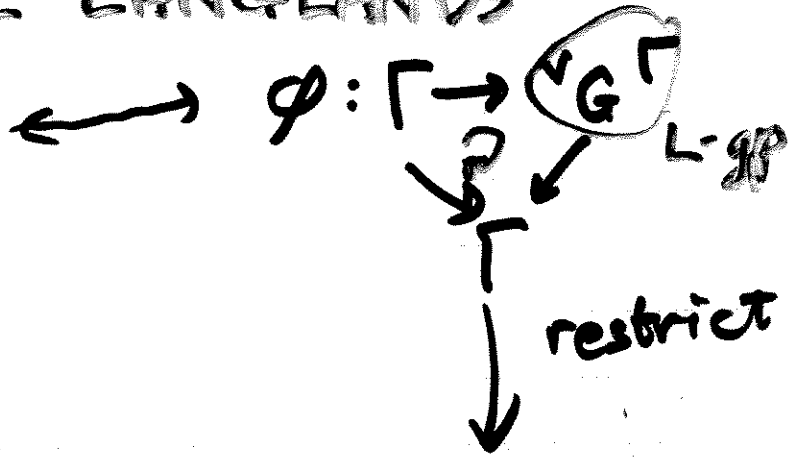
differ by $-w_0$.

Our $\nu\gamma$ are the same.

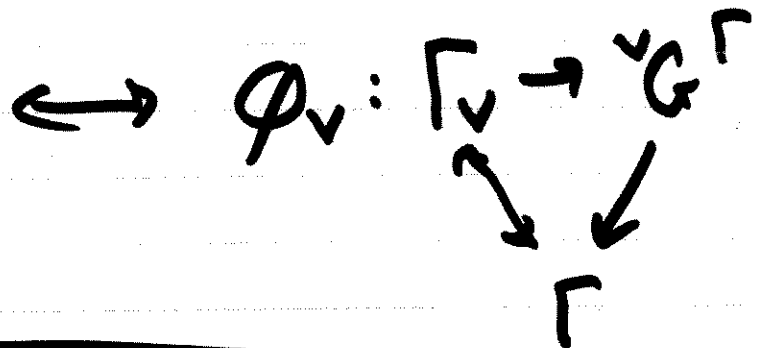
II

UNSUCCESSFUL LANGLANDS
 aut rep π
 of G

factor
 $\pi = \otimes \pi_v$

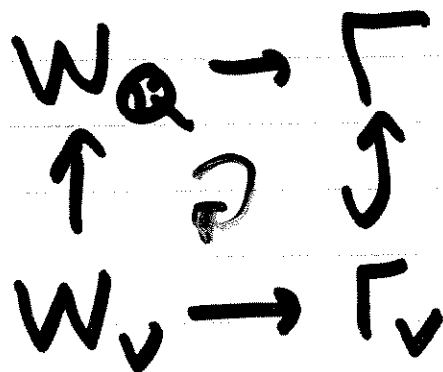


irr rep π_v of
 $G(\mathbb{Q}_v)$



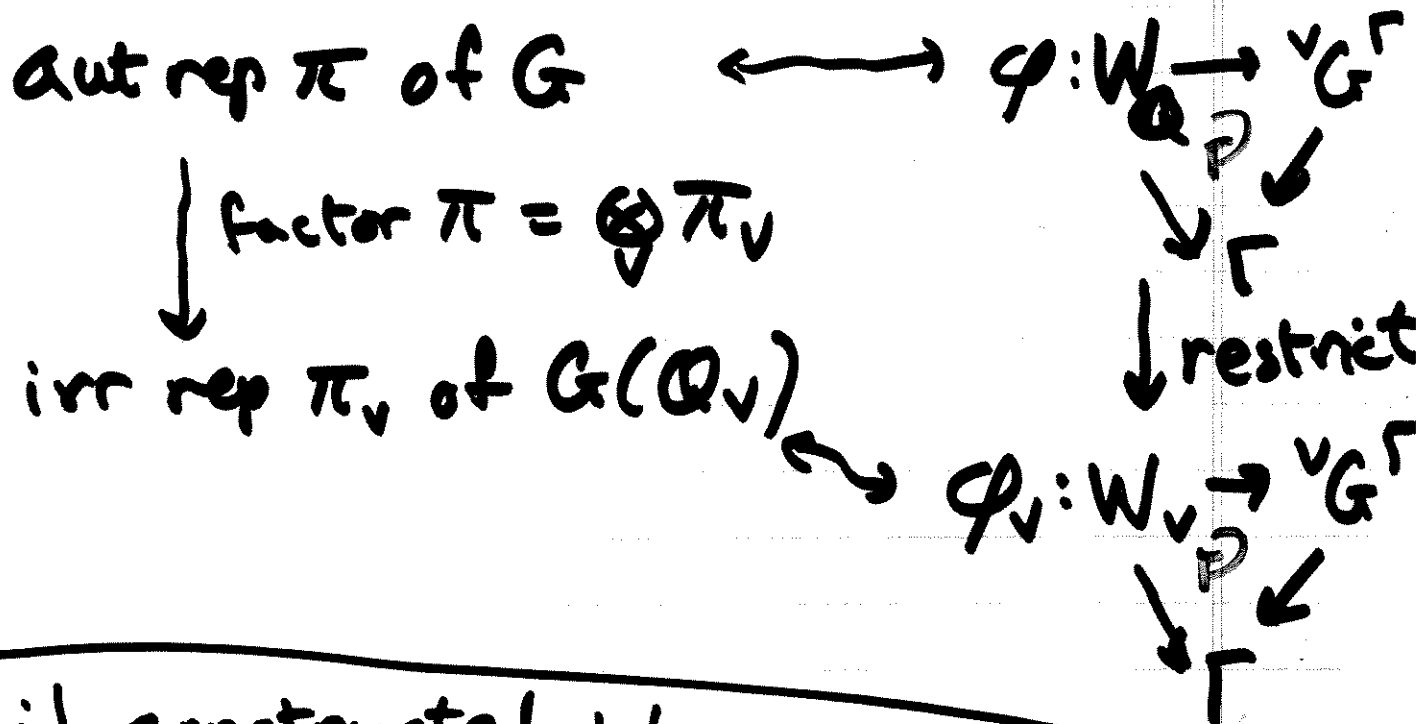
Idea (going back to class field theory
 case $G = GL_1$): FATTEN Γ
 to a "Weil group" $W_{\mathbb{Q}}, W_v$. Want...

