1) Arthur's conjectures for unitary representations

JOINT WORK WITH
- Jeff Adams
- Dan Barbasch

References ABV - "The Langlands classification and irr chars for real reductive groups" Birkhäuser 1992

V - "The local Langlands conjecture" in Repn theory of groups + algebras, AMS 199

Ostensible goal: formulate clearly and completely Arthur's conjectures (describing unitary reps which can contribute to the residual spectrum of L^2-automorphic forms)

Preliminary goals: - recall Langlands' philos.

- recall what philosophy says about local reps.
- interweave with "strong ratl forms"

What to remember from page 1: spelling of "ostens.

2) A bit about Galois groups and local fields

Recall that a <u>place</u> of $\mathbb{Q}$ (or a global field) is an inclusion of $\mathbb{Q}$ in a locally compact field with dense image:

$$\mathbb{Q} \hookrightarrow \underset{\text{loc. cpt.}}{\mathbb{Q}_v}$$

v will generally be a place

The possibilities are $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{Q}_p$ for any prime number p.

Write $\overline{\mathbb{Q}}$ for an algebraic closure,

$$\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

(pro-finite group = inverse limi of Galois groups of finite Galoi extensions)

Fix a place v, and pick an alg closure $\overline{\mathbb{Q}}_v$ for $\mathbb{Q}_v$, put

$$\Gamma_v = \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$$

<u>CHOOSE</u> an inclusion

$$\begin{tikzcd} \overline{\mathbb{Q}} \arrow[r, hook] \arrow[d, dash] & \overline{\mathbb{Q}}_v \arrow[d, dash] \\ \mathbb{Q} \arrow[r, hook] & \mathbb{Q}_v \end{tikzcd}$$

(unique up to action of Γ on $\overline{\mathbb{Q}}$)

GET AN INCLUSION

$$\Gamma_v \hookrightarrow \Gamma$$

(unique up to conj. by Γ)

IMAGE IS CLOSED.

③ What do the local subgroups $\Gamma_v \hookrightarrow \Gamma$ tell you about Γ?

Answer - more or less everything ...

The subgroups Γ_v meet a dense set of conjugacy classes in Γ; so an irr rep φ of Γ is determined by knowing all its restrictions $\varphi_v = \varphi|_{\Gamma_v}$

the
goo
new

Given a collection of reps φ_v of Γ_v, it isn't easy to say when they're the restrictions of a common φ

the
bad
news

Model problem: classify reductive groups over $\mathbb{Q}$.

① Fix a group G over $\overline{\mathbb{Q}}$ (= root datum!)

② Fix an inner class of $\mathbb{Q}$-forms = homomorphism $\gamma: \Gamma \to \mathrm{Out}(G)$

③ Form $G^\Gamma = \underset{\overset{\uparrow}{\overline{\mathbb{Q}}\text{-points}}}{G} \rtimes \Gamma$

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Prop Rational forms of G in inner class $\gamma \longleftrightarrow$ homomorphisms

$$\begin{tikzcd} \bar{\gamma}: \Gamma \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & \bar{G}^\Gamma \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

$\bar{G} = \mathrm{Ad}(G)$
$= G/Z(G)$

4) Locally... $\gamma : \Gamma \to Out(G)$ restricts to
$\gamma_v : \Gamma_v \to Out(G)$, giving G^{Γ_v}

<u>Prop</u> $\mathbb{Q}_v$-forms of G in inner class $\gamma_v \longleftrightarrow$ homs.

$$\begin{tikzcd} \bar{x}_v : \Gamma_v \arrow[r] \arrow[dr] \arrow[dr, phantom, "\circlearrowright" near end, shift left=3] & \bar{G}^{\Gamma_v} \arrow[d] & \bar{\mathbb{Q}}_v\text{-points} \arrow[l] \\ & \Gamma_v & \end{tikzcd}$$

= EXACTLY THE SAME THING A

$$\boxed{\begin{tikzcd} \bar{x}_v : \Gamma_v \arrow[r] \arrow[dr] & \bar{G}^{\Gamma} \arrow[d] \\ & \Gamma \end{tikzcd}}$$

up to $\bar{G}(\bar{\mathbb{Q}}_v)$ conjugacy, we can force $\bar{x}_v$ to take values in $\bar{G}(\bar{\mathbb{Q}})$.

To classify reductive gps over $\mathbb{Q}$, left with two problems:

$\boxed{\text{\underline{local}: for each place } v\text{, classify } \bar{x}_v \text{ (the } \mathbb{Q}_v\text{-forms)}}$

Collection of one $\bar{x}_v$ for each v determines at most one $\bar{x}$ [modulo conjugacy].

$\boxed{\text{\underline{global}: which } \{\bar{x}_v\} \text{ can be assembled to } \bar{x}\text{?}}$

5) DIGRESSION: hint of how to solve local + global problems above

homs.
$$\boxed{\begin{tikzcd} \Gamma \arrow[rr, "x"] \arrow[dr, dashed] & \arrow[d, phantom, "\circlearrowleft"] & G^{\Gamma} \arrow[dl, dashed] \\ & \Gamma & \end{tikzcd}}$$
modulo G conjugacy $\longleftrightarrow$ $\boxed{H^1(\Gamma, G)}$

Roughly: take G simply connected, try to lift

$$\begin{tikzcd} \Gamma_v \arrow[rr, "\bar{x}_v"] \arrow[dr] & & \bar{G}^{\Gamma_v} \arrow[dl] \\ & \Gamma_v & \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} \Gamma_v \arrow[rr, "x_v"] \arrow[dr] & & G^{\Gamma_v} \arrow[dl] \\ & \Gamma_v & \end{tikzcd}$$

Obstruction to existence of x_v is

$$\xi(\bar{x}_v) \in H^2(\Gamma_v, Z(G))$$

Classes $\xi(\bar{x}_v)$ classify $\mathbb{Q}_v$-forms $\bar{x}_v$ in p-adic case. (Not obvious.)

Various $\bar{x}_v$ fit together in some $\bar{x}$ iff obstructions $\xi(\bar{x}_v)$ fit together in a global cohom class

$$\boxed{\xi(\bar{x}) \in H^2(\Gamma, Z(G))}$$

MORAL: correct solutions to local problems tell you how to piece them together to solve global problems.

