Twisted Endoscopy

6/21 : a slightly cleaned-up version

- a generalization which makes the methods of endoscopy applicable to a wider family of trace formulas.

Setting

G connected reductive algebraic group defined over $\mathbb{R}$

$\rightarrow$ $G(\mathbb{C})$, connected, reductive, complex Lie group, with Galois action σ

$\rightarrow$ $G(\mathbb{R})$, real reductive Lie group with finitely many components

$$G(\mathbb{R}) = \{g \in G(\mathbb{C}) : \sigma(g) = g\}.$$

θ (algebraic) automorphism of G, defined over $\mathbb{R}$

$\rightarrow$ $\theta : G(\mathbb{C}) \longrightarrow G(\mathbb{C})$, $\theta\sigma = \sigma\theta$

$\rightarrow$ $\theta : G(\mathbb{R}) \longrightarrow G(\mathbb{R})$

ω quasicharacter on $G(\mathbb{R})$ ... "comes for free".

Underlying motivation : (packets of) representations Π of $G(\mathbb{R})$ satisfying

$$\Pi \circ \theta = \omega \otimes \Pi$$

$$\Pi \circ \theta = \{\pi \circ \theta : \pi \in \Pi\}$$

$$\omega \otimes \Pi = \{\omega \otimes \pi : \pi \in \Pi\}$$

... Langlands parameters etc.

Examples

(a) $\theta = \mathrm{id}$ $G = GL(n)$ ... $\pi = \pi \otimes \omega$

(b) θ inner automorphism, ω trivial ... back to ordinary endoscopy

From now on, to save time, suppress ω from consideration ...

(c) $G = GL(3)$, $\theta(g) = J\,{}^t g^{-1} J^{-1}$,

where $J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

... self-contragredient representations of $GL(3,\mathbb{R})$.

(d) "Base change $\mathbb{C}/\mathbb{R}$"

... Galois-invariant representations of $G(\mathbb{C})$.

Work with $G_0 = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}$, an algebraic group defined over $\mathbb{R}$.

Concretely :

$$G_0(\mathbb{C}) = G(\mathbb{C}) \times G(\mathbb{C})$$

$$\sigma_0(g_1, g_2) = (\sigma(g_2), \sigma(g_1))$$

is the Galois action on $G_0(\mathbb{C})$

$$G_0(\mathbb{R}) = \{(g, \sigma(g)) : g \in G(\mathbb{C})\}$$
$$\leftrightarrow G(\mathbb{C})$$

If $\pi_0 \leftrightarrow \pi$

then $\pi_0 \circ \theta \leftrightarrow \pi \circ \sigma$

where θ is the flip $(g_1, g_2) \to (g_2, g_1)$,

an (algebraic) automorphism of G_0.

Endoscopic data for (G, θ, ω)

- includes data for a connected reductive alg. group over $\mathbb{R}$, quasi-split over $\mathbb{R}$

... "endoscopic group" H,

as well as "just enough" data for stabilizing an appropriate(ly) twisted trace formula...

<u>Example</u> For $(GL(3), \theta)$ above, $GL(2) \downarrow$

$H = SL(2)$ or $GL(2)$ (... not $PGL(2)$)

"z-extension"

<u>Why?</u> To set up (... and use in stabilization ...) the <u>transfer factor</u>, an essentially canonical function

$$\Delta : H(\mathbb{R}) \times G(\mathbb{R}) \longrightarrow \mathbb{C}.$$

→ start with the "very regular" set, a dense subset of $H(\mathbb{R}) \times G(\mathbb{R})$, and then fill in ....

(Very regular (γ, δ) : γ is strongly G-regular and δ is strongly θ-regular … (θ-twisted) centralizer reductive, minimal dimension, "as connected as possible")

→ require "coherent" constructions :

(i) define

$$\Delta_v : \big(H(F_v) \times G(F_v)\big)_{\text{very reg}} \longrightarrow \mathbb{C}$$

and require a product formula over all places …

(Global Hypothesis in ordinary endoscopy)

(ii) Examples for $v = \infty$:

- replace ω with $a \in H^1(W_{\mathbb{R}}, \hat{Z})$

($\hat{Z} = Z(\hat{G})$).

- don't write

$2i \sin\theta$

(ii) require coherency among inner forms of G (Local Hypothesis in ordinary endoscopy).

note: transfer factors appear in the precise statement of the fundamental Lemma.

References for the definition of transfer factors.

- Langlands-Sh "On the definition of transfer factors" (ordinary endoscopy)
- Kottwitz-Sh. Foundations of Twisted Endoscopy (general twisted case)

Labesse "Stable twisted trace formula: elliptic terms"

www.TransferFactorResearch.com ...