- $W_{\mathbb{Q}}, W_v$ locally compact groups
- have natural continuous maps

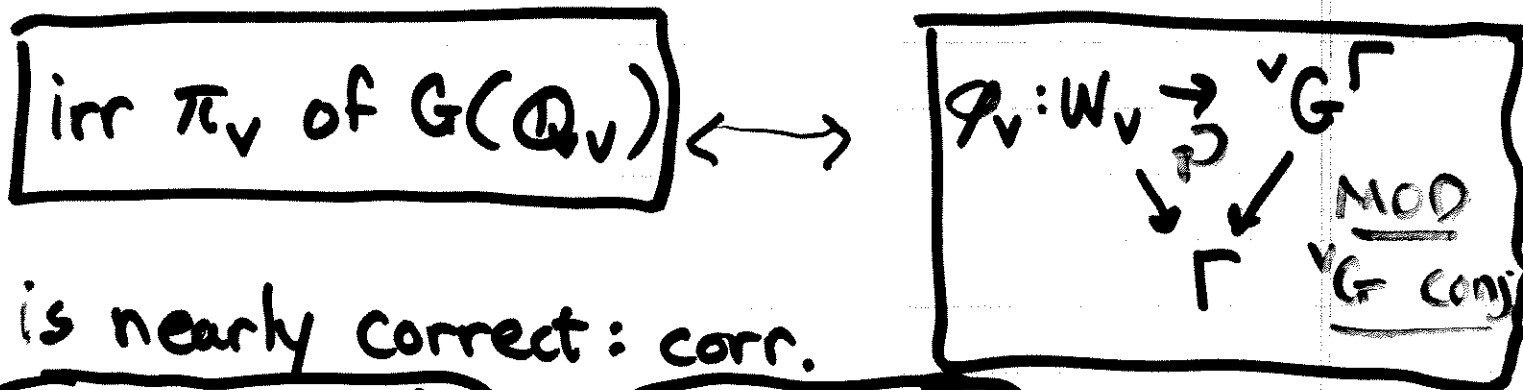


13)

IMPROVED LANGLANDS conj.



Weil constructed $W_{\mathbb{Q}}, W_v$ so that picture above works perfectly for $G=GL_1$.
 Using his W 's, the local conjecture



is nearly correct: corr.

right → left is one → finite

STILL WANT TO UNDERSTAND FIBERS!

The global map right → left is conj. to exist, but NOT surjective

3) HENCEFORTH IGNORE the global case, except to keep in mind that whatever we learn about the local case has to be "restriction" of something about global case.

p-adic case: def of W_p

$$\begin{array}{l} \mathbb{Q}_p \subset \overline{\mathbb{Q}_p} \\ \cup \\ \mathbb{Z}_p = \mathcal{O}_p \subset \overline{\mathbb{Z}_p} \\ \cup \\ p\mathbb{Z}_p = \mathfrak{p} \subset \overline{\mathfrak{p}} \end{array} \quad \begin{array}{l} \Gamma_p = \text{Galois group} \\ \\ = \text{integral closure of} \\ \mathbb{Z}_p \text{ in } \overline{\mathbb{Q}_p} \\ \text{maximal ideal} \end{array}$$

$$\mathbb{Z}_p/p\mathbb{Z}_p = \mathcal{O}_p/\mathfrak{p} = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$$

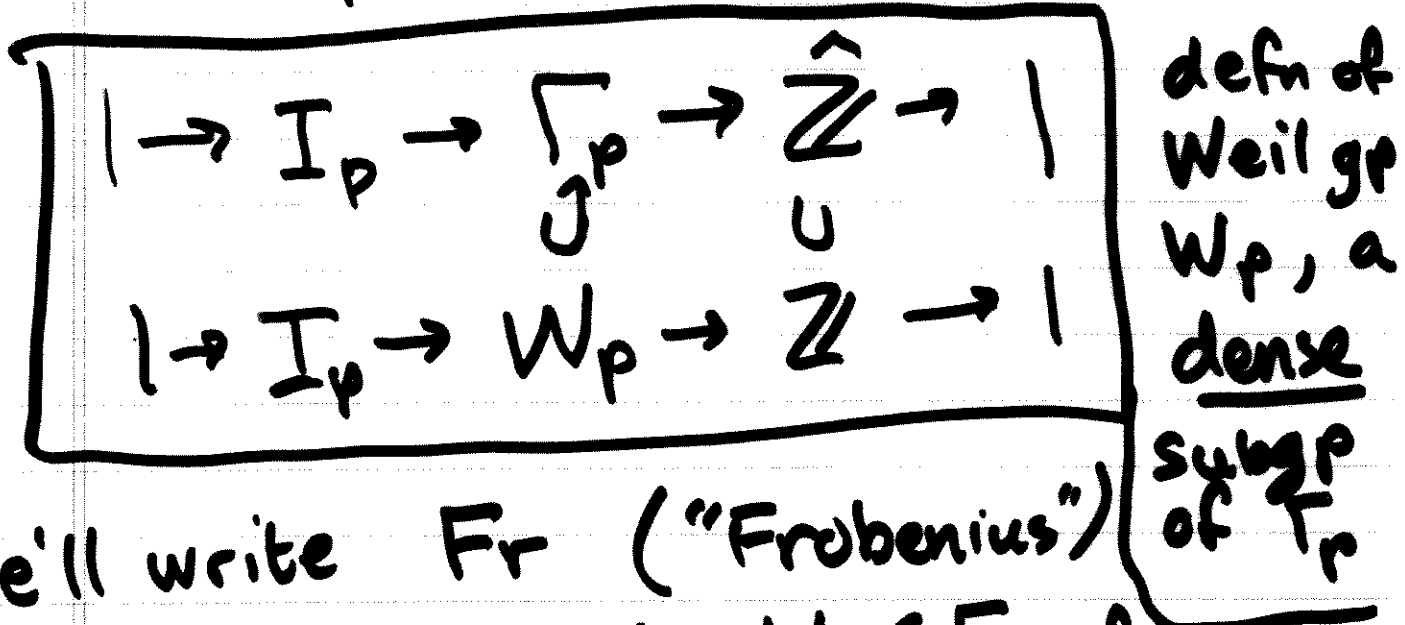
$$\overline{\mathcal{O}_p/\mathfrak{p}} = \overline{\mathbb{F}_p} \text{ an alg closure of } \mathbb{F}_p$$

Γ_p preserves $\overline{\mathcal{O}_p}, \overline{\mathfrak{p}}$, so get

$$\Gamma_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$$

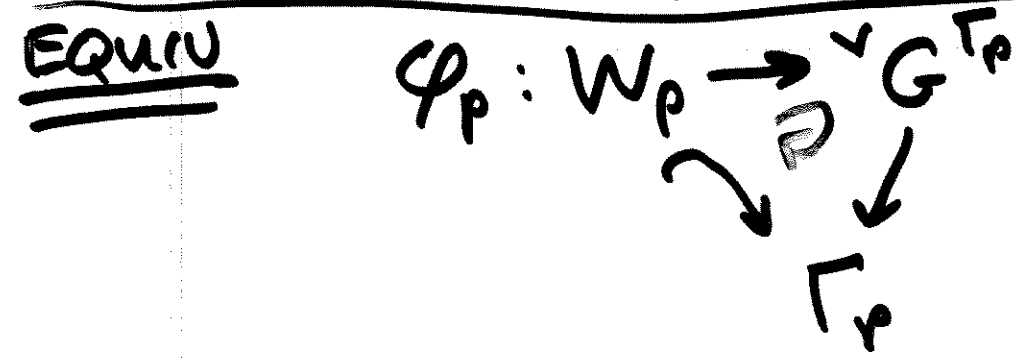
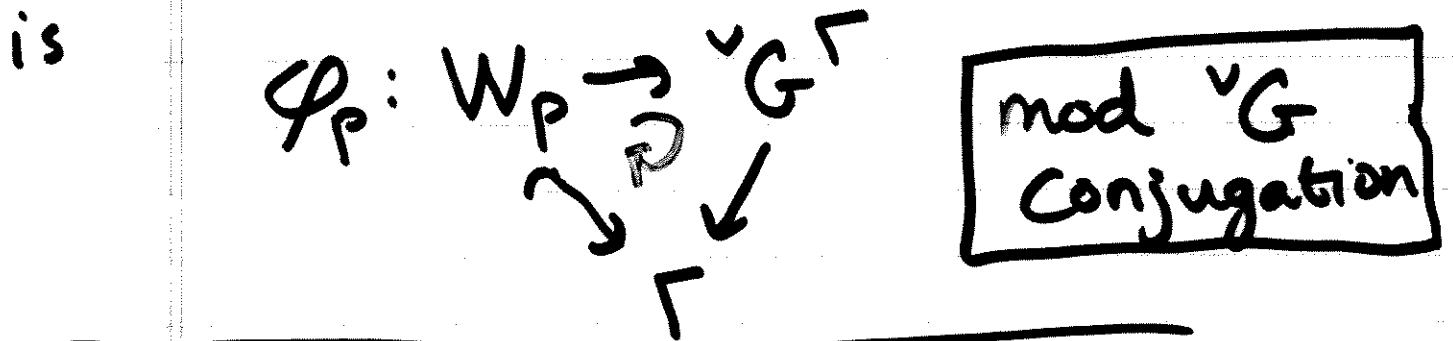
"ZZ"

