Algorithms for Representation Theory of Real Reductive Groups I
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1 Introduction

These are notes to accompany lectures at the conference in honor of Bill Casselman and Dragan Milicic, held at Snowbird, June 4-8, 2006.

Suppose G is a real reductive group, such as $SL(2,\mathbb{R}), GL(n,\mathbb{R})$ or $SO(p,q)$. The irreducible admissible representations have been classified, by work of Langlands, Knapp, Zuckerman and Vogan. This classification is somewhat involved, and requires a substantial number of prerequisites. See [6] for a reasonably accessible treatment. It is fair to say that it is difficult for a non-expert to understand any non-trivial case, not to mention a group like E_8.

The purpose of these notes is to describe an algorithm to compute the irreducible admissible representations of a real reductive group. This algorithm has been implemented on a computer by Fokko du Cloux. An early version of the software, and some other documentation and information, may be found on the web page of the Atlas of Lie Groups and Representations, at `www.liegroups.org`.

The subject is incomprehensible without looking at numerous examples. Some examples are included at the end of these notes, others are found on the Atlas web site, and I will do a number of examples in my lectures.

Here is a little more detail on what what the algorithm does.

(1) Allow the user to define

(a) A reductive algebraic group G,

(b) An inner class of real forms of G

(c) A particular real form $G(\mathbb{R})$ of G

Fix $G(\mathbb{R})$ and let K be the corresponding (complexified) maximal compact subgroup

(2) Compute the component group of $G(\mathbb{R})$

(3) Enumerate the Cartan subgroups of $G(\mathbb{R})$, and describe them as real tori,

(4) For any Cartan subgroup H compute the "real" Weyl group $W(G(\mathbb{R}), H(\mathbb{R}))$

(5) Describe the flag variety $K\backslash G/B$

(6) Compute a set $\mathcal{Z}$ parametrizing the irreducible representations of $G(\mathbb{R})$ with regular integral infinitesimal character

(7) Compute the cross action and Cayley transforms

(8) Compute Kazhdan-Lusztig polynomials

In fact the proper setting for all of the preceding computations is not a single real group $G(\mathbb{R})$, but an entire "inner class" of real forms, as described in Sections 2–4.

The approach used in these notes most closely follows [1]. This reference has the advantage over [2], which later supplanted it, in that it focusses on the case of regular integral infinitesimal character, and avoids some technical complications arising from the general case. There are a few changes in terminology between these references which are discussed in the Remarks. We also use many results from [5].

2 Reductive Groups and Root Data

We first describe the parameters for a connected algebraic group G. These are provided by *root data* and *based root data*. A good reference is Springer's book [13], or Humphreys [4].

We begin with a pair $X, X^\vee$ of free abelian groups of finite rank, together with a perfect pairing $\langle\,,\,\rangle : X \times X^\vee \to \mathbb{Z}$. Suppose $\Delta \subset X, \Delta^\vee \subset X^\vee$ are finite sets, equipped with a bijection $\alpha \to \alpha^\vee$. For $\alpha \in \Delta$ define the reflection $s_\alpha \in \mathrm{Hom}(X, X)$:

$$s_\alpha(x) = x - \langle x, \alpha^\vee\rangle\alpha \quad (x \in X)$$

and $s_{\alpha^\vee} \in \mathrm{Hom}(X^\vee, X^\vee)$ similarly.

A *root datum* is a quadruple

$$D = (X, \Delta, X^\vee, \Delta^\vee) \tag{2.1}$$

where $X, X^\vee$ are free abelian groups of finite rank, in duality via a perfect pairing $\langle\,,\,\rangle$, and $\Delta, \Delta^\vee$ are finite subsets of $X, X^\vee$, respectively.

We assume there is a bijection $\Delta \ni \alpha \to \alpha^\vee \in \Delta^\vee$ such that for all $\alpha \in \Delta$,

$$\langle \alpha, \alpha^\vee\rangle = 2,\ s_\alpha(\Delta) = \Delta,\ s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee.$$

By [3, Lemma VI.1.1] (applied to $\mathbb{Z}\langle\Delta\rangle$ and $\mathbb{Z}\langle\Delta^\vee\rangle$) the conditions determine the bijection uniquely once Δ and $\Delta^\vee$ are given. In particular $(X, \Delta, X^\vee, \Delta^\vee)$ is determined by (X, Δ) if $\mathbb{R}\langle\Delta\rangle = X$. This condition holds if and only if the corresponding group is semisimple.

Suppose $D_i = (X_i, \Delta_i, X_i^\vee, \Delta_i^\vee)$ $(i = 1, 2)$ are root systems. They are isomorphic if there exists $\phi \in \mathrm{Hom}(X_1, X_2)$ satisfying $\phi(\Delta_1) = \Delta_2$ and $\phi^t(\Delta_2^\vee) = \Delta_1^\vee$. Here $\phi^t \in \mathrm{Hom}(X_2^\vee, X_1^\vee)$ is defined by

$$\langle \phi(x_1), x_2^\vee\rangle_2 = \langle x_1, \phi^t(x_2^\vee)\rangle_1 \quad (x_1 \in X_1, x_2^\vee \in X_2^\vee). \tag{2.2}$$

Let G be a connected reductive algebraic group and choose a Cartan subgroup H of G. Let $X^*(H), X_*(H)$ be the character and co-character groups of H respectively. Let $\Delta = \Delta(G, H)$ be the set of roots of H in G, and $\Delta^\vee = \Delta^\vee(G, H)$ the corrresponding co-roots. Associated to (G, H) is the root datum

$$(X^*(H), \Delta, X_*(H), \Delta^\vee).$$

If H' is another Cartan subgroup then the corresponding root datum is isomorphic to the given group, and this isomorphism is canonical up to the Weyl group $W(G, H)$.

Now suppose Δ^+ is a set of positive roots of Δ, with corresponding set of simple roots Π. Then $\Pi^\vee = \{\alpha^\vee \mid \alpha \in \Pi\}$ is a set of simple roots of $\Delta^\vee$, and

$$D_b = (X, \Pi, X^\vee, \Pi^\vee)$$

is a *based root datum.* Given G and H choose a Borel subgroup of G containing H. This defines a set of simple roots Π of Δ, and also a set of simple coroots $\Pi^\vee$. We obtain a based root datum $D_b = (X, \Pi, X^\vee, \Pi^\vee)$. Given another choice of $H' \subset B'$ there is a *canonical* isomorphism of based root data.

The root datum or based root datum of G determine G up to isomorphism [13].

If $D = (X, \Delta, X^\vee, \Delta^\vee)$ is a root datum then the dual root datum is $D^\vee = (X^\vee, \Delta^\vee, X, \Delta)$. Given G with root data $D = (X, \Delta, X^\vee, \Delta^\vee)$ the *dual group* is the group $G^\vee$ defined by $D^\vee$. We define duality of based root data similarly.

2.1 Automorphisms

There is an exact sequence

(2.3)(a) $$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

where $\mathrm{Int}(G)$ is the group of inner automorphisms of G, $\mathrm{Aut}(G)$ is the automorphism group of G, and $\mathrm{Out}(G) \simeq \mathrm{Aut}(G)/\mathrm{Int}(G)$ is the group of outer automorphisms. All automorphisms here are algebraic, or equivalently automorphisms of $G(\mathbb{C})$ as a complex Lie group.

A *splitting datum* or *pinning* for G is a set $S = (B, H, \{X_\alpha\})$ where B is a Borel subgroup, H is a Cartan subgroup contained in B and $\{X_\alpha\}$ is a set of root vectors for the simple roots defined by B.