We return you now to our regularly scheduled program...

G reductive over $\mathbb{Q}$ $\quad$ v any place of $\mathbb{Q}$

$$G(\mathbb{Q}) \hookrightarrow G(\mathbb{Q}_v)$$

$G(\mathbb{Q})$: home of arithmetic questions

$G(\mathbb{Q}_v)$: locally compact group, home of analytic questions

adele group of G is

$$G(\mathbb{Q}) \hookrightarrow G(\mathbb{A}) = \prod_v{}' G(\mathbb{Q}_v)$$

DISCRETE SUBGROUP, **LARGE**

$$X = G(\mathbb{Q}) \backslash G(\mathbb{A}) \circlearrowleft G(\mathbb{A})$$ acts by right mult

AUTOMORPHIC REPRESENTATION

is an irr rep π of $G(\mathbb{A})$ appearing in functions on X

Any nice π of $G(\mathbb{A})$ is

$$\pi = \bigotimes_v \pi_v \qquad \pi_v \text{ irr of } G(\mathbb{Q}_v)$$

7) Means: we can look at functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ almost as if they factored:

$$f(g_\infty, g_2, g_3, g_5 \dots)$$
$$= f_\infty(g_\infty) f_2(g_2) f_3(g_3) f_5(g_5) \dots$$

(Although they <u>don't</u> factor in this way.)

GOALS (LANGLANDS)

- describe automorphic reps π of G in some way involving $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
- describe <u>all</u> reps π_v of local groups $G(\mathbb{Q}_v)$ using $\Gamma_v = \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$

SUBJECT TO

- if π automorphic corresponds to a "Γ-thing" φ, then $\pi = \otimes \pi_v$, where π_v corresponds to "Γ_v-thing"

$$\varphi_v = \varphi \text{ "restricted to" } \Gamma_v \subset \Gamma$$

⑧ When you're a hammer, the whole world looks like a nail...

One possible "Γ-thing" that can be restricted to subgroups is a <u>representati</u>

Here's the most naive version of the Langlands conjecture.

$$\begin{tikzcd}
\begin{array}{l}\text{automorphic}\\ \text{rep of } GL_n\\ \text{(certain function spaces)}\\ \text{on } GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})\end{array} \ \pi = \otimes \pi_v \arrow[r, leftrightarrow] \arrow[d, "\text{FACTOR}"'] & \begin{array}{l}\text{n-diml}\\ \text{complex}\\ \text{rep of } \Gamma\\ \left(\begin{array}{l}\text{hom. of } \Gamma \text{ to}\\ GL_n(\mathbb{C}), \text{ mod}\\ \text{conjugacy}\end{array}\right)\end{array} \arrow[d, "\text{RESTRICT}"] \\
\begin{array}{l}\text{irr rep } \pi_v \text{ of}\\ GL_n(\mathbb{Q}_v)\end{array} \arrow[r, leftrightarrow] & \begin{array}{l}\text{n-diml}\\ \text{complex rep}\\ \text{of } \Gamma_v\end{array}
\end{tikzcd}$$

Arrows $\leftarrow$ are actually supposed to exist (known for $\mathbb{Q}_v$, not for $\mathbb{Q}$). But they're very far from <u>ONTO</u>

Want (locally) - ~~some~~

1) Extension of naive idea from GL_n to other groups

$$GL_2 \rightsquigarrow G_2$$

$G/\mathbb{Q} \rightsquigarrow$ root datum

$$(X^*, \underset{\substack{\uparrow \\ \text{roots}}}{R}, X_*, \underset{\substack{\uparrow \\ \text{coroots}}}{R^\vee})$$

$$\boxed{\gamma : \Gamma \to \mathrm{Out}(G)}$$

DUAL GROUP: ${}^\vee G$ = <u>complex</u> gp, root datum $(X_*, R^\vee, X^*, R)$

$$\boxed{{}^\vee\gamma : \Gamma \to \mathrm{Out}({}^\vee G)}$$

L-group of G: ${}^\vee G^\Gamma \underset{\text{def}}{=} {}^\vee G \rtimes \Gamma$

G version of GL_n naive: replace "n-diml reps of Γ" by SPLITTINGS

$$\varphi : \begin{tikzcd} \Gamma \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & {}^\vee G^\Gamma \arrow[dl] \\ & \Gamma & \end{tikzcd} \Big/ {}^\vee G \text{ conjugatin}$$

CORRECTIONS/emendations

① Proof that an irr rep of $G(\mathbb{A})$ is of the form $\bigotimes_v \pi_v$ is ~~due~~ ~~to~~ expisited by <u>D. FLATH</u> (Corvallis volumes)

② My group G^Γ is a little different from Jeff Adams': Γ action on G related to rational structure, so <u>not</u> an algebraic action; Jeff's related to Cartan involution. For Jeff, passage $\gamma \to {}^\vee\gamma$ is a little subtle, involving $-w_0$; for me it's just inverse transpose

Jeff's $\Gamma(\mathbb{C}/\mathbb{R}) \to \mathrm{Out}(G)$

my $\Gamma(\mathbb{C}/\mathbb{R}) \to \mathrm{Out}(G)$

differ by $-w_0$:

Our ${}^\vee\gamma$ are the same.

UNSUCCESSFUL LANGLANDS

$$\begin{tikzcd} \text{aut rep } \pi \text{ of } G \arrow[r, leftrightarrow] \arrow[ddd, "\text{factor}", "\pi = \otimes \pi_v"'] & \phi : \Gamma \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & {}^\vee G^\Gamma \ \text{L-gp} \arrow[dl] \\ & & \Gamma \arrow[dd, "\text{restrict}"] & \\ & & & \\ \text{irr rep } \pi_v \text{ of } G(\mathbb{Q}_v) \arrow[r, leftrightarrow] & \phi_v : \Gamma_v \arrow[rr] \arrow[dr, hook] & & {}^\vee G^\Gamma \arrow[dl] \\ & & \Gamma & \end{tikzcd}$$

Idea (going back to class field theory case $G = GL_1$): FATTEN Γ to a "Weil group" $W_{\mathbb{Q}}$, W_v. Want...