14) Define $I_p = \text{INERTIA GROUP}$
 $= \ker(\Gamma_p \rightarrow \hat{\mathbb{Z}})$
 $= \{ \sigma \in \Gamma_p \mid \sigma x \equiv x \pmod{\bar{p}}, \text{ all } x \in \bar{\mathcal{O}}_p \}$



We'll write Fr ("Frobenius") for any preimage in $W_p \subset \Gamma_p$ of the can. generator $1 \in \mathbb{Z} \subset \hat{\mathbb{Z}} = \text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$

LANGLANDS PARAMETER for $G(\mathbb{Q}_p)$



5) φ_p comes from (= "extends continuously to")

$$\varphi_p^{\text{red}}: \Gamma_p \rightarrow \bigvee_G \Gamma_p$$

\downarrow \downarrow
 Γ_p Γ_p

if and only if

requires
a little
fuss since
 φ_p doesn't go
directly to \bigvee_G

$\varphi_p(\text{Fr})$ "has finite order"
(in \bigvee_G)

generic (large p) case:

DEF φ_p is unramified if (up to const.)

$$\varphi_p(i) = (1 \times i), \quad i \in I_p$$

$$\varphi_p(\text{Fr}) = (g \times \text{Fr})$$

g can be any elt of I_p -invs in \bigvee_G

\uparrow reductive gp

For large p , I_p acts triv. on

\bigvee_G , so unramified Langlands params \leftrightarrow \bigvee_G -conj. classes in \bigvee_G

9) JORDAN DECOMP

Deligne idea: better to separate unipotent conjugacy class in a more subtle way than Jordan decomp.

G red alg / \mathbb{C}

JORDAN: $g = s \cdot u$ s semisimple
 u unipotent
 s, u commute

There's a canonical s_p semisimple s.t.

$$s_p s = s s_p$$

$$s_p u s_p^{-1} = u^p, \quad \underbrace{(s s_p)}_{s'} u (s s_p)^{-1} = u^p$$

PROP Conj classes in $G \leftrightarrow$

a) pairs (s, u) s semisimple
 u unipotent $s u s^{-1} = u$ / G conj

b) pairs (s', u) s' semisimple
 u unipotent $s' u s'^{-1} = u^p$ / G conj

17)

WEIL-DELIGNE GROUP

Def (p prime) The Weil-Deligne group W'_p is the semidirect product

$$W'_p = \mathbb{C} \rtimes W_p$$

\uparrow
 normal

Here I_p acts trivially on \mathbb{C} , and any Frobenius elt Fr acts on \mathbb{C} by mult. by p . Have $W'_p \rightarrow \Gamma_p$ triv on \mathbb{C}

A Deligne-Langlands parameter is

$\varphi': W'_p \rightarrow {}^v G^\Gamma$
SUBJECT TO

\downarrow
 \downarrow

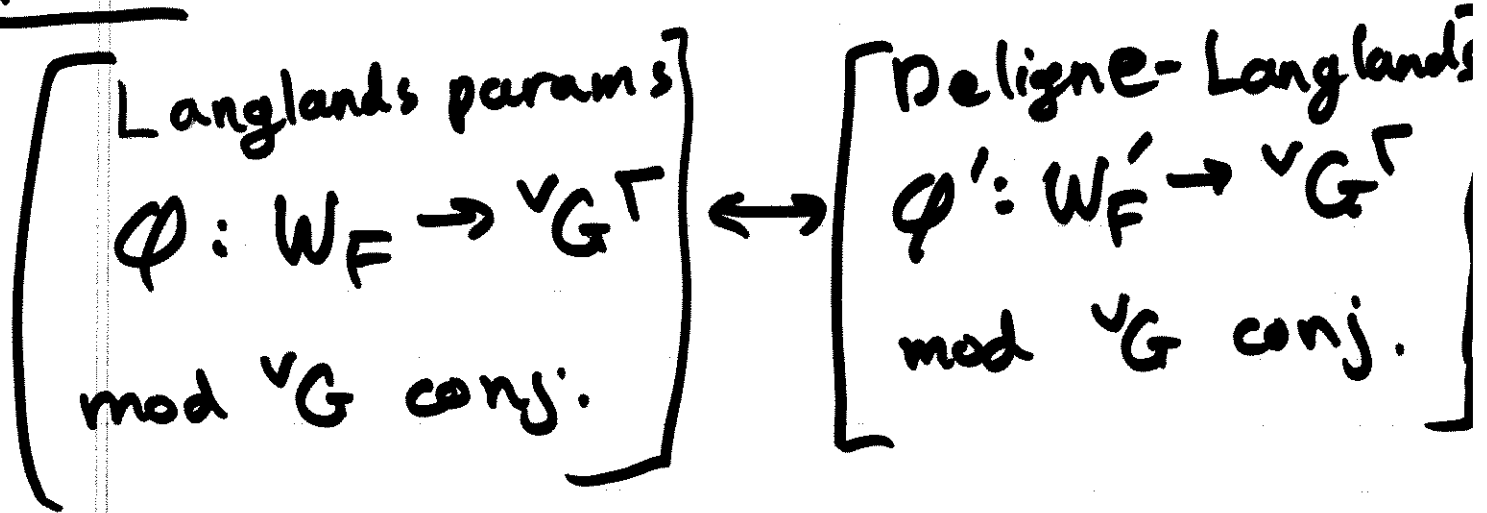
Γ

cont. hom.

a) $\varphi'|_{W_p}$ consists of semisimple elts

b) $\varphi'|_{\mathbb{C}}$ consists of unipotent elts.