An automorphism of G is said to be *distinguished* if it preserves a splitting datum. The only inner automorphism which is distinguished is the identity, and the group $\mathrm{Int}(G)$ acts simply transitively on the set of splitting data. Given a splitting datum $S = (B, H\{X_\alpha\})$ this gives an isomorphism

(2.3)(b) $$\phi_S : \mathrm{Out}(G) \simeq \mathrm{Stab}_{\mathrm{Aut}(G)}(S) \subset \mathrm{Aut}(G)$$

and this is a splitting of the exact sequence (2.3)(a). We obtain isomorphisms

(2.3)(c) $$\mathrm{Out}(G) \simeq \mathrm{Aut}(D_b) \simeq \mathrm{Aut}(D)/W.$$

If G is semisimple and simply connected or adjoint then $\mathrm{Out}(G)$ is isomorphic to the automorphism group of the Dynkin diagram of G.

If $\tau \in \mathrm{Aut}(D)$ then $-\tau^t \in \mathrm{Aut}(D^\vee)$ (cf. 2.2). Now suppose $\tau \in \mathrm{Aut}(D_b)$. While $-\tau^t$ is probably not in $\mathrm{Aut}(D_b^\vee)$, if we let w_0 be the long element of the Weyl group we have $-w_0\tau^t \in \mathrm{Aut}(D_b^\vee)$.

Definition 2.4 *Suppose $\tau \in Aut(D_b)$. Let $\tau^\vee = -w_0\tau^t \in Aut(D_b^\vee)$. This defines a bijection $Aut(D_b) \leftrightarrow Aut(D_b^\vee)$. By* (2.3)(c) *we obtain a bijection $Out(G) \leftrightarrow Out(G^\vee)$ by composition:*

$$Out(G) \leftrightarrow Aut(D_b) \leftrightarrow Aut(D_b^\vee) \leftrightarrow Out(G^\vee). \tag{2.5}$$

For $\gamma \in Out(G)$ we write $\gamma^\vee$ for the corresponding element of $Out(G^\vee)$. The map $\gamma \to \gamma^\vee$ is a bijection of sets.

Remark 2.6 This is not necessarily an isomorphism of groups. For example the identity goes to the image of $-w_0$ in $\mathrm{Out}(G)$, which is the identity if and only if $-1 \in W(G,H)$.

Example 2.7 Let $G = PGL(n)$ $(n \geq 3)$. Then $G^\vee = SL(n)$ and $\mathrm{Out}(G) \simeq \mathrm{Out}(G^\vee) \simeq \mathbb{Z}/2\mathbb{Z}$. If $\gamma = 1 \in \mathrm{Out}(G)$ then $\gamma^\vee$ is the non-trivial element of $\mathrm{Out}(G^\vee)$. It is represented by the automorphism $\tau^\vee \colon g \to {}^tg^{-1}$ of $G^\vee$. Note that $\tau^\vee(g) = g^{-1}$ for g in the diagonal Cartan subgroup of $G^\vee$.

3 Involutions of Reductive Groups

Suppose σ is an anti-holomorphic involution of $G(\mathbb{C})$. We say σ or $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ is a real form of $G(\mathbb{C})$.

We prefer to work with holomorphic involutions of $G(\mathbb{C})$, or equivalently involutions in $\mathrm{Aut}(G)$. This is the Cartan involution.

Definition 3.1 *An involution of G is an element $\theta \in Aut(G)$ satisfying $\theta^2 = 1$. Two involutions are equivalent if they are conjugate by an element of $Int(G)$.*

Thus two involutions θ, θ' are equivalent if $\theta' = \mathrm{int}(h) \circ \theta \circ \mathrm{int}(h^{-1})$ for some $h \in G$, i.e.

$$\theta'(g) = h\theta(h^{-1}gh)h^{-1} \quad \text{for all } g \in G. \tag{3.2}$$

This differs from the usual terminology in one subtle but important way: the usual definition allows conjugacy by all of $\mathrm{Aut}(G)$ rather than $\mathrm{Int}(G)$.

Suppose σ is a real form of $G(\mathbb{C})$ with $G(\mathbb{R}) = G(\mathbb{C})^\sigma$. Then there is an involution $\theta \in \mathrm{Aut}(G)$ such that $\theta\sigma = \sigma\theta$ and $G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$. Conversely given an involution $\theta \in \mathrm{Aut}(G)$ choose σ_0 so that $\theta\sigma_0 = \sigma_0\theta$ and $G(\mathbb{C})^{\sigma_0}$ is compact. Then $\sigma = \theta\sigma_0$ is a real form of $G(\mathbb{C})$, and if $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ then $G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$.

This defines a bijection between $G(\mathbb{C})$-conjugacy classes of anti-holomorphic involutions σ and $G(\mathbb{C})$-conjugacy classes of holomorphic involutions θ. See [7, Section VI.2].

Definition 3.3 *An involution $\theta \in Aut(G)$ (Definition 3.1) is in the inner class of $\gamma \in Out(G)$ if θ maps to γ in the exact sequence* (2.3)(a)*. If θ, θ' are involutions of G we say θ is inner to θ' if θ and θ' have the same image in $Out(G)$.*

This corresponds to the usual notion of inner form [13, 12.3.7].

Remark 3.4 The results of [2] are stated in terms of real forms (i.e. anti-holomorphic involutions. See Remark 5.11.

4 Basic Data

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Based on the preceding considerations, our basic data will be:

(1) A connected reductive algebraic group G,

(2) Cartan and Borel subgroups $H \subset B \subset G$,

(3) An involution $\gamma \in \mathrm{Out}(G)$.

Alternatively we may choose

(a) A based root datum $D_b = (X, \Pi, X^\vee, \Pi^\vee)$,

(b) An involution γ of D_b.

The relationship of (1-3) and (a-b) is given in Section 2.

A key to our algorithm is that H and B are fixed once and for all. This enables us to do all of our constructions on a fixed Cartan subgroup. We let $W = W(G, H)$ be the Weyl group.

By duality we obtain the dual based root datum $D_b^\vee$, $\gamma^\vee$ (cf. Definition 2.4), and the dual group $G^\vee$. In particular $G^\vee$ comes equipped with fixed Cartan and Borel subgroups $H^\vee \subset B^\vee$. We have $X^*(H) = X = X_*(H^\vee)$

and $X_*(H) = X^\vee = X^*(H^\vee)$. These are canonical identifications. They induce identifications $\mathfrak{h} = \mathfrak{h}^{\vee *}$ and $\mathfrak{h}^\vee = \mathfrak{h}^*$. We are also given the bijection $\Delta = \Delta(G, H) \ni \alpha \to \alpha^\vee \in \Delta^\vee = \Delta(G^\vee, H^\vee)$. We identify $W(G, H)$ with $W(G^\vee, H^\vee)$ by the map $W(G, H) \ni w \to w^t \in W(G^\vee, H^\vee)$ (2.2).

The weight lattice is

$$P = \{\lambda \in X^*(H) \otimes \mathbb{C} \,|\, \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\} \tag{4.1}$$

and dually the *co-weight lattice* is

$$P^\vee = \{\lambda^\vee \in X_*(H) \otimes \mathbb{C} \,|\, \langle \alpha, \lambda^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}. \tag{4.2)(a}$$

These are actually lattices only if G is semisimple. We write $P(G, H)$ and $P^\vee(G, H)$ to indicate the dependence on G and H. We may identify $2\pi i X_*(H)$ with the kernel of $\exp : \mathfrak{h} \to H(\mathbb{C})$. Under this identification

$$P^\vee = \{\lambda^\vee \in \mathfrak{h} \,|\, \exp(\lambda^\vee) \in Z(G)\} \tag{4.2)(b}$$

We also define

$$P_{\text{reg}} = \{\lambda \in P \,|\, \langle \lambda, \alpha^\vee \rangle \neq 0 \text{ for all } \alpha \in \Delta\} \tag{4.2)(c}$$

$$P^\vee_{\text{reg}} = \{\lambda^\vee \in P^\vee \,|\, \langle \alpha, \lambda^\vee \rangle \neq 0 \text{ for all } \alpha \in \Delta\} \tag{4.2)(d}$$

5 Extended Groups and Strong Involutions

Fix basic data (G, γ) as in Section 4. Let $\Gamma = \{1, \sigma\}$ be the Galois group of $\mathbb{C}/\mathbb{R}$.