- $W_{\mathbb{Q}}$, W_v locally compact groups
- have natural continuous maps

$$\begin{tikzcd} W_{\mathbb{Q}} \arrow[r] & \Gamma \\ W_v \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \Gamma_v \arrow[u, hook] \end{tikzcd}$$

IMPROVED LANGLANDS conj.

$$\begin{tikzcd}
\text{aut rep } \pi \text{ of } G \arrow[r, leftrightarrow] \arrow[dd, "\text{factor } \pi = \bigotimes_v \pi_v"] & W_{\mathbb{Q}} \arrow[r, "\varphi"] \arrow[dr] \arrow[r, phantom, shift right=3, "\circlearrowright"] & {}^{\vee}G^{\Gamma} \arrow[d] \\
 & & \Gamma \arrow[d, "\text{restrict}"] \\
\text{irr rep } \pi_v \text{ of } G(\mathbb{Q}_v) \arrow[r, leftrightarrow] & W_v \arrow[r, "\varphi_v"] \arrow[dr] \arrow[r, phantom, shift right=3, "\circlearrowright"] & {}^{\vee}G^{\Gamma} \arrow[d] \\
 & & \Gamma
\end{tikzcd}$$

Weil constructed $W_{\mathbb{Q}}$, W_v so that picture above works perfectly for $G = GL_1$. Using his W's, the local conjecture

$$\begin{tikzcd}
\boxed{\text{irr } \pi_v \text{ of } G(\mathbb{Q}_v)} \arrow[r, leftrightarrow] & W_v \arrow[r, "\varphi_v"] \arrow[dr] \arrow[r, phantom, shift right=3, "\circlearrowright"] & {}^{\vee}G^{\Gamma} \arrow[d] & \underline{\text{MOD}} \\
 & & \Gamma & \underline{{}^{\vee}G \text{ conj}}
\end{tikzcd}$$

is nearly correct: corr. right → left is one → finite

STILL WANT TO UNDERSTAND FIBERS!

The global map right → left is conj. to exist, but <u>NOT</u> surjective

3) HENCEFORTH IGNORE the global case, except to keep in mind that whatever we learn about the local case has to be "restriction" of something about global case.

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p-adic case: def of W_p

$$\begin{array}{ccc} \mathbb{Q}_p & \subset & \overline{\mathbb{Q}}_p \\ \cup & & \cup \\ \mathbb{Z}_p = \mathcal{O}_p & \subset & \overline{\mathbb{Z}}_p \\ \cup & & \cup \\ p\mathbb{Z}_p = \mathfrak{p} & \subset & \overline{\mathfrak{p}} \end{array}$$

Γ_p = Galois group

$\overline{\mathbb{Z}}_p$ = integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$

$\overline{\mathfrak{p}}$ maximal ideal

$$\mathbb{Z}_p/p\mathbb{Z}_p = \mathcal{O}_p/\mathfrak{p} = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$$

$$\cap$$

$$\overline{\mathcal{O}}_p/\overline{\mathfrak{p}} = \overline{\mathbb{F}}_p \text{ an alg closure of } \mathbb{F}_p$$

Γ_p preserves $\overline{\mathcal{O}}_p$, $\overline{\mathfrak{p}}$, so get

$$\boxed{\Gamma_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \underset{\hat{\mathbb{Z}}}{\underset{\shortparallel}{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}}}$$

Define I_p = INERTIA GROUP

$$= \ker(\Gamma_p \to \hat{\mathbb{Z}})$$

$$= \{\sigma \in \Gamma_p \mid \sigma x \equiv x \mod \bar{\mathfrak{p}}, \text{ all } x \in \bar{\mathcal{O}}_p\}$$

$$\boxed{\begin{tikzcd} 1 \arrow[r] & I_p \arrow[r] & \Gamma_p \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \\ 1 \arrow[r] & I_p \arrow[r] & W_p \arrow[r] \arrow[u, hook] & \mathbb{Z} \arrow[r] \arrow[u, phantom, "\cup"] & 1 \end{tikzcd}}$$

defn of Weil gp W_p, a **dense** subgp of Γ_p

We'll write Fr ("Frobenius") for any preimage in $W_p \subset \Gamma_p$ of the can. generator $1 \in \mathbb{Z} \subset \hat{\mathbb{Z}} = \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$

LANGLANDS PARAMETER ~~for~~ $G(\mathbb{Q}_p)$ is

$$\begin{tikzcd} \varphi_p : W_p \arrow[rr] \arrow[dr] \arrow[r, phantom, shift right=4, "\circlearrowright"] & & {}^\vee G^{\Gamma} \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

$$\boxed{\text{mod } {}^\vee G \text{ conjugation}}$$

EQUIV

$$\begin{tikzcd} \varphi_p : W_p \arrow[rr] \arrow[dr] \arrow[r, phantom, shift right=4, "\circlearrowright"] & & {}^\vee G^{\Gamma_p} \arrow[dl] \\ & \Gamma_p & \end{tikzcd}$$

5) φ_p comes from (= "extends continuously to")

$$\begin{tikzcd}\varphi_p^{\mathrm{Gal}} : \Gamma_p \arrow[r] \arrow[dr] & {}^\vee G^{\Gamma_p} \arrow[d] \\ & \Gamma_p\end{tikzcd}$$

if and only if

requires a little fuss since φ_p doesn't go directly to ${}^\vee G$ $\rightarrow$ $\boxed{\varphi_p(\mathrm{Fr}) \text{ "has finite order"}}$ (in ${}^\vee G$)

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generic (large p) case:

<u>DEF</u> φ_p is <u>unramified</u> if (up to conj.)

$$\varphi_p(i) = \overset{\in {}^\vee G \rtimes \Gamma}{(1 \rtimes i)}, \quad i \in I_p$$

$$\varphi_p(\mathrm{Fr}) = (g \rtimes \mathrm{Fr})$$

g can be any elt of $\boxed{I_p\text{-invts in } {}^\vee G}$

$\uparrow$ reductive gp

<u>For large p, I_p acts triv. on</u> ${}^\vee \underline{G}$, so $\left[\begin{matrix}\text{unramified}\\ \text{Langlands params}\end{matrix}\right] \leftrightarrow \left[\begin{matrix}{}^\vee G\text{-conj. classes}\\ \text{in } {}^\vee G\end{matrix}\right]$

JORDAN DECOMP

LANGLAND'S

Deligne idea: better to separate unipotent conjugacy class in a more subtle way than Jordan decomp.