18) PROP There's a 1-1 corr

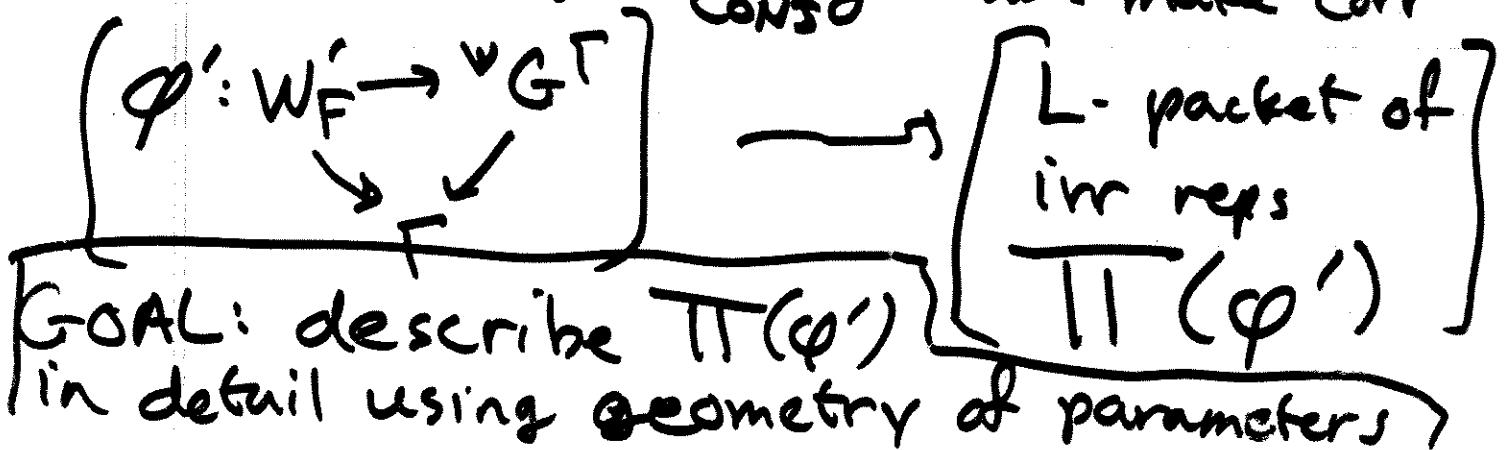


$$\varphi(\text{Fr}) = (ss)u \iff \varphi' \left(\underset{\uparrow}{1}, \underset{\uparrow}{z} \right) = u^z$$

Improvement over Jordan:

$\varphi' |_{W_F} = \varphi$ [semisimple part of φ]
 • [canonical ss elt related to $\varphi(\text{Fr})$]
 $u = \text{unipotent part of } \varphi(\text{Fr})$
 carries more rep-theor. inform than Jordan decomp.
 Think of φ' as "infl char" of corr. reps

p-adic Deligne-Langlands: make corr



Reps of $G(\mathbb{Q}_p)$ at "Fixed in Ad char"
 means: all L-packets whose ~~params~~
 params $\varphi'_p: W'_p \rightarrow {}^v G^\Gamma$ have FIXED
 restr. to W_F \uparrow (up to ${}^v G$ conj.)

FIX $\varphi_p: W_p \rightarrow {}^v G^\Gamma$ Langlands
 param
 (SS image)

$X(\varphi_p) =$ all extensions of φ_p to
 $\varphi'_p: W'_F \rightarrow {}^v G^\Gamma$

Collection of ^{all} algebraic group homs

$$\psi: \mathbb{C} \rightarrow {}^v G$$

SUCH THAT

$$\varphi_p(\text{Fr}) \psi(z) \varphi_p(\text{Fr})^{-1} = \psi(pz) = \psi(z)^p$$

$${}^v \mathfrak{g} = \text{Lie}({}^v G)$$

$$X(\varphi_p) = \left\{ N \in {}^v \mathfrak{g} \mid \begin{array}{l} \text{fixed by } \text{Ad}^\psi[\varphi_p(\text{Fr})] \\ \text{- } p\text{-eigenspace of} \\ \text{Ad}(\varphi_p(\text{Fr})) \end{array} \right\}$$

linear subspace

Group $H = \text{Cent}_G(\phi(W_F))$ acts
 [red alg. gp, maybe disconn.]

on (vector space) $X(\phi_p)$, finitely many orbits

PROP $\forall G$ -conj classes of Deligne-Langlands params of infl char ϕ_p
 \longleftrightarrow $\forall H$ orbits on vector space $X(\phi_p)$

Ex $G = GL_n$ $\forall G = GL_n(\mathbb{C})$
 $\forall G^F = GL_n(\mathbb{C}) \times \Gamma$

$$\phi_p(i) = (1 \times i), i \in \mathbb{I}_p$$

$$\phi_p(Fr) = \begin{pmatrix} p & & & \\ & p^{n-1} & & \\ & & \ddots & \\ & & & p \end{pmatrix} \cdot Fr$$

$$X(\phi_p) = \left\{ \begin{pmatrix} 0 & w_1 & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 & \\ & & & & & & w_{n-1} \end{pmatrix} \right\} \cong \mathbb{C}^{n-1}$$

$$\forall H = \text{diag torus} = (\mathbb{C}^*)^n$$

$$z = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \cdot w$$

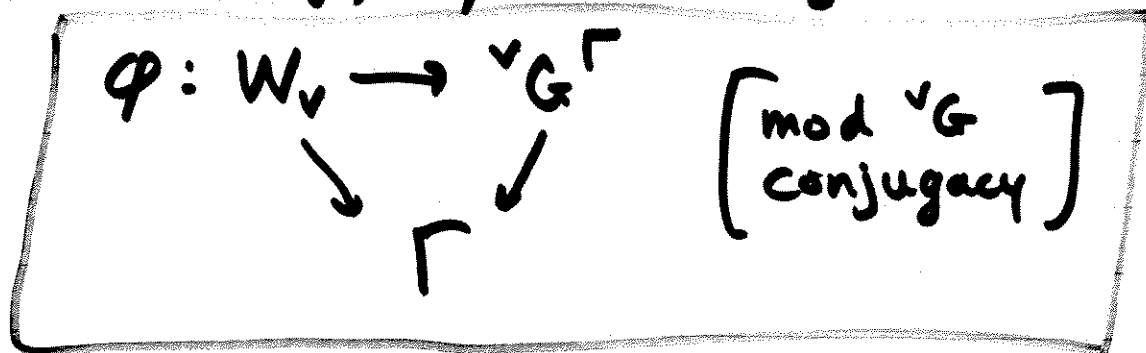
$$(z \cdot w)_j = z_j^{-1} z_{j+1}^{-1} w_j$$

2^{n-1} orbits, given by which coords of w vanish

SO FAR:

(20.10)

- want to parametrize irr reps of local groups $G(\mathbb{Q}_v)$ by something like

$$\varphi: W_v \rightarrow {}^v G^\Gamma \quad \left[\text{mod } {}^v G \text{ conjugacy} \right]$$


$\varphi \rightsquigarrow \Pi(\varphi) = \text{set of irr reps of } G(\mathbb{Q}_v)$

TODAY 1st: describe irr reps in $\Pi(\varphi)$ in more detail using geometry related to φ

TODAY 2nd: describe geometry for Arthur packets / Arthur conjectures

Deligne-Langlands parameters for p -adic G

come in finite families param. by

$$\varphi: W_p \rightarrow {}^v G^\Gamma \quad (\text{semisimple image})$$

$X(\varphi)$ = all extensions of φ to D -L param

$$\varphi': W_p' \rightarrow {}^v G^\Gamma$$

$$(W_p' = \mathbb{C} \rtimes W_p)$$

\cong p -eigenspace of $\text{Ad}(F_\Gamma)$
on ${}^v \mathfrak{g} [\Gamma\text{-invt}]$

Have action of reductive
(maybe disconn.)

$${}^v H = \{g \in {}^v G \mid g \text{ centralizes } \text{im}(\varphi)\}$$

DL param = one orbit of ${}^v H$ on $X(\varphi)$

Tale of two categories...