Definition 5.1 *A weak extended group for (G, γ) is a semi-direct product $G \rtimes \Gamma$ where the automorphism int(σ) of G is distinguished (cf. Section 2.1), and the image of int(σ) in Out(G) is γ.*

An extended group for (G, γ) is a weak extended group G^Γ, together with a G-conjugacy class of splittings of the exact sequence

$$1 \to G \to G^\Gamma \to \Gamma \to 1. \tag{5.2}$$

The following Lemma follows immediately from the definitions.

Lemma 5.3 *A weak extended group G^Γ for (G,γ) contains an element δ satisfying*

(1) $G^\Gamma = \langle G, \delta\rangle$,

(2) $\delta^2 = 1$,

(3) int(δ) stabilizes a splitting datum $(B, H, \{X_\alpha\})$,

(4) the image of int(δ) in Out(G) is γ.

We say an element satisfying (1-4) is distinguished. An extended group is, in addition, a choice of G-conjugacy class of distinguished elements δ.a

The weak extended group for (G,γ) is unique up to isomorphism. If $\langle G, \delta\rangle, \langle G, \delta'\rangle$ are extended groups there is such an isomorphism ϕ, uniquely determined up to conjugation by G by the condition that $\phi(\delta)$ is G-conjugate to δ'. Thus we refer to *the* (weak) extended group G^Γ for (G,γ).

Example 5.4 Suppose $\gamma = 1$. Then the weak extended group is $G^\Gamma = G\times\Gamma$. The extended group is, in addition, a choice of element $\delta = (z,\sigma) \in G\times\Gamma$ where $z \in Z(G)$ and $z^2 = 1$.

The weak extended group G^Γ encapsulates all of the real forms of G in the inner class defined by γ. That is if $x \in G^\Gamma\backslash G$ satisfies $x^2 \in Z(G)$ then int(x) is in the inner class of γ. Conversely, if θ is in the inner class of γ, then there is an $x \in G^\Gamma\backslash G$ with $x^2 \in Z(G)$ and $\theta = \text{int}(x)$. For the algorithm it is important to keep track of x, and not just $\theta = \text{int}(x)$.

Definition 5.5 *A strong involution for (G,γ) is an element $x \in G^\Gamma\backslash G$ such that $x^2 \in Z(G)$. Two strong involutions are said to be equivalent if they are conjugate by an element of G. If x is a strong involution let $\theta_x = $ int(x). We say x is distinguished if θ_x is distinguished (cf. Section 2.1).*

Let $\mathcal{I}(G,\gamma)$ be the strong involutions for (G,γ).

If γ is understood we will say that x is a strong involution of G, and let $\mathcal{I} = \mathcal{I}(G,\gamma)$. The next Lemma is immediate from the definitions.

Lemma 5.6 *The map $\mathcal{I}(G,\gamma) \ni x \to \theta_x$ is a surjection from $\mathcal{I}(G,\gamma)$ to the set of involutions of G in the inner class of γ. It factors to a surjection*

$$\mathcal{I}(G,\gamma)/G \twoheadrightarrow \{\textit{involutions of } G \textit{ in the inner class of } \gamma\}/G \tag{5.7}$$

If G is adjoint this is a bijection, and the right hand side is in bijection with the set of real forms of G (cf. Section 3).

We will make frequent use of the following construction. Choose a set of representatives $\{x_i \,|\, i \in I\}$ of the set of equivalence classes of strong involutions. That is

(5.8)(a) $$\{x_i \,|\, i \in I\} = \mathcal{I}(G,\gamma)/G.$$

(Note that G does not act on $\mathcal{I}(G,\gamma)$; the right hand side means $\mathcal{I}(G,\gamma)$ modulo the equivalence relation: $x \simeq x'$ if x' is G-conjguate to x.) If G is semisimple I is a finite set. For $i \in I$ let

(5.8)(b) $$\theta_i = \mathrm{int}(x_i), \quad K_i = G^{\theta_i}.$$

Then the stabilizer of x_i in G is K_i and we have

(5.8)(c) $$\mathcal{I}(G,\gamma) \simeq \cup_{i\in I} G/K_i.$$

The automorphisms $\theta_\delta = \mathrm{int}(\delta)$ for δ as in Lemma 5.3 constitute a single G-conjugacy class of distinguished automorphisms. This is the Cartan involution of the "maximally compact" real form in the inner class of γ.

We say an involution of G is *quasisplit* if it is the Cartan involution of a quasisplit group. For a characterization of quasisplit involutions see [5, Proposition 6.24] (where they are called "principal"). By [5, Theorem 6.14] there is a unique conjugacy class of quasisplit involutions in this inner class.

Lemma 5.9 *Suppose G^Γ is a weak extended group for (G,γ). (1) There exists a strong involution $x \in G^\Gamma$ such that θ_x is distinguished. The involution θ_x is unique up to conjugacy by G.*
(2) There exists a strong involution x so that θ_x is a quasisplit involution. The involution θ_x is unique up to conjugacy by G.

Remark 5.10 The extended group G^Γ defined in [5, Definition 9.6] is defined in terms of a quasisplit involution, rather than a distinguished one. The equivalence of the two definitions is the content of [5, (9.7)]. This discussion also shows that, applied to $(G^\vee, \gamma^\vee)$, the group $G^{\vee\Gamma}$ is isomorphic to the L-group of G.

Remark 5.11 Since [2] works with real forms instead of involutions (cf. Remark 3.4), the extended groups G^Γ defined in [2, Chapter 3] are in terms of an anti-holomorphic involution. The resulsts are equivalent, but some translation is necessary between the two pictures.

6 Representations

Fix basic data (G, γ) as in Section 2. A representation of a strong involution x is a pair (x, π) where x is a strong involution of G and π is a $(\mathfrak{g}, K_x(\mathbb{C}))$-module. We say (x, π) is isomorphic to (x', π') if there exists $g \in G$ such that $gxg^{-1} = x'$ and $\pi^g \simeq \pi$.

The next example is a key one for understanding the formalism of representations of strong involutions.

Example 6.1 Suppose $G = SL(2)$. Then $\gamma = 1$, δ acts trivially on G and we may drop it from the notation. Let $x = \mathrm{diag}(i, -i)$. Then $K_x(\mathbb{C}) \simeq \mathbb{C}^*$ and the corresponding real group is isomorphic to $SL(2, \mathbb{R})$. Let π be a $(\mathfrak{g}, K_x(\mathbb{C}))$-module in the discrete series with the same infinitesimal character as the trivial representation. Then (x, π) is a representation of the strong involution x of G.

Now consider the representation (x, π^*) where π^* is the contragredientn representation. Then $\Pi = \{(x, \pi), (x, \pi^*)\}$ may be thought of as an L-packet of discrete series representations of $SL(2, \mathbb{R})$. The key point is that there exists $g \in G$ conjugating x to $-x$. By our notion of equivalence of representations of strong real forms we say (x, π^*) and $(-x, \pi^{*g})$ are equivalent. Now $\pi^{*g} \simeq \pi$ as a $(\mathfrak{g}, K_x)$-module. So another way to write our L-packet is $\Pi = \{(x, \pi), (-x, \pi)\}$. In this way Π is parametrized by the set $\pm x$.

More generally we also allow $x = \pm I$. Note that these element are not conjugate, and in each case $K_x = G$, so the corresponding involution is compact. Let σ be the trivial representation of G. Then (I, σ) and $(-I, \sigma)$ are trivial representations of distinct strong involutions of G.