G red alg / $\mathbb{C}$

JORDAN: $g = s \cdot u$ $\quad$ s semisimple, u unipotent, s, u commute

There's a canonical s_p semisimple s.t.

$$s_p s = s s_p$$

$$s_p u s_p^{-1} = u^p, \quad \underbrace{(s s_p)}_{s'} u (s s_p)^{-1} = u^p$$

PROP Conj classes in $G \longleftrightarrow$

a) pairs (s, u) $\quad$ s semisimple, u unipotent, $sus^{-1} = u$ $\quad$ / G conj

b) pairs (s', u) $\quad$ s' semisimple, u unipotent, $s'us'^{-1} = u^p$ $\quad$ / G conj

WEIL-DELIGNE GROUP

Def (p prime) The **Weil-Deligne group** W_p' is the semidirect produc

$$W_p' = \underset{\substack{\uparrow \\ \text{normal}}}{\mathbb{C}} \rtimes W_p$$

Here I_p acts trivially on $\mathbb{C}$, and any Frobenius elt Fr acts on $\mathbb{C}$ by mult. by p. Have $W_p' \to \Gamma_p$ triv on $\mathbb{C}$

A **Deligne-Langlands parameter** is

$$\begin{tikzcd} \phi' : W_p' \arrow[rr] \arrow[dr] & & {}^\vee G^\Gamma \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

cont. hom.

SUBJECT TO

a) $\phi'|_{W_p}$ consists of semisimple elts

b) $\phi'|_{\mathbb{C}}$ consists of unipotent elts.

18)

PROP There's a 1-1 corr

$$\left[\begin{matrix}\text{Langlands params}\\ \varphi : W_F \to {}^\vee G^\Gamma \\ \text{mod } {}^\vee G \text{ conj.}\end{matrix}\right] \longleftrightarrow \left[\begin{matrix}\text{Deligne-Langlands}\\ \varphi' : W_F' \to {}^\vee G^\Gamma \\ \text{mod } {}^\vee G \text{ conj.}\end{matrix}\right]$$

$$\varphi(Fr) = (ss)\,u \longleftrightarrow \varphi'(\underset{\uparrow}{1}, \underset{\uparrow}{z}) = u^z$$

Improvement over Jordan:

$$\boxed{\varphi'|_{W_F}} = [\text{semisimple part of } \varphi] \cdot [\text{canonical ss elt related to } u = \text{unipotent part of } \varphi(Fr]$$

$\uparrow$ carries more rep-theor. inform than Jordan decomp.

Think of φ' as "infl char" of corr. reps

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p-adic Deligne-Langlands CONJ: make corr

$$\left[\begin{tikzcd} W_F' \arrow[rr, "\varphi'"] \arrow[dr] & & {}^\vee G^\Gamma \arrow[dl] \\ & \Gamma & \end{tikzcd}\right] \longrightarrow \left[\begin{matrix}\text{L- packet of}\\ \text{irr reps}\\ \overline{\Pi(\varphi')}\end{matrix}\right]$$

GOAL: describe $\Pi(\varphi')$ in detail using geometry of parameters

9) Reps of $G(\mathbb{Q}_p)$ at "fixed infl char"

means: all L-packets whose ~~[illegible]~~ params

$$\begin{tikzcd} W_p' \arrow[r, "\varphi_p'"] \arrow[dr] & {}^\vee G^\Gamma \arrow[d] \\ & \Gamma \end{tikzcd}$$

have $\underline{\text{FIXED}}$ restr. to W_F (up to ${}^\vee G$ conj.)

$\underline{\text{FIX}}$ $\varphi_p : W_p \to {}^\vee G^\Gamma$ Langlands param (ss image)

$$\boxed{\begin{array}{l} X(\varphi_p) = \text{all extensions of } \varphi_p \text{ to} \\ \begin{tikzcd} W_F' \arrow[r, "\varphi_p'"] \arrow[dr] & {}^\vee G^\Gamma \arrow[d] \\ & \Gamma \end{tikzcd} \end{array}}$$

$\|$

collection of all algebraic group homs

$\psi : \mathbb{C} \to {}^\vee G$

SUCH THAT

$$\boxed{\begin{aligned} \varphi_p(Fr)\,\psi(z)\,\varphi_p(Fr)^{-1} &= \psi(pz) \\ &= \psi(z)^p \end{aligned}}$$

${}^\vee\mathfrak{g} = \mathrm{Lie}({}^\vee G)$

$$X(\varphi_p) = \underbrace{\left\{ N \in {}^\vee\mathfrak{g} \;\middle|\; \begin{array}{l} \cdot\ \text{fixed by } \mathrm{Ad}[\varphi_p(I_p)] \\ \cdot\ p\text{-eigenspace of } \mathrm{Ad}(\varphi_p(Fr)) \end{array} \right\}}_{\text{linear subspace}}$$

Group ${}^\vee H = \mathrm{Cent}_{{}^\vee G}(\phi(W_F))$ acts
[red alg. gp, maybe disconn.]
on (vector space) $X(\phi_p)$, finitely many orbits

PROP ${}^\vee G$-conj classes of Deligne-Langlands params of infl char ϕ_p $\xleftrightarrow[1\text{-}1]{}$ ${}^\vee H$ orbits on vector space $X(\phi_p)$

Ex $G = GL_n$ $\quad {}^\vee G = GL_n(\mathbb{C})$

${}^\vee G^\Gamma = GL_n(\mathbb{C}) \times \Gamma$

$\phi_p(i) = (1 \times i)$, $i \in I_F$

$$\phi_p(Fr) = \begin{pmatrix} p^{n} & & & \\ & p^{n-1} & & \\ & & \ddots & \\ & & & p \end{pmatrix} \cdot Fr$$

$$X(\phi_p) = \left\{ \begin{pmatrix} 0 & w_1 & & 0 \\ & & \ddots & w_{n-1} \\ & 0 & & 0 \end{pmatrix} \right\} \cong \mathbb{C}^{n-1}$$

${}^\vee H$ = diag torus $= (\mathbb{C}^\times)^n$

$$z = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \cdot w$$

$$\boxed{(z \cdot w)_j = z_j z_{j+1}^{-1} w_j}$$

2^{n-1} orbits, given by which coords of w vanish

SO FAR:

- want to parametrize irr reps of local groups $G(\mathbb{Q}_v)$ by something like

$$\begin{tikzcd}
\varphi : W_v \arrow[r] \arrow[dr] & {}^{\vee}G^{\Gamma} \arrow[d] \\
& \Gamma
\end{tikzcd}$$

[mod ${}^{\vee}G$ conjugacy]

$\varphi \rightsquigarrow \Pi(\varphi)$ = set of irr reps of $G(\mathbb{Q}_v)$

TODAY 1st: describe irr reps in $\Pi(\varphi)$ in more detail using geometry related to φ

TODAY 2nd: describe geometry for Arthur packets/Arthur conjectures

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Deligne-Langlands parameters for p-adic G
Come in finite families param. by

$$\varphi : W_p \longrightarrow {}^\vee G^\Gamma \quad \text{(semisimple image)}$$

$X(\varphi)$ = all extensions of φ to D-L param

$$\varphi' : W_p' \to {}^\vee G^\Gamma$$

$(W_p' = \mathbb{C} \rtimes W_p)$

$\boxed{\cong \text{ p-eigenspace of } Ad(Fr) \text{ on } {}^\vee\mathfrak{g} \, [I_p\text{-invts}]}$

Have action of reductive
(maybe disconn.

$${}^\vee H = \{ {}^\vee g \in {}^\vee G \mid {}^\vee g \text{ centralizes } im(\varphi$$

$\boxed{\text{DL param = one orbit of } {}^\vee H \text{ on } X(\varphi)}$

Tale of two categories...

$\mathcal{M}({}^\vee G, \varphi)$
$\shortparallel$
${}^\vee H$-eqvt D-modules
on $X(\varphi)$

$\mathcal{M}(G, \varphi)$
$\shortparallel$
adm reps of some ratl form $G(\mathbb{Q}_p)$, all comp factors in L-packets ↔ $X(\varphi)$

Deligne-Langlands-Lusztig parameter:

${}^\vee H$ orbit $Z \subseteq X(\varphi)$
+
${}^\vee H$-eqvt irr local system on Z

(φ, ξ)

} parametrize irr. ${}^\vee H$-eqvt D-modules on $X(\varphi)$ ← Deligne-Langlands-Lusztig conjec.
AND
irrs in $\mathcal{M}(G, \varphi)$

21 D)

p-adic local Langlands Conjecture (Deligne-Langlands-Lusztig)

Fix $\varphi : W_p \to {}^\vee G^\Gamma$ classical Langlands parameter (semisimple image)

[illegible]

${}^\vee H$ = centralizer of im φ in ${}^\vee G$

rehom. ector space for ${}^\vee H$

$X(\varphi)$ = all extensions of φ to $\varphi' : W_p' \to {}^\vee G^\Gamma$ Deligne-Langlands param

complete geometric parameter =: irr ${}^\vee H$-eqvt local system on a ${}^\vee H$ orbit in $X(\varphi)$

$\updownarrow$ conjectural bijection

exists splitting $\Gamma \to G^\Gamma$ $\leftrightarrow$ pure

rep of pure inner form of $G(\overline{\mathbb{Q}}_p)$ = irr adm. rep of pure inner form $G(\mathbb{Q}_p)$ $/ (\overline{\mathbb{Q}}_p$ conj

Jeff: $x^2 = 1$

21C) EXAMPLE

$$G = PGL(2, \overline{\mathbb{Q}}_p)$$

(split inner class)

ratl forms:

- $PGL_2(\mathbb{Q}_p) = GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times$
- $D_p^\times/\mathbb{Q}_p^\times$, D_p = 4-dial div. alg. over $\mathbb{Q}_p$
  COMPACT

Look at three reps:

$\text{trivial}^{\text{split}}$ = triv rep of $PGL_2(\mathbb{Q}_p)$

$\text{Steinberg}^{\text{split}}$ = non-cuspidal discrete series of $PGL_2(\mathbb{Q}_p)$

$\text{trivial}^{\text{division}}$ = trivial of $D_p^\times/\mathbb{Q}_p^\times$

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local Langlands:

$\text{trivial}^{\text{split}} \longleftrightarrow \xi^{\text{triv}}_0$

$\text{Steinberg}^{\text{split}} \longleftrightarrow \xi^{\text{triv}}_{\mathbb{C}^\times}$

$\text{trivial}^{\text{division}} \longleftrightarrow \xi^{\text{twisted}}_{\mathbb{C}^\times}$

$^\vee G = SL(2, \mathbb{C})$ [illegible]

Trivially Langlands duality → Langlands param with [illegible] into $SL(2, \mathbb{C})$

Fix $\varphi : W_p \to SL(2, \mathbb{C})$

$\varphi(I_p) = 1$

$$\varphi(Fr) = \begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2} \end{pmatrix}$$

$^\vee H$ = diagonal matrices

$$= \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \middle| z \in \mathbb{C}^\times \right\} \simeq \mathbb{C}^\times$$

$$X(\varphi) \leftrightarrow \left\{ \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \right\} \simeq \mathbb{C}$$

$$\varphi' \to d\varphi'|_{\mathbb{C}}$$

$^\vee H$ action on $X(\varphi)$

$$\boxed{z \cdot w = z^2 w}$$

Three geom params:

$\xi^{\text{triv}}_0 = \{\text{pt } 0, \text{ triv local}\}$

$\xi^{\text{triv}}_{\mathbb{C}^\times} = \{\mathbb{C}^\times, \text{ triv local}\}$

$\xi^{\text{twisted}}_{\mathbb{C}^\times} = \{\mathbb{C}^\times, \sqrt{w} \text{ local system}\}$

② Irr reps of $G(\mathbb{Q}_p)$:

$$\text{irr } L(\varphi,\xi) \xleftarrow[\text{Langlands}]{} \underbrace{\text{std } M(\varphi,\xi)}_{\text{induced from tempered, strictly pos continuous parameter.}}$$

BASIC PROBLEM: decompose each standard repn $M(\varphi,\xi)$ into irrs: (in Gr. group)

$$[M(\varphi,\xi)] = \sum_{\xi'} \underbrace{m_r(\xi',\xi)}_{\text{non-neg integer}} [L(\varphi,\xi')]$$