$\mathcal{M}({}^v G, \varphi)$
"
 ${}^v H$ -equiv D-modules
on $X(\varphi)$

$\mathcal{M}(G, \varphi)$
"
adm reps of some
ratl form $G(\mathbb{Q}_p)$,
all comp factors in
L-packets $\rightarrow X(\varphi)$

Deligne-Langlands-Lusztig
parameter:

${}^v H$ orbit $Z \subseteq X(\varphi)$
+
 ${}^v H$ -equiv irr local system
on Z
(φ, φ')

parametrize
irr. ${}^v H$ -equiv D-modules
on $X(\varphi)$
AND Deligne-Langlands-Lusztig conjet.
irrs in $\mathcal{M}(G, \varphi)$

21 (b)

p-adic local Langlands Conjecture (Deligne-Langlands-Lusztig)

Fix $\varphi: W_p \rightarrow {}^v G^\Gamma$ classical Langlands parameter (semisimple image)

${}^v H =$ centralizer of $\text{im } \varphi$ in ${}^v G$

rehom. vector space for ${}^v H$

$X(\varphi) =$ all extensions of φ to $\varphi': W_p' \rightarrow {}^v G^\Gamma$

Deligne-Langlands param

complete geometric parameter

$=$ irr ${}^v H$ -equiv local system on a ${}^v H$ orbit in $X(\varphi)$

exists splitting $\Gamma \rightarrow G^\Gamma$

conjectural bijection

rep of pure mod. form of $G(\mathbb{Q}_p)$

irr s.d. rep of pure mod. form $G(\mathbb{Q}_p)$

Jeff: $\chi^2 = 1$

$G(\mathbb{Q}_p)$ / cong

21C) EXAMPLE

$$G = \mathrm{PGL}(2, \overline{\mathbb{Q}}_p)$$

(split inner class)

ratl forms:

$$- \mathrm{PGL}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p) / \mathbb{Q}_p^\times$$

$$- D_p^\times / \mathbb{Q}_p^\times, \quad D_p = 4\text{-dim div. alg. over } \mathbb{Q}_p$$

COMPACT

Look at three reps:

trivial^{split} = triv rep of $\mathrm{PGL}_2(\mathbb{Q}_p)$

Steinberg^{split} = non-cuspidal discrete series of $\mathrm{PGL}_2(\mathbb{Q}_p)$

trivial^{division} = trivial of $D_p^\times / \mathbb{Q}_p^\times$

local Langlands:

trivial^{split} \longleftrightarrow $\begin{cases} \text{triv} \\ \mathbb{Q} \end{cases}$

Steinberg^{split} \longleftrightarrow $\begin{cases} \text{triv} \\ \mathbb{Q}^\times \end{cases}$

trivial^{division} \longleftrightarrow $\begin{cases} \text{twisted} \\ \mathbb{Q}^\times \end{cases}$

$$G = \mathrm{SL}(2, \mathbb{C})$$

Trivial, Langlands
Deligne \rightarrow Langlands
forms with \dots
into $\mathrm{SL}(2, \mathbb{C})$

Fix $\varphi: W_p \rightarrow \mathrm{SL}(2, \mathbb{C})$

$$\varphi(I_p) = 1$$

$$\varphi(\mathrm{Fr}) = \begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2} \end{pmatrix}$$

\vee $H =$ diagonal matrices
 $= \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^\times \right\}$
 $\cong \mathbb{C}^\times$

$$X(\varphi) \leftrightarrow \left\{ \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \right\} \subseteq \mathbb{C}$$

$\varphi' \rightarrow d\varphi' \Big|_{\mathbb{C}}$

\vee H action on $X(\varphi)$

$$z \cdot w = z^2 w$$

Three geom params:

- $\begin{cases} \text{triv} \\ \mathbb{Q} \end{cases} = \{ \text{pt } 0, \text{ triv local} \}$
- $\begin{cases} \text{triv} \\ \mathbb{C}^\times \end{cases} = \{ \mathbb{C}^\times, \text{ triv local} \}$
- $\begin{cases} \text{twisted} \\ \mathbb{C}^\times \end{cases} = \{ \mathbb{C}^\times, \text{ tw local system} \}$

(2) Irr reps of $G(\mathbb{Q}_p)$:

irr $L(\varphi, \xi) \leftarrow$ Langlands $\underbrace{\text{std } M(\varphi, \xi)}$
 induced from tempered, strictly pos continuous parameter.

BASIC PROBLEM: decompose each standard repn $M(\varphi, \xi)$ into irrs: (in Gr. group)

$$[M(\varphi, \xi)] = \sum_{\xi'} \underbrace{m(\xi', \xi)}_{\text{non-neg integer}} [L(\varphi, \xi')] \quad \text{Expect: non-zero only if orbit for } \xi' \subset \text{closure of orbit for } \xi$$

irr \mathcal{H} -equiv D-modules

$L(\varphi, \xi) \xrightarrow{\text{Riemann-Hilbert corresp.}} \text{irr } \mathcal{H}\text{-equiv perverse sheaf on } X(\varphi) \xrightarrow{\text{disassemble into constructible sheaves}}$

$$\sum_{\xi'} \underbrace{m(\xi', \xi)}_{\substack{\text{integer,} \\ \text{negative if sheaf appears in odd degree} \\ \text{columns}}} [\text{sheaf } \xi' \text{ ext by zero}] \quad \text{"standard" } \mathbb{D} \text{ module}$$

NON-ZERO only if orbit for $\xi' \subset$ closure of orbit for ξ

23) p-adic KL conjecture (Zelevinsky) (1981)

$$m(\xi', \xi) = (-1)^{\text{codim}(\xi \text{ in } \xi')} \cdot m_g(\xi, \xi')$$

mult. of rep. corr to ξ' in std corr. to ξ

coeff of std sheaf ξ in irr perverse sheaf corr. to ξ'

algorithm by Lusztig 2006

Two Grothendieck groups...

$K_r(\varphi) = \text{Gr. gp } \mathcal{M}(G, \varphi)$ basis $\{M(\varphi, \xi)\}$ <u>standard</u> <u>OR</u> basis $\{L(\varphi, \xi)\}$ <u>irr</u>	$K_g(\varphi) = \text{Gr gp for } \mathcal{M}(G, \varphi)$ basis $\{\xi\}$ <u>local system</u> <u>OR</u> basis $\{P(\xi)\}$ <u>irr perverse</u>
--	--

Free \mathbb{Z} -modules of finite ranks

KL conjecture says

CONJ There's perfect pairing $\langle, \rangle : K_r \times K_g \rightarrow \mathbb{Z}$ satisfying both

$$\langle M(\varphi, \xi), \xi' \rangle = (-1)^{\dim \xi} \delta_{\xi, \xi'}$$

and

$$\langle L(\varphi, \xi), P(\xi') \rangle = (-1)^{\dim \xi} \delta_{\xi, \xi'}$$

(Given Deligne-Langlands-Lusztig conj, can define pairing by 1st formula. Then second formula is equiv to KL conj.)