Thus $\mathbf{\Pi} = \{(x, \pi), (-x, \pi), (I, \sigma), (-I, \sigma)\}$ is the set of discrete series representations of strong involutions of G, with trivial infinitesimal character, up to equivalence. Note that it is parametrized naturally by $\{x \in H \mid x^2 \in Z(G)\}$.

An infinitesimal character for $\mathfrak{g}$ may be identified, via the Harish-Chandra homomorphism, with a G-orbit of semisimple elements in $\mathfrak{g}^*$. Recall we have a fixed Cartan subgroup $H \subset G$. The $\mathfrak{g}^*/G$ is in bijection with $\mathfrak{h}^*/W$. For $\lambda \in \mathfrak{h}^*$ we write χ_λ for the corresponding infinitesimal character; note that $\chi_\lambda = \chi_{w\lambda}$ for all $w \in W$.

Definition 6.2 *We say λ and χ_λ are integral if $\lambda \in P$, and regular if $\langle \lambda, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta$.*

Given a strong involution x and λ we let $\mathcal{M}(x,\lambda)$ be the category of $(\mathfrak{g}, K_x(\mathbb{C}))$-modules with infinitesimal character λ.

Lemma 6.3 *Suppose λ, λ' are regular and $\lambda - \lambda' \in X^*(H)$. Then there is a canonical translation functor $\Psi_\lambda^{\lambda'} : \mathcal{M}(x,\lambda) \to \mathcal{M}(x,\lambda')$ which is an equivalence of categories. If (x,π) is an irreducible or standard module then so is $(x, \Psi_\lambda^{\lambda'}(\pi))$.*

Let W_λ be the integral Weyl group defined by λ, i.e. the Weyl group of the root system $\{\alpha \,|\, \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}$. After acting by W_λ we may assume $\langle \lambda, \alpha^\vee \rangle \in \mathbb{N}$ if and only if $\langle \lambda', \alpha^\vee \rangle \in \mathbb{N}$. Then $\Psi_\lambda^{\lambda'}$ is a standard translation functor. See [16].

Definition 6.4 *Fix a regular element $\lambda_0 \in \mathfrak{h}^*$. Let*

(6.5)(a) $$\mathcal{T}(\lambda_0) = \{\lambda \in \lambda_0 + X^*(H) \,|\, \lambda \textit{ is regular}\}$$

A translation family based at λ_0 is a set of representations $\{\pi(\lambda) \,|\, \lambda \in \mathcal{T}(\lambda_0)\}$ satisfying

(1) $\pi(\lambda) \in \mathcal{M}(x,\lambda) \quad (\lambda \in \mathcal{T}(\lambda_0))$

(2) $\Psi_\lambda^{\lambda'}(\pi(\lambda)) = \pi(\lambda') \quad (\lambda, \lambda' \in \mathcal{T}(\lambda_0))$.

We let $\mathcal{M}_t(x,\lambda_0)$ be the set of all translation familes based at λ_0.

7 L-data

Fix basic data (G,γ), with corresponding dual data $(G^\vee, \gamma^\vee)$ (cf. Section 4). Let $G^{\vee\Gamma}$ be the weak extended group associated to $(G^\vee,\gamma^\vee)$ (Definition 5.1). We begin by parametrizing admissible maps of the Weil group into $G^{\vee\Gamma}$.

So let $W_\mathbb{R}$ be the Weil group of $\mathbb{R}$. That is $W_\mathbb{R} = \langle \mathbb{C}^*, j\rangle$ where $jzj^{-1} = \overline{z}$ and $j^2 = -1$. An admissible homomorphism $\phi : W_\mathbb{R} \to G^\Gamma$ is a continuous homomorphism such that $\phi(\mathbb{C}^*)$ consists of semsimple elements and $\phi(j) \in G^\Gamma \backslash G$.

Suppose $\phi : W_\mathbb{R} \to G^\Gamma$ is an admissible homomorphism. Then $\phi(\mathbb{C}^*)$ is contained in a Cartan subgroup $H_1(\mathbb{C})$ and $\phi(z) = \exp(2\pi i(\lambda z + \nu\overline{z}))$ for some $\lambda, \nu \in \mathfrak{h}_1$. We define the *infinitesimal character* of ϕ to be the G-orbit of λ. We say the infinitesimal character of ϕ is integral if $\lambda \in P$ (4.2)(a).

Definition 7.1 *(1) A one-sided L-datum for* $(G^\vee,\gamma^\vee)$ *is a pair* $(y,B_1^\vee)$ *where* y *is a strong involution of* $G^\vee$ *(Definition 5.5) and* $B_1^\vee$ *is a Borel subgroup of* $G^\vee$*.*

(2) A complete one-sided L-datum for (G,γ) *is a set* $(y,B_1^\vee,\lambda)$ *with* $(y,B_1^\vee)$ *as in (1) and* $\lambda\in P^\vee_{reg}$ *(cf.* (4.2)(d)*) satisfies* $\exp(2\pi i\lambda)=y^2$*.*

Let

$$\begin{aligned}\mathcal{P}(G^\vee,\gamma^\vee)&=\{\textit{one-sided L-data}\}/G^\vee\\ \mathcal{P}_c(G^\vee,\gamma^\vee)&=\{\textit{complete one-sided L-data}\}/G^\vee\end{aligned}\tag{7.2}$$

If $S=(y,B_1^\vee)$ is a one-sided L-datum we let $z(S)=y^2\in Z(G^\vee)$. This gives a well defined map

$$\mathcal{P}(G^\vee,\gamma^\vee)\ni S\to z(S)\in Z(G^\vee).\tag{7.3}$$

Fix a complete one-sided L-datum $(y,B_1^\vee,\lambda)$. By [9], also see [2, Lemma 6.18] there is a θ_y-stable Cartan subgroup $H_1^\vee$ of $B_1^\vee$, unique up to conjugacy by $K_y^\vee\cap B_1^\vee$. Choose $g\in G^\vee$ such that $gH^\vee g^{-1}=H_1^\vee$ and $\langle\mathrm{Ad}(g)\lambda,\alpha^\vee\rangle\ge 0$ for all $\alpha\in\Delta(B_1^\vee,H_1^\vee)$. Let $\lambda_1=\mathrm{Ad}(g)\lambda\in\mathfrak{h}_1^\vee$.

Define $\phi:W_{\mathbb{R}}\to G^\Gamma$ by:

$$\begin{aligned}\phi(z)&=z^{\lambda_1}\overline{z}^{y\lambda_1}\\ \phi(j)&=\exp(-\pi i\lambda_1)y\end{aligned}\tag{7.4}$$

The first statement is shorthand for $\phi(e^z)=\exp(z\lambda_1+\mathrm{Ad}(y)\overline{z}\lambda_1)$. It is easy to see ϕ is an admissible homomorphism, and the $G^\vee$ conjugacy class of ϕ is independent of the choice of H_1 and g. The next result follows easily.

Proposition 7.5 *There is a natural bijection between* $\mathcal{P}_c(G^\vee,\gamma^\vee)$ *and* $G^\vee$*-conjugacy classes of admissible homomorphisms* $\phi:W_{\mathbb{R}}\to G^{\vee\Gamma}$ *with integral infinitesimal character (Definition 6.2).*

By [8] $G^\vee$-conjugacy classes admissible homomorphisms $\phi:W_{\mathbb{R}}\to G^{\vee\Gamma}$ parametrize "L-packets" of representations of real forms of G. As in (5.8) choose a set $\{x_i\,|\,i\in I\}$ of representatives of equivalence classes of strong involutions. We define an L-packet to be the union, over $i\in I$ of L-packets for the strong involution x_i (which may be empty). We obtain:

Corollary 7.6 *There is a natural bijection between* $\mathcal{P}(G^\vee,\gamma^\vee)$ *and the set of translation families of L-packets of representations of strong involutions of* G *with regular integral infinitesimal character.*

For the purposes of this Proposition we have defined one-sided L-data $\mathcal{P} = \mathcal{P}(G^\vee, \gamma^\vee)$ for $(G^\vee, \gamma)$. It is evident that the definition is symmetric, and applies equally to (G, γ).