The transfer factors defined in K-S have a simple form <u>locally</u> on the very regular set:

(Ignore the discriminant term Δ_{IV})

for F nonarchimedean they are locally constant inside the very regular set, and

for F archimedean they are locally constant up to "essentially" a local character on $H(\mathbb{R})$.

(See sheet #18)

What is complicated (and long) is the definition of $\Delta(\gamma,\delta)/\Delta(\gamma',\delta')$ for arbitrary pairs (γ,δ), (γ',δ') in the very regular set.

(Galois hypercohomology for $X \xrightarrow{1-\theta} X$)

Back to endoscopic data

... focus on the particular set.

First transport the F-automorphism $\theta : G \longrightarrow G$ to an (algebraic) automorphism $\hat{\theta} : \hat{G} \longrightarrow \hat{G}$ ($\hat{G}$ = complex dual group of G) commuting with the Galois action on $\hat{G}$:

Fix an inner form G^* of G and an $\mathbb{R}$-splitting spl_{G^*} of G^*: $spl_{G^*} = (B^*, T^*, \{X_\alpha\})$, where $B^* \supset T^*$ are both defined over $\mathbb{R}$ and $\sigma_{G^*}(X_\alpha) = X_{\sigma_{G^*}\alpha}$, for $\alpha \in \underset{\text{simple roots}}{\Sigma(B^*, T^*)}$.

Use an inner twist $\psi : G \to G^*$ to pull θ across to θ^* an automorphism of G^* preserving spl_{G^*}

(i.e choose $g_0 \in G^*(\mathbb{C})$ such that

$\theta^* = \mathrm{Int}\, g_0 \circ \psi \circ \theta \circ \psi^{-1}$ preserves spl_{G^*}). Then θ^* must be defined over $\mathbb{R}$ ($\sigma \theta^* \sigma^{-1}$ has the same property and differs from θ^* by an inner automorphism).

ψ gives $\hat{G} \cong \hat{G}^*$.

Fix a Γ-splitting $spl_{\hat{G}}$ of $\hat{G}$

$spl_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}_{\alpha^\vee}\})$

Define $\hat{\theta} : \hat{G} \to \hat{G}$ dual to θ^* and preserving $spl_{\hat{G}}$.

We let $W_{\mathbb{R}}$ act on $\hat{G}$ thru $W_{\mathbb{R}} \to \Gamma$ and extend $\hat{\theta}$ to ${}^L\theta$:

$$g \times w \mapsto \hat{\theta}(g) \times w ,$$

an automorphism of ${}^LG = \hat{G} \rtimes W_{\mathbb{R}}$.

• here is where w/a comes in ...

To define the <u>basic set of endoscopic data</u> for (G, θ) consider

the fixed points of $\hat{\theta}$ – this is a reductive group, not necessarily connected.

Set $\hat{H} = (\hat{G}^{\hat{\theta}})^{\circ}$.

From the $\underbrace{\text{the}\ \Gamma}_{\hat{\theta}\text{-stable}}$-splitting $\mathrm{spl}_{\hat{G}} = (\mathcal{B}, \ldots)$ we get the Γ-splitting

$spl_{\hat{H}} = (\mathcal{B} \cap \hat{H}, T \cap \hat{H}, \{X_{\alpha^\vee_{res}}\})$

of $\hat{H}$. H = dual quasi-split group over $\mathbb{R}$

(Reference for automorphisms of algebraic groups : Steinberg AMS Memoir # 80)

$i : h \times w \longrightarrow h \times w$

embeds ${}^L H$ in ${}^L G$.

- $(H, {}^L H, 1, i)$ is the basic set of endoscopic data for (G, θ). Write $H = H_{basic}$

(1 = identity element of G)

<u>Examples</u>

(a) θ inner

$H_{basic} = G^*$, quasi-split inner form

(b) θ = base change

$$\hat{G}_0 = \hat{G} \times \hat{G}$$

$$\hat{\theta} = \text{flip}$$

$$\hat{H} = \hat{G} \text{ embedded diagonally in } \hat{G} \times \hat{G}$$

$$H_{\text{basic}} = G^*$$

(c) $(GL(3), \theta)$ $\quad H_{\text{basic}} = SL(2)$

For ~~more~~ general endoscopic groups/data consider

$$\hat{H}_s = (\text{fixed points of } \mathrm{Int}\, s \circ \hat{\theta})^\circ$$

$$= (\hat{\theta}\text{-twisted centralizer of } s)^\circ$$

$$= \{ g \in \hat{G} : g^{-1} s \hat{\theta}(g) = s \}^\circ$$

We call s $\hat{\theta}$-<u>semisimple</u> if $\hat{H}_s$ is reductive or, equivalently, if $\mathrm{Int}\, s \circ \hat{\theta}$ preserves some pair $(\mathcal{B}', \mathcal{T}')$ ($\mathcal{B}'$ is a Borel subgroup containing the maximal torus $\mathcal{T}'$).