24)

The big picture (conjecturally, that is):

$$\left[\begin{array}{l} \mathbb{Z}\text{-comb of irr reps} \\ \text{in } L\text{-packets related} \\ \text{to } \varphi \end{array} \right] = \left[\begin{array}{l} \mathbb{Z}\text{-linear functional} \\ \text{on Grothendieck group} \\ \text{of } \mathbb{H}\text{-eqvt } D\text{-modules} \\ \text{on } X(\varphi) \end{array} \right]$$

interesting combs. of reps = interesting linear functionals!

Ex

um of all std. reps in L-packet param by φ'

$\varphi' \in X(\varphi)$ ^{D-L} parameter

linear functional $\mathcal{M} \rightarrow$ [alt. sum of dims of local soln sheaf stalks at φ']

STABLE CHARACTER (and every stable char is a lin comb of these)

Alternate formulation: $\mathcal{O}(\varphi) = \mathbb{V}G\text{-conj. class of } \varphi: W_p \rightarrow \mathbb{V}G^*$

$$X(\mathcal{O}(\varphi)) = \left\{ \begin{array}{l} \text{all extensions to} \\ \psi': W_p' \rightarrow \mathbb{V}G^* \text{ of } \\ \psi \in \mathcal{O}(\varphi) \end{array} \right\} \cong \mathbb{V}G \times_{\mathbb{V}H} X(\varphi)$$

$$\mathbb{V}G\text{-eqvt } D\text{-modules on } X(\mathcal{O}(\varphi)) \cong \mathbb{V}H\text{-eqvt } D\text{-modules on } X(\varphi)$$

25) Real groups version

$$W_{\mathbb{R}} = \langle j, \mathbb{C}^{\times} \rangle \left\langle \begin{array}{l} j^2 = -1 \in \mathbb{C}^{\times} \\ j = j^{-1} = \bar{j} \end{array} \right\rangle \rightarrow \Gamma_{\mathbb{R}}$$

$j \longmapsto \text{cplx conj}$

$$\varphi: W_{\mathbb{R}} \rightarrow {}^v G^{\Gamma}$$

\downarrow
pair (y, λ)

$y \in {}^v G^{\Gamma}$, component corr. to complex conj.
 $\lambda \in {}^v \mathfrak{g}$ semisimple
 $y^2 = e(\lambda)$ $[\lambda, \lambda^{\theta}] = 0$

Problem with imitating p-adic case: everything here is semisimple, ${}^v G$ orbits of such

$$e(\lambda) = \exp(2\pi i \lambda)$$

φ are closed; can't detect relations between reps by closure relations.

SO... (idea of Martin Andler)

REPLACE Langlands parameters by something different.

\equiv infl char for $G(\mathbb{R})$
reps

$\mathcal{O} = {}^v G$ orbit of semisimple elements in ${}^v \mathfrak{g}$

This is a smooth affine alg variety. Want to fiber it by linear ^{affine} subspaces of ${}^v \mathfrak{g}$

$$\lambda \in \mathcal{O} \quad {}^v \mathfrak{n}(\lambda) = \bigoplus_{m=1}^{\infty} \mathfrak{m}\text{-eigenspace of } \text{ad}(\lambda) \text{ on } {}^v \mathfrak{g}$$

CANONICAL FLAT $\Delta(\lambda) \stackrel{\text{def}}{=} \lambda + {}^v \mathfrak{n}(\lambda)$

(2) Real local Langlands conjecture

$\mathcal{O} \subset \mathcal{V}_{\mathcal{G}}$ semisimple $\mathcal{V}_{\mathcal{G}}$ orbit
infl^l char for reps
of $G(\mathbb{R})$

$e(\mathcal{O}) \subset \mathcal{V}_{\mathcal{G}}$ semisimple conjugacy class
↑
exp(2πi·)

FIX $\boxed{z \in e(\mathcal{O})}$ $\mathcal{V}H = \text{Cent}_{\mathcal{V}_{\mathcal{G}}}(z)$
def

possibly disconnected
reductive alg. group

$\boxed{\mathcal{O}_z = \{\lambda \in \mathcal{O} \mid e(\lambda) = z\}}$ = ONE $\mathcal{V}H$ conj.
class in $\mathcal{V}_{\mathcal{G}}$

NOTE: $\lambda \in \mathcal{O}_z \Rightarrow$ flat $\Lambda(\lambda) \subset \mathcal{O}_z$

REAL GEOMETRIC PARAM.

is a pair

(y, Λ)

$y \in \mathcal{V}G^{\Gamma}$, coset corr. to
complex conj.

$y^2 = e(\Lambda)$

$\Lambda \subset \mathcal{O}$ a canonical
flat

$\boxed{X(\mathcal{O}) = \text{geom params}}$

Prop $\mathcal{V}G$ acts alg. on $X(\mathcal{O})$ with finite
number of orbits. The map

$(y, \lambda) \mapsto (y, \Lambda(\lambda))$

is B.I.J.

on $\mathcal{V}G$
orbits

from Langlands params to geom params

Prop The canonical flat $\Lambda(\lambda)$ is contained in the orbit $\mathcal{O} = {}^v G \cdot \lambda$.
 The exponential is constant on $\Lambda(\lambda)$:
 $e(\lambda') = e(\lambda)$, all $\lambda' \in \Lambda(\lambda)$
 $\exp(2\pi i \lambda')$

Ex. ${}^v G = SL(4, \mathbb{C})$

$$\lambda = \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & -4 \end{pmatrix} \quad {}^v n(\lambda) = \begin{pmatrix} 0 & * & * & * \\ & 0 & 0 & * \\ & & 0 & * \\ 0 & & & 0 \end{pmatrix}$$

↑
dim 5

dim $\mathcal{O} = 10$

$$\Lambda(\lambda) = \left\{ \begin{pmatrix} 2 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & -4 \end{pmatrix} \right\}$$

$e(\lambda') = 1$, all $\lambda' \in \Lambda(\lambda)$

28)

real local Langlands conjecture
(theorem of Langlands, Knapp...)

$\mathcal{O} \subset \mathcal{V}_{\mathfrak{g}}$ semisimple \mathcal{V}_G orbit

$X(\mathcal{O}) =$ geometric parameter space

COMPLETE GEOM
PARAMETER

= irr. \mathcal{V}_G -eqvt local system on a \mathcal{V}_G orbit in $X(\mathcal{O})$

complete geom parameter

$L(\mathfrak{g})$
irr. adm rep of pure strong real form of G

Conj. by G

"pure" real forms of G : $\chi^2 = 1$ in Jeff Adams' setting

NOTE: to bring in "non-pure" real forms, replace " \mathcal{V}_G -eqvt local system" by " $\mathcal{V}_{\tilde{G}}$ -eqvt local system," where $\mathcal{V}_{\tilde{G}}$ is a certain covering of \mathcal{V}_G .