Definition 7.7 *An L-datum for* (G, γ) *is a set*

$$(x, B_1, y, B_1^\vee) \tag{7.8}$$

where x is a strong real form of G, y is a strong real form of $G^\vee$, B_1 is a Borel subgroup of G and $B_1^\vee$ is a Borel subgroup of $G^\vee$, satisfying

$$\theta^t_{x,\mathfrak{h}} = -\theta_{y,\mathfrak{h}^\vee} \tag{7.9}$$

A complete L-datum is a set

$$(x, B_1, y, B_1^\vee, \lambda) \tag{7.10}$$

where the same conditions hold, $\lambda \in P^\vee_{reg}$ and $\exp(2\pi i\lambda) = y^2$.

Let

$$\begin{aligned} \mathcal{L} &= \{\textit{L-data}\}/G \times G^\vee \\ \mathcal{L}_c &= \{\textit{complete L-data}\}/G \times G^\vee \end{aligned} \tag{7.11}$$

Note that

$$\begin{aligned} \mathcal{L} &\subset \mathcal{P}(G, \gamma) \times \mathcal{P}(G^\vee, \gamma^\vee) \\ \mathcal{L}_c &\subset \mathcal{P}_c(G, \gamma) \times \mathcal{P}_c(G^\vee, \gamma^\vee). \end{aligned} \tag{7.12}$$

If $S = (x, B_1, y, B_1^\vee)$ is an L-datum we let $S_G = (x, B_1)$ and $S_G^\vee = (y, B_1^\vee)$. These are one-sided L-data for (G, γ), and $(G^\vee, \gamma^\vee)$ respectively.

Suppose $S_c = (x, B_1, y, B_1^\vee, \lambda)$ is a set of complete L-data for (G, γ). By [1, Theorem 2.12] associated to S_c is a $(\mathfrak{g}, K_x)$-module $I(S_c)$. This is standard module, with regular integral infinitesimal character λ, and has a unique irreducible quotient $J(S_c)$.

Theorem 7.13 *The map*

$$\mathcal{L}_c \ni S_c \to J(S_c) \tag{7.14}$$

is a bijection between $\mathcal{L}_c$ and representations of strong involutions of G with regular integral infinitesimal character (Definition 6.2).

We give several alternative formulations of this result.

Suppose $S = (x, B_1, y, B_1^\vee)$ is a set of L-data. Note that $S_c = (S, \lambda)$ is a set of complete L-data provided $\exp(2\pi i\lambda) = z(S_G^\vee)$ (cf. (7.3)). Consider the map

$$\mathcal{L} \ni S \to \{J(S,\lambda) \mid \exp(2\pi i\lambda) = z(S_{G^\vee}), \lambda \text{ regular}\}. \tag{7.15}$$

This is a translation family of representations (Definition 6.4).

Corollary 7.16 *(1) The map* (7.15) *induces a bijection between $\mathcal{L}$ and translation families of irreducible representations of strong involutions of G with regular integral infinitesimal character.*

(2) Fix a set $\Lambda \subset P^\vee_{reg}$ of representatives of $P/X^*(H)$. The map (7.15) *induces a bijection between $\mathcal{L}$ and the union, over $\lambda \in \Lambda$, of irreducible representations of strong real forms of G with infinitesimal character λ.*

(3) Suppose G is semisimple and simply connected . Then the map (7.15) *induces a bijection between $\mathcal{L}$ and the irreducible representations of strong involutions of G with infinitesimal character the same as that of the trivial representation.*

(4) Suppose G is adjoint, and fix a set $\Lambda \subset P_{reg}$ of representatives of P/R. Then the map (7.15) *induces a bijection between $\mathcal{L}$ and the and the irreducible representations of real forms of G, with infinitesimal character in Λ.*

8 Relation with the flag variety

The one-sided parameter space $\mathcal{P} = \mathcal{P}(G, \gamma)$ has a natural interpretation in terms of the flag variety. We see this by conjugating any pair (x, B_1) to one with x in a set of representatives of strong involutions. In the next section we will instead conjugate B_1 to B, and thereby obtain a combinatorial model of $\mathcal{P}$.

Now let $\mathcal{B}$ be the set of Borel subgroups of G. Then the set of one-sided L-data for (G, γ) is $\mathcal{I} \times \mathcal{B}$, and

$$\mathcal{P} = (\mathcal{I} \times \mathcal{B})/G. \tag{8.1}$$

Every Borel subgroup is conjugate to B, so $\mathcal{B} \simeq G/B$.

As in (5.8) choose a set $\{x_i \mid i \in I\}$ of representatives of equivalence classes of strong involutions. Then

$$\mathcal{P} \simeq \cup_{i\in I}(G/K_i \times G/B)/G \tag{8.2}$$

with G acting by multiplication on G/K_i and G/B. It is now an elementary exercise to see the map

$$(x, B_1) = (gx_ig^{-1}, hBh^{-1}) \to K_i(g^{-1}h)B \quad (g, h \in G) \tag{8.3}$$

is a bijection:

Proposition 8.4 *There is a canonical bijection*

$$\mathcal{P} \leftrightarrow \cup_{i\in I} K_i\backslash G/B. \tag{8.5}$$

Let us apply this Proposition to $(G^\vee, \gamma^\vee)$.

9 The One Sided Parameter Space

Fix basic data (G, γ), and let G^Γ be a weak extended group (Definition 5.1) as usual. We turn now to the question of formulating an effective algorithm for computing $\mathcal{P} = \mathcal{P}(G, \gamma)$. We begin by looking for a normal form for one-sided L-data.

Recall $\mathcal{P} = (\mathcal{I} \times \mathcal{B})/G$. Since every Cartan subgroup is conjugate to B, every element of $\mathcal{I} \times \mathcal{B}$ may be conjugated to one of the form (x, B). Thatis the map

$$\mathcal{I} \ni x \to (x, B) \in (\mathcal{I} \times \mathcal{B})/G = \mathcal{P} \tag{9.1)(a}$$

is surjective. Since B is its own normalizer, we see (x, B) is G-conjugate to (x', B) if and only if x is B-conjugate to B'. So we obtain a bijection

$$\mathcal{I}/B \in x \to (x, B) \in \mathcal{P}. \tag{9.1)(b}$$

Suppose $x \in \mathcal{I}$. By [9] $x \in \mathrm{Norm}_{G^\Gamma}(H_1)$ for some Cartan subgroup $H_1 \subset B$. There exists $b \in B$ such that $bH_1b^{-1} = H$, so $bxb^{-1} \in N^\Gamma$. If b_1 is another such element then $b_1 = hb$ with $h \in H$, and $b_1 x b_1^{-1} = h(bxb^{-1})h^{-1}$. Therefore

$$\text{(9.1)(c)} \qquad \mathcal{I}/B \simeq (\mathcal{I} \cap N^\Gamma)/H$$

and by (a) and (c) we have

$$\text{(9.1)(d)} \qquad \mathcal{P} \simeq (\mathcal{I} \cap N^\Gamma)/H$$

This gives our primary combinatorial construction:

Definition 9.2 *Let*

$$\text{(9.3)} \qquad \begin{aligned} \mathcal{X}(G,\gamma) &= (\mathcal{I} \cap N^\Gamma)/H \\ &= \{x \in \mathrm{Norm}_{G^\Gamma \backslash G}(H) \,|\, x^2 \in Z(G)\}/H \end{aligned}$$

the set of strong involutions normalizing H, modulo conjugation by H. By $\mathrm{Norm}_{G^\Gamma \backslash G}(H)$ we mean $\{g \in \mathrm{Norm}_{G^\Gamma}(H) \,|\, g \in G^\Gamma \backslash G\}$. If (G,γ) are understood we write $\mathcal{X} = \mathcal{X}(G,\gamma)$.