Expect: non-zero only if orbit for $\xi \subset$ closure of orbit for ξ'

$m(\xi,\xi) = 1$

---

irr ${}^\vee H$-eqvt D-modules

$$\mathcal{L}(\varphi,\xi) \xrightarrow[\text{Riemann-Hilbert corresp.}]{} \text{irr } {}^\vee H\text{-eqvt perverse sheaf on } X(\varphi) \longrightarrow \text{disassemble into constructible sheaves}$$

$$\sum_{\xi'} \underbrace{{}^\vee m(\xi',\xi)}_{\text{integer, negative if sheaf appears in odd degree cohom}} \underbrace{\begin{bmatrix}\text{sheaf } \xi' \\ \text{ext by zero}\end{bmatrix}}_{\text{"standard" } \mathcal{D} \text{ module}}$$

$(\xi,\xi) = 1$

NON-ZERO only if orbit for $\xi' \subset$ closure of orbit for ξ

(3) p-adic KL conjecture (Zelevinsky 1981)

$$\underbrace{m_r(\xi', \xi)}_{\substack{\text{mult of rep.}\\ \text{corr to } \xi' \text{ in}\\ \text{std corr. to } \xi}} = (-1)^{\text{codim}(\xi \text{ in } \xi')} \underbrace{{}^\vee m_g(\xi, \xi')}_{\substack{\text{coeff of std sheaf } \xi\\ \text{in irr perverse sheaf}\\ \text{corr. to } \xi'}} \leftarrow \text{algorithm by Lusztig 2006}$$

Two Grothendieck groups...

$K_r(\varphi) = $ Gr. gp $\mathcal{M}(G, \varphi)$

basis $\{M(\varphi, \xi)\}$ $\boxed{\text{standard}}$

OR basis $\{L(\varphi, \xi)\}$ $\boxed{\text{irr}}$

$K_g(\varphi) = $ Gr gp for $\mathcal{M}(\mathfrak{G}, \varphi)$

basis $\{\xi\}$ $\boxed{\text{local system}}$

OR

basis $\{P(\xi)\}$ $\boxed{\text{irr perverse}}$

free $\mathbb{Z}$-modules of finite rank

KL conjecture says

CONJ There's perfect pairing $\langle\, ,\, \rangle : K_r \times K_g \to \mathbb{Z}$

satisfying both $\langle M(\varphi, \xi), \xi' \rangle = (-1)^{\dim \xi} \delta_{\xi, \xi'}$

and $\langle L(\varphi, \xi), P(\xi') \rangle = (-1)^{\dim \xi} \delta_{\xi, \xi'}$

(Given Deligne-Langlands-Lusztig conj, can define pairing by 1st formula. Then second formula is equiv to KL conj.)

The big picture (conjecturally, that is):

$\left[\text{$\mathbb{Z}$-comb of irr reps in L-packets related to } \varphi\right]$ = $\boxed{\text{$\mathbb{Z}$-linear functional on Grothendieck group of ${}^\vee H$-equt D-modules on } X(\varphi)}$

interesting combs. of reps = Interesting linear functionals!

Ex

$\varphi' \in X(\varphi)$ D-L parameter

→ linear functional

$\mathcal{M} \longrightarrow$ [alt. sum of dims of local soln sheaf stalks at φ']

← um of all std. -eps in L-packet param by φ'

↑ STABLE CHARACTER (and every stable char is a lin comb of these)

Alternate formulation: $\mathcal{O}(\varphi) = {}^\vee G$-conj. class of $\varphi : W_p \to {}^\vee G^\Gamma$

$$\boxed{X(\mathcal{O}(\varphi))} = \left\{ \begin{array}{l} \text{all extensions to} \\ \psi' : W_p' \to {}^\vee G^\Gamma \text{ of} \\ \psi \in \mathcal{O}(\varphi) \end{array} \right\} \cong \boxed{{}^\vee G \times_{{}^\vee H} X(\varphi)}$$

${}^\vee G$-equt D-modules on $X(\mathcal{O}(\varphi))$ ≅ ${}^\vee H$-equt D-modules on $X(\varphi)$

Real groups version

$$\boxed{W_{\mathbb{R}} = \left\langle j, \mathbb{C}^\times \right\rangle \Big/\!\!\Big/ \left\langle \begin{matrix} j^2 = -1 \in \mathbb{C}^\times \\ jzj^{-1} = \bar{z} \end{matrix} \right\rangle \to \Gamma_{\mathbb{R}}}$$

$j \longmapsto$ cplx conj

$$\begin{tikzcd} \varphi : W_{\mathbb{R}} \arrow[r] \arrow[d, leftrightarrow] & {}^\vee G^\Gamma \\ \text{pair } (y, \lambda) & \end{tikzcd}$$

$$\boxed{\begin{array}{l} y \in {}^\vee G^\Gamma, \text{ component corr. to complex conj.} \\ \lambda \in {}^\vee\mathfrak{g} \text{ semisimple} \\ y^2 = e(\lambda) \quad [\lambda, \lambda^y] = 0 \end{array}}$$

$e(\lambda) = \exp(2\pi i \lambda)$

Problem with imitating p-adic case: everything here is semisimple, ${}^\vee G$ orbits of such φ are <u>closed</u>; can't detect relations between reps by closure relations.

SO ... (idea of Martin Andler)

REPLACE Langlands parameters by something different.

≡ infl char for $G(\mathbb{R})$ reps

$\mathcal{O} = {}^\vee G$ orbit of semisimple elements in ${}^\vee\mathfrak{g}$.