SEE ABV book

24)

EXAMPLE

$G = PGL(2, \bar{\mathbb{R}})$
(split inner class)

real forms:

- $PGL_2(\mathbb{R}) = GL_2(\mathbb{R}) / \mathbb{R}^*$

- $\mathbb{H}^* / \mathbb{R}^* \cong SO(3)$
↑
quaternions

$PGL_2(\mathbb{R})$ is disconnected, so has two one-dim reps

trivial^{split}, sign^{split}

emp. ds^{split} = first discrete series of $PGL_2(\mathbb{R})$

emp. trivial division = trivial of $\mathbb{H}^* / \mathbb{R}^* = SO(3)$

local Langlands:

trivial^{split} \leftrightarrow NP^{trivial}

sign^{split} \leftrightarrow SP^{trivial}

ds^{split} \leftrightarrow REST^{trivial}

trivial^{div} \leftrightarrow REST^{twisted}

$\vee G^\Gamma = SL(2, \mathbb{C}) \times \Gamma$

$\mathcal{O} = \text{conj class of } \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$e(\mathcal{O}) = -I \in SL(2)$

POSSIBLE y :
(conj. class of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$), cplx conj.

$\cong \vee G / \vee T$

Flats in \mathcal{O}
 \leftrightarrow Borel subgps of $\vee G$
 $\leftrightarrow \vee G / \vee B$

$X(\mathcal{O}) = [\vee G / \vee T] \times [\vee G / \vee B]$

as $\vee G$ space

orbits: $\mathbb{Z}\mathbb{Z}$

NP } poles
SP } local systems
trivial

REST
two eqvt local systems triv, twisted

$\vee G / \vee B$
as $\vee T$ -space

30)

UNDERSTANDING GEOM PARAMETER SPACE

$\mathcal{O} \subset \mathcal{V}\mathfrak{g}$, $z \in e(\mathcal{O}) \subset \mathcal{V}G$ $\mathcal{V}H = \text{Cent}_{\mathcal{V}G}(z)$
 FIX $y \in \mathcal{V}G^\Gamma$, complex conj coset $y^2 = z$ } Finitely many possible y up to $\mathcal{V}H$ conj.

$\mathcal{V}K = \text{Cent}_{\mathcal{V}G}(y)$, symmetric subgp of $\mathcal{V}H$

$$\mathcal{O}_z = \{ \lambda \in \mathcal{O} \mid e(\lambda) = z \} \subset \mathcal{V}\mathfrak{g}$$

$\mathcal{P} =$ canonical flats in \mathcal{O}_z
 \cong partial flag variety for $\mathcal{V}H$

$$X(\mathcal{O}_z, y) = \{ \text{pairs } (y, \lambda) \in X(\mathcal{O}) \} \cong \mathcal{P}$$

STUDY $\mathcal{V}K$ -equiv \mathcal{D} -modules on $\mathcal{P} \cong X(\mathcal{O}_z, y)$

$$X(\mathcal{O}) = \coprod_{\substack{\text{conj. classes } \mathcal{C} \\ \text{of } y\text{'s}}} X(\mathcal{O}, \mathcal{C})$$

$\mathcal{V}G$ -equiv \mathcal{D} -modules on $X(\mathcal{O}, \text{class of } y)$ \cong $\mathcal{V}K$ -equiv \mathcal{D} -modules on $X(\mathcal{O}_z, y)$

31) Real local Langlands

Recall bijection

complete geom params

\mathfrak{L}
local system
on orbit

\rightarrow

irr rep of pure
real form

$L(\mathfrak{L})$

$P(\mathfrak{L}) =$ corr irr perverse
sheaf / simple \mathcal{D} -module

$M(\mathfrak{L}) =$ standard
rep with
Langlands quotient
 $L(\mathfrak{L})$

$$P(\mathfrak{L}) = \sum_{\mathfrak{L}'} m_g(\mathfrak{L}', \mathfrak{L}) \cdot \mathfrak{L}'$$

express irr perverse/ \mathcal{D} -module
in terms of standard (in
Groth. gp)

$$M(\mathfrak{L}) = \sum_{\mathfrak{L}'} m_r(\mathfrak{L}', \mathfrak{L}) L(\mathfrak{L}')$$

express std
in terms of irrs
(in Groth gp.)

THEOREM (Kazhdan-Lusztig conj for real groups -
see ABV book)

$$m_r(\mathfrak{L}', \mathfrak{L}) = (-1)^{\text{codim}(\mathfrak{L} \text{ in } \mathfrak{L}')} \cdot m_g(\mathfrak{L}, \mathfrak{L}')$$

Cor Perfect pairing (making stds = dual bases
AND irrs = dual bases)

[Groth. gp of reps of
pure real forms, infl
char \mathbb{Q}]

[Groth gp of G -equiv
 \mathcal{D} -modules on
 $X(\mathbb{Q})$]

examples of interesting (31B)
 \mathbb{Z} -linear functors on Gr. group
of eqvt \mathcal{D} -modules on $X(\mathcal{O})$:

take point in $X(\mathcal{O})$, Euler char

↓
Langlands param

of soln sheaves at point.

Corr rep :