From the preceding discussion we conclude:

Proposition 9.4 *The map*

$$\text{(9.5)} \qquad \mathcal{X} \ni x \to (x, B) \in \mathcal{P}$$

is a bijection.

Let $I \leftrightarrow \mathcal{I}/G, \theta_i$ and K_i be as in (5.8). By Proposition 8.4 we obtain a combinatorial description of the flag variety:

Corollary 9.6 *There is a canonical bijection*

$$\text{(9.7)} \qquad \cup_i K_i \backslash G/B \leftrightarrow \mathcal{X}$$

From Corollary 7.6 we see

Proposition 9.8 *There is a canonical bijection between $\mathcal{X}(G^\vee, \gamma^\vee)$ and the set of translation families of L-packets for strong involutions of G with regular integral infinitesimal character.*

We need to understand the structure of $\mathcal{X}(G,\gamma)$ in some detail. We now give more information about it. At the same time we reiterate some earlier definitions and introduce the twisted involutions in the Weyl group.

We fix (G,γ) throughout and drop them from the notation.

Let

(9.9)(a) $$N^\Gamma = \mathrm{Norm}_{G^\Gamma}(H), \quad N = \mathrm{Norm}_G(H)$$

and

(9.9)(b) $$W^\Gamma = N^\Gamma/H, \quad W = N/H.$$

The group N^Γ acts on H by conjugation, and this action factors to W^Γ. Restricted to $W \subset W^\Gamma$ this is the usual Weyl group action. We have a natural embedding

(9.9)(c) $$W^\Gamma \backslash W \hookrightarrow \mathrm{Aut}(H).$$

and an exact sequence

(9.9)(d) $$1 \to H \to N^\Gamma \to W^\Gamma \to 1.$$

This is equivariant for the action of N; on H and W^Γ this action factors to W.

Recall (Definition 5.5)

(9.9)(e) $$\mathcal{I} = \{x \in G^\Gamma \backslash G \mid x^2 \in Z(G)\},$$

and that G acts on $\mathcal{I}$ by conjugation. Let

(9.9)(f) $$\begin{aligned}\widetilde{\mathcal{X}} &= \mathcal{I} \cap N^\Gamma \\ &= \{x \in N^\Gamma \backslash N \mid x^2 \in Z(G)\}.\end{aligned}$$

These are the strong involutions in N^Γ (see Definition 5.5). This carries an action of N by conjugation. Let

(9.9)(g) $$\mathcal{X} = \widetilde{\mathcal{X}}/H$$

as in (9.3). The action of N on $\widetilde{\mathcal{X}}$ factors to an action of W on $\mathcal{X}$. Every strong involution is conjugate to one in $\widetilde{\mathcal{X}}$, and we see

(9.9)(h) $$\widetilde{\mathcal{X}}/N \simeq \mathcal{X}/W.$$

See Proposition 11.9 for an interpretation of this space.

For $\tilde{x} \in \widetilde{\mathcal{X}}$ the restriction of $\theta_{\tilde{x}}$ only depends on the image x of $\tilde{x}$ in $\mathcal{X}$. Therefore we may define

(9.9)(i) $$\theta_{x,H} = \theta_{\tilde{x}} \text{ restricted to } H.$$

Let R_H denote multiplication on the right by H, and let

(9.9)(j) $$\mathcal{I}_W = \widetilde{\mathcal{X}}/R_H = \mathcal{X}/R_H.$$

Let $\tilde{p} : \widetilde{\mathcal{X}} \to \mathcal{I}_W$ and $p : \mathcal{X} \to \mathcal{I}_W$ be the natural maps. Then for $x_1, x_2 \in \mathcal{X}$,

(9.9)(k) $$\theta_{x_1,H} = \theta_{x_2,H} \Leftrightarrow p(x_1) = p(x_2).$$

So for $\tau \in \mathcal{I}_W$ define

(9.9)(l) $$\begin{aligned} \theta_{\tau,H} &= \theta_{x,H} \text{ where } p(x) = \tau \\ &= \theta_{\tilde{x}}|_H \text{ where } \tilde{p}(\tilde{x}) = \tau. \end{aligned}$$

We have $\mathcal{I}_W = (\mathcal{I} \cap N^\Gamma)/H$, which shows

(9.9)(m) $$\mathcal{I}_W = \{w \in W^\Gamma \backslash W \mid w^2 = 1\}.$$

It carries an action of W, and the maps $p, \tilde{p}$ are equivariant for the action of N. By (9.9)(n) $\mathcal{I}_W$ may be thought of as the set of Cartan involutions of H:

(9.9)(n) $$\mathcal{I}_W \leftrightarrow \{\theta_{x,H} \mid x \in \mathcal{X}\}$$

The map $\tilde{x} \to \tilde{x}^2 \in Z(G)$ is constant on fibers of the map $\widetilde{\mathcal{X}} \to \mathcal{X}$. For $x \in \mathcal{X}$ we define $x^2 \in Z(G)$ accordingly.

Fix $z \in Z(G)$. Let

(9.9)(o) $$\mathcal{X}(z) = \{x \in \mathcal{X} \mid x^2 = z\}.$$

Define $\theta \in \mathrm{Aut}(Z)$ to be $\mathrm{int}(\delta)$ for any $\delta \in G^\Gamma \backslash G$; this is independent of the choice of δ. Note that $\mathcal{X}(z)$ is empty unless $z \in Z^\theta$.

We can make these constructions more concrete using the splitting of (5.2), i.e. an element δ as in Lemma 5.3. Let $\theta = \mathrm{int}(\delta)$. Then

$$
\begin{aligned}
\widetilde{\mathcal{X}} &= \{x \in N\delta \,|\, x^2 \in Z(G)\} \\
&= \{g\delta \,|\, g \in N,\, g\theta(g) \in Z(G)\} \\
&\leftrightarrow \{g \in N \,|\, g\theta(g) \in Z(G)\} \\
\mathcal{X} &= \widetilde{\mathcal{X}}/\{g\delta \to hg\theta(h^{-1})\delta \,|\, h \in H\} \\
&\leftrightarrow \{g \in N \,|\, g\theta(g) \in Z(G)\}/\{g \to hg\theta(h^{-1}) \,|\, h \in H\} \\
\mathcal{I}_W &= \{\tau \in W\delta \,|\, \tau^2 = 1\} \\
&= \{w\delta \,|\, w \in W, w\theta(w) = 1\} \\
&\leftrightarrow \{w \in W \,|\, w\theta(w) = 1\}
\end{aligned} \tag{9.10}
$$

10 Fibers of the map $\phi : \mathcal{X} \to \mathcal{I}$

We continue with our basic data (G, γ) and $\mathcal{X}$. It is important to understand the fibers of $\widetilde{\phi}$ and ϕ. For $\tau \in \mathcal{I}_W$ let $\widetilde{\mathcal{X}}_\tau = \widetilde{\phi}^{-1}(\tau)$ and $\mathcal{X}_\tau = \phi^{-1}(\tau)$. Recall $\theta_x|_H = \tau$, independent of $x \in \mathcal{X}_\tau$.

Proposition 10.1 *Fix $\tau \in \mathcal{I}_W$. Let*

(10.2)(a) $$H_1(\tau) = \{h \in H \,|\, h\tau(h) \in Z(G)\}$$

and

(10.2)(b) $$H_2(\tau) = \{h \in H \,|\, h\tau(h) = 1\} \subset H_1(\tau)$$

Let

(10.2)(c) $$H_2(\tau)^0 = \{h\tau(h^{-1}) \,|\, h \in H\}.$$

This is the identity component of $H_2(\tau)$. Then

(1) $H_1(\tau)$ acts simply transitively on $\widetilde{\mathcal{X}}_\tau$,

(2) $H_1(\tau)/H_2(\tau)^0$ acts simply transitively on $\mathcal{X}_\tau$,

(3) Fix $z \in Z(G)$. If $\mathcal{X}_\tau(x)$ is non-empty then $H_2(\tau)/H_2(\tau)^0$ acts simply transitively on $\mathcal{X}_\tau(z)$. If $z \notin Z(G)^\tau$ then $\mathcal{X}_\tau(z)$ is empty.