... H will be a z-extension of H_s dual to $\hat{H}_s$.

Are these groups the same as the ordinary (untwisted) endoscopic groups for H_{basic}?

- For base-change, yes* (data <u>not</u> same)
- In general, no

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* Up to $\hat{\theta}$-conjugacy, can assume $s = (g, 1)$, then $\mathrm{Cent}_{\hat{\theta}}(s, \hat{G}_0) = \mathrm{Cent}(s\hat{\theta}(s), \hat{H}_{\text{basic}})$

Transfer factors for any finite cyclic extension E/F of local fields of char. zero are closely related to ordinary transfer factors.

The transfer factors for $\mathbb{C}/\mathbb{R}$ "work as intended" (Sh, Bouaziz) at the tempered level. In the case of the basic endoscopic group this means the following:

- There is a natural norm mapping $\{\delta\}_{\theta\text{-st}} \longrightarrow \{\gamma\}_{st}$ of stable θ-twisted conjugacy classes of strongly θ-regular elements in $G_0(\mathbb{R})$ to stable conjugacy classes of strongly regular elements in $H_{basic}(\mathbb{R})$.

(... start with $g \rightarrow g\sigma(g)$ in $G(\mathbb{C})$ ...)

Let C_G be a suitable space of functions on $G(\mathbb{R})$ (... Schwartz ... C_c^∞ ... same mod center, with central character etc.) and let C_H be the corresponding space for $H(\mathbb{R})$.

- For $f \in C_G$ define integral $O_f^\theta(\delta)$ of f along the θ-conjugacy class of strongly θ-regular δ in $G(\mathbb{R})$
- For $f_1 \in C_H$ define integral $O_{f_1}^{st}(\gamma)$ of f_1 along stable conjugacy class of strongly γ in $H(\mathbb{R})$

(... normalize measures carefully)

- For (γ, δ) in the very regular set, define

$$\Delta(\gamma, \delta) = \begin{cases} 1 & \{\gamma\}_{st} \text{ is a norm of } \{\delta\}_\theta \\ 0 & \text{if not.} \end{cases}$$

Theorem 1 ("Geometric side")

For all $f \in C_G$ there exists $f_1 \in C_H$

such that

$$O^{st}_{f_1}(\gamma) = \sum_{\delta} \Delta(\gamma, \delta)\, O^{\theta}_{f}(\delta)$$

for all (γ, δ) in the very regular set.

Remarks on proof:

- the right side is a stable (twisted) orbital integral in this case. I have written it with the transfer factor since that mimics the (almost) general form. (For complete generality ...)

- To prove Thm 1, start with a local analysis of the right side i.e. of twisted orbital integrals, using descent à la Harish-Chandra. Show that $\Phi(\gamma) = \sum_{\delta} \Delta(\gamma, \delta)\, O^{\theta}_{f}(\delta)$ satisfies the necessary & sufficient conditions (Thm 0) to be the family of stable orbital integrals of a <u>Schwartz function</u> on $H(\mathbb{R})$. (have ignored center)

Thm 0 is a straightforward consequence of Harish-Chandra's theory of cusp forms & Eisenstein integrals. Bouaziz's theorem allows passage to C_c^{∞}-functions.

<u>Theorem 2</u> ("<u>Spectral side</u>") due to Bouaziz

- $\text{Stab-tr}\, \Pi_1(f_1) = (\text{const})\, \text{Tw-tr}\, \Pi(f)$
- explicit character identities for tempered spectrum ....

[redacted]

Re sheet #8 :

T_H a maximal torus over $\mathbb{R}$ (CSG) in the endoscopic group H.

We have two ways to [redacted] map $^LT_H = \hat{T}_H \rtimes W_{\mathbb{R}}$ in $^LG = \hat{G} \rtimes W_{\mathbb{R}}$

$$\begin{tikzcd} & {}^LH_{\text{basic}} \arrow[r, hook] & {}^LG \\ {}^LT_H \arrow[ur] \arrow[dr, hook] & & \\ & {}^LH \arrow[uur] & \end{tikzcd}$$

with "L-mappings". Such maps always exist but are severely constrained ...

difference between two of them gives a ρ-shift + <u>small term</u>

small term is "essentially" a linear map on Lie($H(\mathbb{R})$).