This is ~~a~~ smooth affine alg variety. Want to fiber it by $\boxed{\text{affine linear subspaces of } {}^\vee\mathfrak{g}}$

$\lambda \in \mathcal{O} \quad {}^\vee\mathfrak{n}(\lambda) = \bigoplus_{m=1}^{\infty}$ m-eigenspace of $\mathrm{ad}(\lambda)$ on ${}^\vee\mathfrak{g}$

CANONICAL FLAT $\boxed{\Lambda(\lambda) \underset{def}{=} \lambda + {}^\vee\mathfrak{n}(\lambda)}$

Real local Langlands conjecture

$\mathcal{O} \subset {}^\vee\mathfrak{g}$ semisimple ${}^\vee G$ orbit

‖ infl char for reps of $G(\mathbb{R})$

$$\underset{\substack{\uparrow \\ \exp(2\pi i\,\cdot)}}{e}(\mathcal{O}) \subset {}^\vee G \text{ semisimple conjugacy class}$$

FIX $\boxed{z \in e(\mathcal{O})}$ $\quad {}^\vee H \underset{\text{def}}{=} \mathrm{Cent}_{{}^\vee G}(z)$

possibly disconnected reductive alg. group

$$\boxed{\mathcal{O}_z = \{\lambda \in \mathcal{O} \mid e(\lambda) = z\}} = \text{ONE } {}^\vee H \text{ conj. class in } {}^\vee\mathfrak{g}$$

<u>NOTE</u>: $\lambda \in \mathcal{O}_z \Longrightarrow$ flat $\Lambda(\lambda) \subset \mathcal{O}_z$

REAL GEOMETRIC PARAM.

is a pair (y, Λ)

$y \in {}^\vee G^\Gamma$, coset corr. to complex conj.

$y^2 = e(\Lambda)$

$\Lambda \subset \mathcal{O}$ a canonical flat

$$\boxed{X(\mathcal{O}) = \text{geom params}}$$

<u>Prop</u> ${}^\vee G$ acts alg. on $X(\mathcal{O})$ <u>with finite</u> number of orbits. The map

$$(y, \lambda) \mapsto (y, \Lambda(\lambda))$$

from Langlands params to geom params is BIJ. on ${}^\vee G$ orbits

Prop The canonical flat $\Lambda(\lambda)$ is contained in the orbit $\mathcal{O} = {}^\vee G \cdot \lambda$. The exponential is constant on $\Lambda(\lambda)$:

$$\underset{\exp(2\pi i \lambda')}{\overset{\shortparallel}{e(\lambda')}} = e(\lambda), \quad \text{all } \lambda' \in \Lambda(\lambda)$$

Ex. ${}^\vee G = SL(4, \mathbb{C})$

$$\lambda = \begin{pmatrix} 2 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -4 \end{pmatrix} \qquad \underset{\substack{\uparrow \\ \dim 5}}{{}^\vee\mathfrak{n}(\lambda)} = \begin{pmatrix} 0 & * & * & * \\ & 0 & 0 & * \\ & & 0 & * \\ 0 & & & 0 \end{pmatrix}$$

$\dim \mathcal{O} = 10$

$$\Lambda(\lambda) = \left\{ \begin{pmatrix} 2 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & -4 \end{pmatrix} \right\}$$

$e(\lambda') = 1$, all $\lambda' \in \Lambda(\lambda)$

---

real local Langlands conjecture (theorem of Langlands, Knapp...)

$\mathcal{O} \subset {}^\vee\mathfrak{g}$ semisimple ${}^\vee G$ orbit

$X(\mathcal{O})$ = geometric parameter space

COMPLETE GEOM PARAMETER = irr. ${}^\vee G$-equvt local system on a ${}^\vee G$ orbit in $X(\mathcal{O})$

↑ complete geom parameter ξ

$\Big\updownarrow$

$L(\xi)$

irr. adm rep of pure strong real form of G / conj. by G

"pure" real forms of G: $x^2 = 1$ in Jeff Adams' setting

NOTE: to bring in "non-pure" real forms, replace "${}^\vee G$-equvt local system" by "${}^\vee G^\sim$-equvt local system," where ${}^\vee G^\sim$ is a certain covering of ${}^\vee G$.

SEE ABV book

EXAMPLE

$G = PGL(2, \mathbb{R})$

(split inner class)

real forms:

- $PGL_2(\mathbb{R}) = GL_2(\mathbb{R}) / \mathbb{R}^\times$
- $\underset{\substack{\uparrow \\ \text{quaternions}}}{\mathbb{H}^\times} / \mathbb{R}^\times \cong SO(3)$

$PGL_2(\mathbb{R})$ is disconnected, so has two one-diml reps

$\boxed{\text{trivial}^{\text{split}}, \quad \text{sign}^{\text{split}}}$

emp. ds^{split} = first discrete series of $PGL_2(\mathbb{R})$

emp. $\text{trivial}^{\text{division}}$ = trivial of $\mathbb{H}^\times/\mathbb{R}^\times = SO(3)$

---

local Langlands:

$\text{trivial}^{\text{split}} \leftrightarrow NP^{\text{trivial}}$

$\text{sign}^{\text{split}} \leftrightarrow SP^{\text{trivial}}$

$ds^{\text{split}} \leftrightarrow REST^{\text{trivial}}$

$\text{trivial}^{\text{div}} \leftrightarrow REST^{\text{twisted}}$

${}^\vee G^\Gamma = SL(2, \mathbb{C}) \times \Gamma$

$\mathcal{O}$ = conj class of $\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

$e(\mathcal{O}) = -I \in SL(2$

$\boxed{\text{POSSIBLE } y: \left(\text{conj. class of } \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \text{ cplx conj.}\right.}$

$\boxed{\cong {}^\vee G / {}^\vee T}$

$\boxed{\begin{array}{l} \text{Flats in } \mathcal{O} \\ \leftrightarrow \text{Borel subgps of } {}^\vee G \\ \leftrightarrow {}^\vee G / {}^\vee B \end{array}}$

$$X(\mathcal{O}) = \underbrace{\left[{}^\vee G / {}^\vee T\right] \times \left[{}^\vee G / {}^\vee B\right]}_{\text{as } {}^\vee G \text{ space}}$$

orbits: ?? $\boxed{\begin{array}{c} {}^\vee G / {}^\vee B \\ \text{as } {}^\vee T\text{-space} \end{array}}$

$\left.\begin{array}{l} NP \\ SP \end{array}\right\}$ $\underline{\text{poles}}$ local systems trivial

$\underbrace{\text{REST}}$ two eqvt local systems triv, twisted

UNDERSTANDING GEOM PARAMETER SPACE

$\mathcal{O} \subset {}^\vee\mathfrak{g}$, $z \in e(\mathcal{O}) \subset {}^\vee G$ ${}^\vee H = \mathrm{Cent}_{{}^\vee G}(z)$

FIX $y \in {}^\vee G^\Gamma$, complex conj coset
$y^2 = z$

} Finitely many possible y up to ${}^\vee H$ conj.