In particular for each $z \in Z(G)$, $|\mathcal{X}_\tau(z)|$ is a power of 2. If G has no compact central torus then then $\mathcal{X}$ is a finite set.

Proof. Choose $\tilde{x} \in \widetilde{\mathcal{X}}_\tau$. Then $\widetilde{\mathcal{X}}_\tau = \{h\tilde{x} \,|\, h \in H, (h\tilde{x})^2 \in Z(G)\} = \{h\tilde{x} \,|\, h\tau(h)\tilde{x}^2 \in Z(G)\}$. The first claim follows.

Because H is connected the image of the map $h \to h\tau(h^{-1})$ is contained in $H_2(w)^0$. On the other hand if $h \in H_2(w)$ then $h\tau(h^{-1}) = h^2$, so the image of this map is all of $H_2(\tau)^0$.

For $h \in H$ we have $h\tilde{x}h^{-1} = h\tau(h^{-1})x$. This shows that the stabilizer in $H_1(\tau)$ of the image of $\tilde{x}$ in $\mathcal{X}$ is $\{h\tau(h^{-1}) \,|\, h \in H\} = H_2(\tau)^0$. This proves (2), and (3) follows immediately from the fact that $(hx)^2 = h\tau(h)x^2$. The final assertions are clear, since $H_2(\tau)/H_2(\tau)^0$ is an elementary abelian two-group. qed

Remark 10.3 We introduce some alternative notation, which will play a role in Section 14. Let

(10.4)
$$\begin{aligned} H^\tau &= \{h \in H \,|\, \tau(h) = h\} \\ H^{-\tau} &= \{h \in H \,|\, \tau(h) = h^{-1}\} = H_2(\tau) \\ T_\tau &= (H^\tau)^0 \\ A_\tau &= (H^{-\tau})^0 \end{aligned}$$

Note that $H = T_\tau A_\tau$ and $A_\tau \cap T_\tau$ is an elementary abelian two group. Then the group in (3) is

(10.5)
$$(H^{-\tau})^0/H^{-\tau} \simeq T_\tau(2)/A_\tau \cap T_\tau$$

If we write the real torus corresponding to τ as $\mathbb{R}^{\times a} \times S^{1b} \times \mathbb{C}^{\times c}$ then this is isomorphic to $\mathbb{Z}/2\mathbb{Z}^b$.

Remark 10.6 We give two alternative descriptions of the set in (3). Let Γ act on H with the non-trivial element acting by τ. Let $H^\vee$ be the dual torus of H, and let $H^\vee(\mathbb{R})$ be the involution corresponding to $-\tau^\vee$. Then

(10.7)
$$\begin{aligned} H_2(\tau)/H_2(\tau)^0 &\simeq H^1(\Gamma, H) \\ &\simeq (H^\vee(\mathbb{R})/H^\vee(\mathbb{R})^0)^\wedge \end{aligned}$$

10.1 Root Systems and the Weyl group

It is convenient to collect some definitions and terminology.

Fix $\tau \in \mathcal{I}_W$. Let

$$
\begin{aligned}
\Delta_i &= \{\alpha \in \Delta \mid \tau(\alpha) = \alpha\} \text{ (the imaginary roots)}\\
\Delta_r &= \{\alpha \in \Delta \mid \tau(\alpha) = -\alpha\} \text{ (the real roots)}\\
\Delta_{cx} &= \{\alpha \in \Delta \mid \tau(\alpha) \neq \pm\alpha\} \text{ (the complex roots)}\\
\Delta_i^+ &= \Delta_i \cap \Delta^+,\ \Delta_r^+ = \Delta_r \cap \Delta^+\\
W_i &= W(\Delta_i)\\
W_r &= W(\Delta_r)
\end{aligned} \tag{10.8}
$$

We also let $\rho_i = \frac{1}{2}\sum_{\alpha\in\Delta_i^+}\alpha$, and $\rho_r^\vee = \frac{1}{2}\sum_{\alpha\in\Delta_r^+}\alpha^\vee$. As in [15, Proposition 3.12] let

$$
\begin{aligned}
\Delta_C &= \{\alpha \in \Delta \mid \langle\rho_i, \alpha^\vee\rangle = \langle\alpha, \rho_r^\vee\rangle\} = 0 \subset \Delta_{cx}\\
W_C &= W(\Delta_C)
\end{aligned} \tag{10.9}
$$

Now τ acts on W, and we let W^τ be the fixed points. By [15, Proposition 3.12]

$$W^\tau = (W_C)^\tau \ltimes (W_i \times W_r). \tag{10.10}$$

Suppose $\alpha \in \Delta_i$ and choose $\tilde{x} \in \widetilde{\mathcal{X}}_\tau$. If X_α is an α root vector $\theta_{\tilde{x}}(X_\alpha)$ only depends on the image x of $\tilde{x}$ in $\mathcal{X}$. We say $\mathrm{gr}_x(\alpha) = 0$ if $\theta_x(X_\alpha) = X_\alpha$ and 1 otherwise. This is a $\mathbb{Z}/2\mathbb{Z}$-grading of Δ_i in the sense that if $\alpha, \beta, \alpha+\beta \in \Delta_i$ then $\mathrm{gr}_x(\alpha+\beta) = \mathrm{gr}_x(\alpha) + \mathrm{gr}_x(\beta)$.

Let $W(K,H) = \mathrm{Norm}_K(H)/H\cap K$. This is isomorphic to $W(G(\mathbb{R}), H(\mathbb{R}))$ where $G(\mathbb{R})$ is the real form of G corresponding to θ, and we call it the real Weyl group (as opposed to the Weyl group of the real roots). Clearly $W(K,H) \subset W^\theta$. Let A be the identity component of $\{h \in H \mid \tau(h) = h^{-1}\}$ and $M = \mathrm{Cent}_G(A)$. By [15, Proposition 4.16],

$$W(K,H) = (W_C)^\theta \ltimes (W(M\cap K, H) \times W_r) \tag{10.11}$$

Furthermore

$$W(M\cap K, H) \simeq W_{i,c} \ltimes \mathcal{A}(H) \tag{10.12}$$

where $W_{i,c}$ is the Weyl group of the compact imaginary roots and $\mathcal{A}(H)$ is a certain two-group [15]. This describes $W(K, H)$ in terms of the Weyl groups $(W_C)^\theta$, W_r and $W_{i,c}$, which are straightforward to compute, and the two-group $\mathcal{A}(H)$. For more information on $\mathcal{A}(H)$ see Proposition 11.12 and Remark 11.14.

11 Action of W on $\mathcal{X}$

We now study the action of W on $\mathcal{X}$, which plays an important role.