sum of std reps in
L-packet \Rightarrow pt

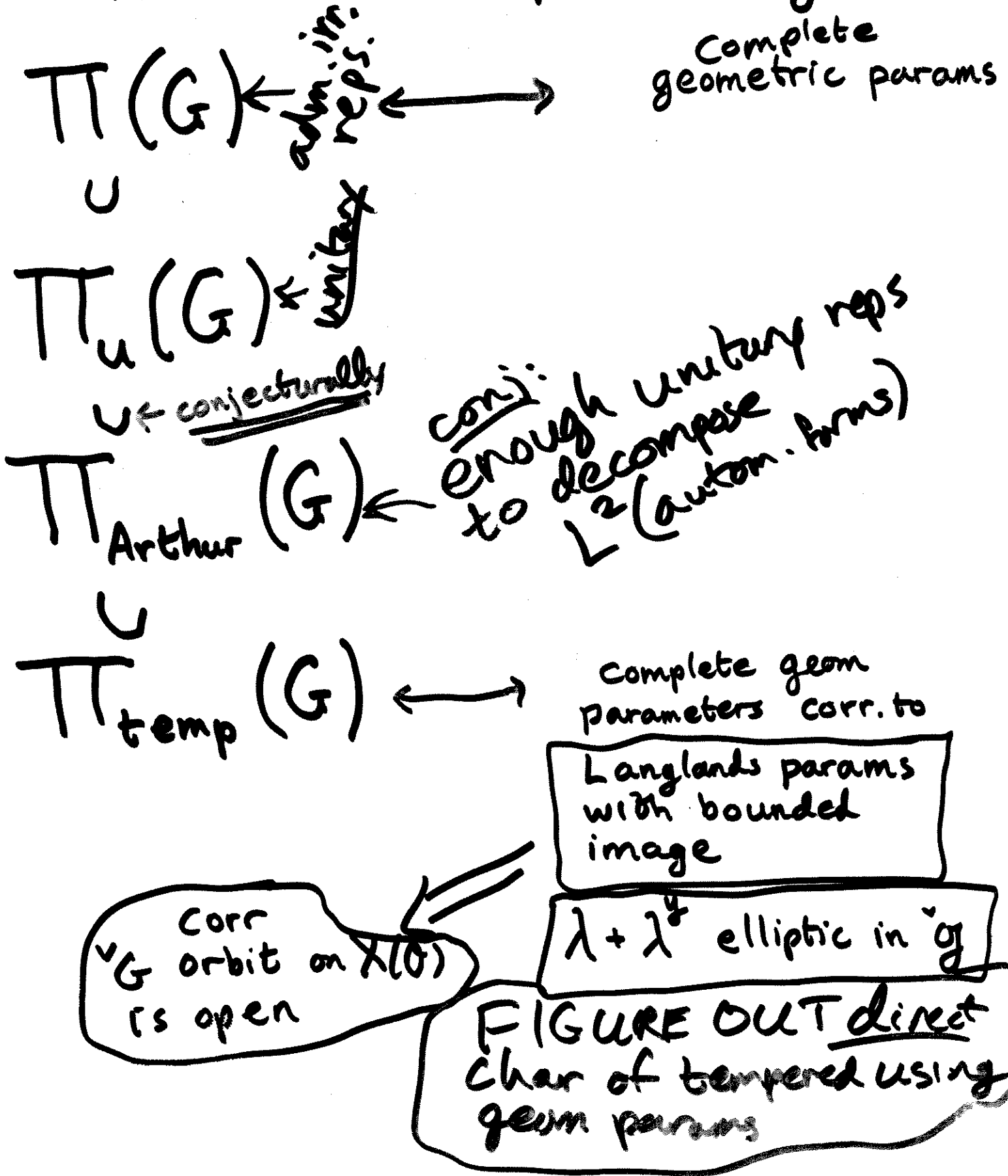
all stable

sums are

in combs of these.

→ STABLE

32) Harmonic analysis for algebraists



Arthur idea (fix $G \rightarrow \text{infl char}$)

$X = \text{geom param space}$

$$= \coprod G\text{-orbits } Z_1, \dots, Z_n$$

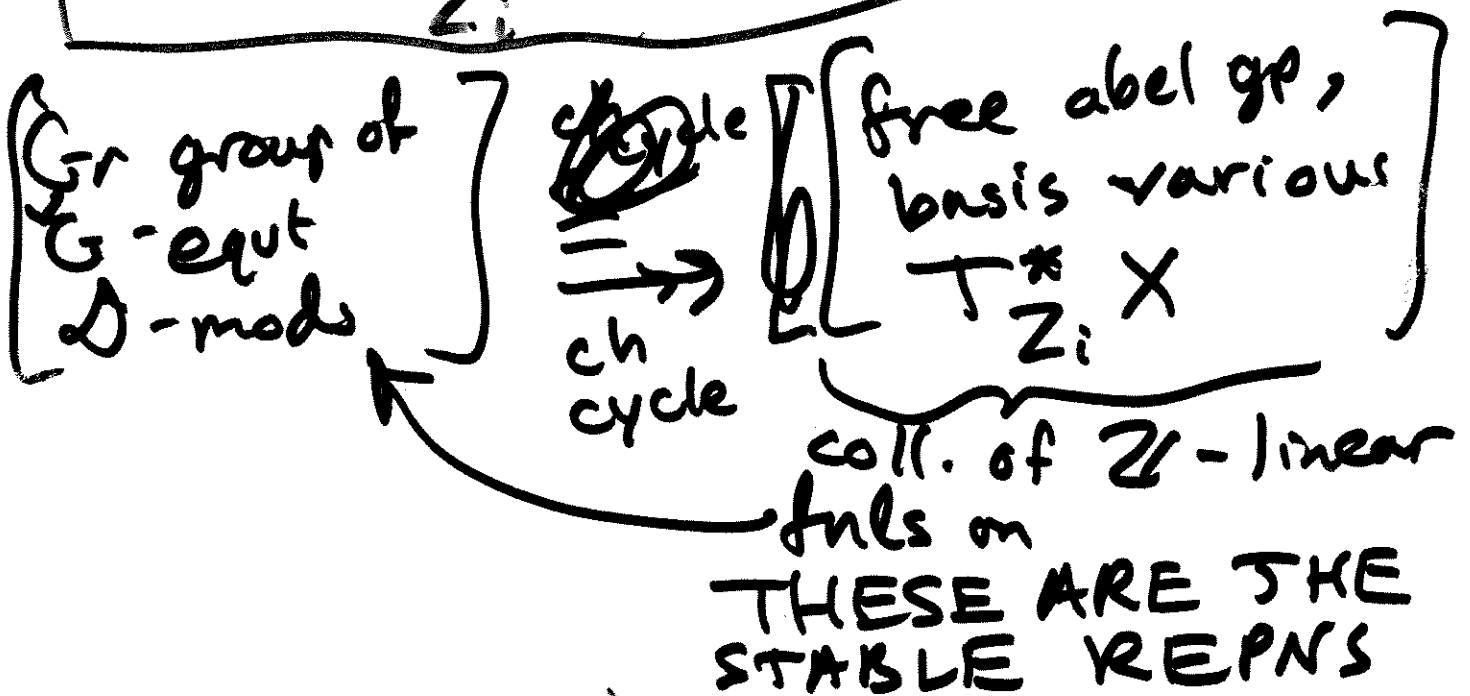
G -equiv \mathcal{D} -modules on X

\leftrightarrow coh. sheaves on "conormal to
" G action"" $\subset T^*X$

$$\sim \coprod_i \underbrace{T^*_{Z_i} X}_{\text{lagr. subvar of } T^*X}$$

Arthur picks one
 Z_i . Look
at lin \mathcal{D} at
"mult of
 $T^*_{Z_i}$ in ch cycle"

$$\begin{aligned} \dim X &= m \\ \dim T^*X &= 2m \\ \dim T^*_{Z_i} X &= m \end{aligned}$$



Obvious consequences...

irr. \mathcal{D} -module \mathcal{M}
 \leftrightarrow orbit Z

\Rightarrow mult of T^*Z in $ch(\mathcal{M})$
is ≥ 0
(= dim of local system)

Consequence:

"Arthur packet"

"

all irr reps whose char cycle contains T^*Z

INCLUDES L -packet for Z ,

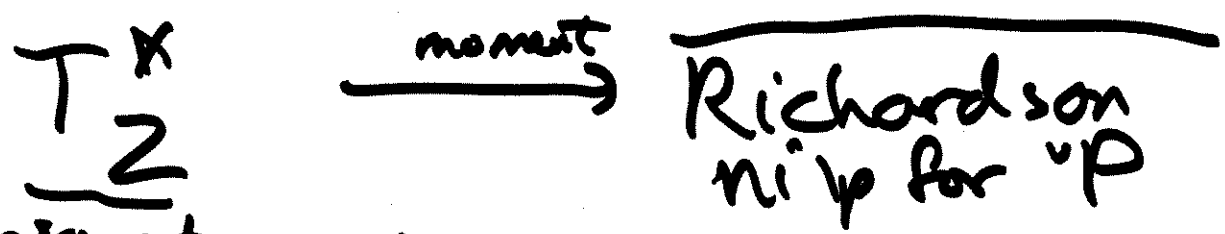
Most interesting case
of Arthur params: UNIPOTENT
(Weil group part trivial on \mathbb{C}^*)

$\checkmark H = \text{cent}_{\checkmark G}$ (image of -1 in $SL(2)$)

$\checkmark K \rightarrow \checkmark K$ action on $\mathcal{P} = \checkmark H / \checkmark \rho$
flag var.

orbit for Arthur:

CLOSED $\checkmark K$ orbit $Z \subset \mathcal{P}$



big cotangent direction since Z small

Arthur packet \Leftrightarrow all \mathcal{A} modules whose corr $(\checkmark \sigma, \checkmark K)$ modules are as big as possible