${}^\vee K = \mathrm{Cent}_{{}^\vee G}(y)$, symmetric subgp of ${}^\vee H$

$\mathcal{O}_z = \{\lambda \in \mathcal{O} \mid e(\lambda) = z\} \subset {}^\vee\mathfrak{h}$

$\mathcal{P}$ = canonical flats in $\mathcal{O}_z$
≅ partial flag variety for ${}^\vee H$

$X(\mathcal{O}_z, y) = \{\text{pairs } (y, \Lambda) \in X(\mathcal{O})\} \cong \mathcal{P}$

STUDY ${}^\vee K$-eqvt $\mathcal{D}$-modules on
$\mathcal{P} \cong X(\mathcal{O}_z, y)$

$$X(\mathcal{O}) = \coprod_{\substack{\text{conj. classes } \sigma \\ \text{of } y\text{'s}}} X(\mathcal{O}, \sigma)$$

${}^\vee G$-eqvt $\mathcal{D}$-modules on $X(\mathcal{O}$, class of $y)$ $\cong$ ${}^\vee K$-eqvt $\mathcal{D}$-modules on $X(\mathcal{O}_z, y)$

31)

Real local Langlands

Recall bijection

complete geom params ξ ← local system on orbit → irr rep of pure real form $L(\xi)$

$P(\xi)$ = corr irr perverse sheaf / simple $\mathcal{D}$-module

$M(\xi)$ = standard rep with Langlands quotient $L(\xi)$

$$P(\xi) = \sum_{\xi'} {}^\vee m_g(\xi';\xi) \cdot \xi$$

express irr perverse/$\mathcal{D}$-module in terms of standard (in Groth. gp)

$$M(\xi) = \sum_{\xi'} m_r(\xi';\xi) L(\xi$$

express std in terms of irrs (in Groth gp.)

THEOREM (Kazhdan-Lusztig conj for real groups - see ABV book)

$$m_r(\xi';\xi) = (-1)^{\mathrm{codim}(\xi \text{ in } \xi')}\, {}^\vee m_g(\xi,\xi')$$

Cor Perfect pairing (making stds = dual bases AND irrs = dual bases)

[Groth. gp of reps of pure real forms, infl char $\mathcal{O}$]

[Groth gp of ${}^\vee G$-eqvt $\mathcal{D}$-modules on $X(\mathcal{O})$]

examples of interesting $\mathbb{Z}$-linear fnls on Gr. group of eqvt $\mathcal{D}$-modules on $X(\mathcal{O})$:

take $\underbrace{\text{point in } X(\mathcal{O})}_{\updownarrow \text{ Langlands param}}$, Euler char of soln sheaves at point.

Corr rep : $\boxed{\begin{array}{l}\text{sum of std reps in}\\ \text{L-packet} \Leftrightarrow \text{pt}\end{array}}$

all stable sums are lin combs of these. → STABLE

Harmonic analysis for algebraists

$\Pi(G) \leftarrow$ adm. irr. reps $\longleftrightarrow$ Complete geometric params

$\cup$

$\Pi_u(G) \leftarrow$ unitary

$\cup \leftarrow$ conjecturally

$\Pi_{\mathrm{Arthur}}(G) \leftarrow$ conj: enough unitary reps to decompose L^2(autom. forms)

$\cup$

$\Pi_{\mathrm{temp}}(G) \longleftrightarrow$ complete geom parameters corr. to

Langlands params with bounded image

$\lambda + \lambda^{y}$ elliptic in ${}^\vee\mathfrak{g}$

corr ${}^\vee G$ orbit on $X(\mathcal{O})$ is open

FIGURE OUT direct char of tempered using geom params

Arthur idea (fix $\mathcal{O} \leftrightarrow$ infl char)

X = geom param space

$= \coprod {}^{\vee}G$-orbits $Z_1, \ldots Z_n$

${}^{\vee}G$-equt $\mathcal{D}$-modules on X

$\longleftrightarrow$ coh. sheaves on "conormal to ${}^{\vee}G$ action" $\subset T^*X$

$$\sim \coprod_i \underbrace{T^*_{Z_i} X}_{\text{lagr. subvar of } T^*X}$$

$\dim X = m$

$\dim T^*X = 2m$

$\dim T^*_{Z_i} X = m$

Arthur: picks one Z_i. Look at lin fnl "mult of $T^*_{Z_i}$ in ch cycle"

$$\begin{bmatrix} \text{Gr group of} \\ {}^{\vee}G\text{-equt} \\ \mathcal{D}\text{-mods} \end{bmatrix} \xrightarrow[\text{ch cycle}]{} \underbrace{\begin{bmatrix} \text{free abel gp,} \\ \text{basis various} \\ T^*_{Z_i} X \end{bmatrix}}_{\text{coll. of } \mathbb{Z}\text{-linear fnls on}}$$

THESE ARE THE STABLE REPNS

Obvious consequences..

irr. $\mathcal{D}$-module $\mathcal{M}$ ⟷ orbit Z $\Longrightarrow$ mult of T^*_Z in ch($\mathcal{M}$) is > 0 (= dim of local system)

Consequence:

"Arthur packet"

‖

all irr reps whose char cycle contains T^*_Z

INCLUDES L-packet for Z.

Most interesting case of Arthur params: <u>UNIPOTENT</u>

(Weil group part triv on $\mathbb{C}^*$)

$${}^\vee H = \text{cent}_{{}^\vee G}(\text{image of } -1 \text{ in } SL(2))$$

$$\begin{tikzcd} {}^\vee H \\ {}^\vee K \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\rightsquigarrow$ ${}^\vee K$ action on $\mathcal{B} = {}^\vee H / {}^\vee P$ flag var.

orbit for Arthur:

CLOSED ${}^\vee K$ orbit $Z \subset \mathcal{B}$

$$\underbrace{T^*_Z}_{\text{big cotangent direction since } Z \text{ small}} \xrightarrow{\text{moment}} \overline{\text{Richardson nilp for } {}^\vee P}$$

$$\boxed{\text{Arthur packet} \leftrightarrow \text{all } \mathcal{D} \text{ modules whose corr } ({}^\vee\mathfrak{g}, {}^\vee K) \text{ modules are as big as possible}}$$