Let

(11.1) $$\mathcal{H} = \{(x, H_1) \,|\, x \in \mathcal{I}, H_1 \text{ a } \theta_x - \text{stable Cartan subgroup}\}/G.$$

With $I \leftrightarrow \mathcal{I}/G, \theta_i$ and K_i as in (5.8) we have

(11.2) $$\mathcal{H} = \cup_i \{\theta_i\text{-stable Cartan subgroups of } G\}/K_i.$$

On the other hand every Cartan subgroup is conjugate to H, and the normalizer of H is N, so

(11.3) $$\mathcal{H} \leftrightarrow \mathcal{I} \cap N^\Gamma/N.$$

Recall $\mathcal{I} \cap N^\Gamma = \widetilde{\mathcal{X}}$, so

(11.4) $$(\mathcal{I} \cap N^\Gamma)/N = \widetilde{\mathcal{X}}/N \simeq \mathcal{X}/W.$$

Given $i \in I$ let

(11.5) $$\mathcal{X}_i = \{x \in \mathcal{X} \,|\, x \text{ is } G\text{-conjugate to } x_i\},$$

the strong involutions in $\mathcal{X}$ equivalent to x_i. We conclude

Proposition 11.6 *For each $i \in I$ we have*

(11.7) $$\mathcal{X}_i/W \leftrightarrow \{\theta_i\text{-stable Cartan subgroups of } G\}/K_i.$$

Taking the union over $i \in I$ gives

(11.8) $$\mathcal{X}/W \leftrightarrow \cup_i \{\theta_i\text{-stable Cartan subgroups of } G\}/K_i$$

Recall that by Proposition (8.4) $\mathcal{X}_i \simeq K_i \backslash G/B$.

Proposition 11.9 *The map $p : \mathcal{X}_i/W \to \mathcal{I}_W/W$ is injective. If θ_i is quasisplit it is a bijection.*

Remark 11.10 The map $\mathcal{X}_i/W \to \mathcal{I}_W/W$ is discussed in [11].

Remark 11.11 The Lemma says that the conjugacy classes of Cartan subgroups of a real form of G embed in those of the quasisplit form. See [10].

Proof. For injectivity we have to show that $x, x' \in \mathcal{X}$, $p(x) = p(x')$ and $x' = gxg^{-1}$ ($g \in G$) implies $g' = nxn^{-1}$ for some $n \in N$. The condition $p(x) = p(x')$ implies $x' = hx$ for some $h \in H$, and then $x' = gxg^{-1}\ h = g\theta_x(g^{-1})$. By [11, Proposition 2.3] there exists $n \in N$ satisfying $h = n\theta_x(n^{-1})$, implying $x' = nxn^{-1}$.

We defer the proof of surjectivity in the quasisplit case until we have the machinery of Cayley transforms.
qed

It is easy to interpret the real Weyl group (Section 10.1) in our setting. Fix $x \in \mathcal{X}$ and let $K_x = G^{\theta_x}$.

Proposition 11.12 $W(K_x, H) \simeq \mathit{Stab}_W(x)$

Proof. Choose a pre-image $\widetilde{x} \in \widetilde{\mathcal{X}}$ of $x \in \mathcal{X}$. Then

$$\begin{aligned} W(K,H) &= \mathrm{Norm}_K(H)/H \cap K \\ &= \mathrm{Stab}_N(\widetilde{x})/\mathrm{Stab}_H(\widetilde{x}) \\ &= \mathrm{Stab}_N(\widetilde{x})H/H. \end{aligned} \tag{11.13}$$

It is easy to see that $\mathrm{Stab}_N(\widetilde{x})H = \mathrm{Stab}_N(x)$, so this equals

$$\mathrm{Stab}_N(x)/H \simeq \mathrm{Stab}_{N/H}(x) = \mathrm{Stab}_W(x).$$

qed

Remark 11.14 Recall the computation of $W(K,H)$ comes down to the computation of a certain elementary abelian two-group $\mathcal{A}(H)$ (cf. (10.12)). We use Proposition 11.12 to compute $\mathcal{A}(H)$.

Now fix $\tau \in \mathcal{I}_W$. By Proposition 11.12 and (10.11) we see $(W_C)^\tau$ and W_r act trivially on $\mathcal{X}_\tau$. It is worth noting that we can see this directly.

Proposition 11.15 *Both $(W_C)^\tau$ and W_r act trivially on $\mathcal{X}_\tau$.*

Proof. Fix $\tilde{x} \in \widetilde{\mathcal{X}}_\tau$. The group $(W_C)^\tau$ is generated by elements $s_\alpha s_{\tau\alpha}$ where $\alpha \in \Phi_C$. So suppose $\alpha \in \Phi_C$ and let $\sigma_\alpha \in N$ be a preimage of $s_\alpha \in W$. Let $\sigma_{\tau(\alpha)} = \tilde{x}\sigma_\alpha\tilde{x}^{-1}$. Note that $\alpha + \tau(\alpha)$ is not a root, since it would have to be imaginary, and (by (10.9)) orthogonal to ρ_i. Therefore the root subgroups G_α and $G_{\tau(\alpha)}$ commute. Then $\tilde{x}\sigma_\alpha\sigma_{\tau(\alpha)}\tilde{x}^{-1} = \sigma_{\tau(\alpha)}\sigma_\alpha = \sigma_\alpha\sigma_{\tau(\alpha)}$.

If α is a real root with respect to τ this reduces easily to a computation in $SL(2)$. We omit the details.
qed

Another useful result obtained from the action of W^θ is the computation of strong involutions. Choose a distinguished element $\delta \in \mathcal{X}$ (cf. Lemma 5.3) and let $\tau = p(\delta) \in \mathcal{I}_W$. By Lemma 5.9 the W-conjugacy class of τ is independent of the choice of δ.

Every real form of G in the given inner class contains a unique "fundamental" (most compact) Cartan subgroup. In our setting this amounts to the fact that every $x \in \mathcal{X}$ is G-conjugate to an element of $\mathcal{X}_\tau$.

Proposition 11.16 *There is a canonical bijection between $\mathcal{X}_\tau/W^\tau$ and equivalence classes of strong involutions of G.*

12 The reduced parameter space

If G is adjoint the parameter space $\mathcal{X} = \mathcal{X}(G,\gamma)$ perfectly captures the representation of real forms (equivalently involutions) of G. If G is not adjoint then strong involutions play an essential role, and the difference between involutions and strong involutions is inescapable. Nevertheless in some respects the space $\mathcal{X}$ is larger than necessary, and a satisfactory theory is obtained with a quotient, the *reduced one-sided parameter space*. In particular this is always a finite set.

There is a natural action of $Z = Z(G)$ on $\mathcal{X}$ by left multiplication. This preserves the fibers $\mathcal{X}_\tau$, and commutes with the conjugation action of G. For $x \in \mathcal{X}$ and $z \in Z$ multiplication by z is a bijection between $\{x' \in \mathcal{X} \,|\, x' \text{ is conjugate to } x\}$ and $\{x'' \in \mathcal{X} \,|\, x'' \text{ is conjugate to } zx\}$. In our parametrization of representations (cf. Section 15) this will amount to the same representations of two different strong involutions, corresponding to the same ordinary involution (or real form). In other words the orbit pictures for x and zx are identical.

For example suppose $G = SL(2)$ and take $x = I$ and $zx = -I$. Then x and zx both correspond to the compact group $SU(2)$, and we are simply getting the trivial representation of $SU(2)$, counted twice. See the $SL(2)$ example in Section 16.

Recall (following (9.9)(o)) $\theta \in \mathrm{Aut}(Z)$ is defined and $\mathcal{X}(z)$ is empty for $z \notin Z^\theta$. Note that if $x \in \mathcal{X}(z')$ and $z \in Z$ then

$$zx \in \mathcal{X}(z'z\theta(z). \tag{12.1}$$

It is easy to see that

$$Z^\theta/\{z\theta(z) \mid z \in Z\} \tag{12.2}$$

is a finite set. This comes down to the fact that if Z is a torus then $Z^\theta/\{z\theta(z)\} \simeq \mathbb{Z}/2\mathbb{Z}^n$ where n is the number of $\mathbb{R}^\times$ factors in the corresponding real torus (cf. Remark 10.3).

Definition 12.3 *Choose a set of representatives Z_0 for $Z^\theta/\{z\theta(z)\}$. The reduced parameter space is*

$$\mathcal{X}_0(G,\gamma) = \{\mathcal{X}(z) \mid z \in Z_0\} \tag{12.4}$$

Remark 12.5 This is not the same thing $\mathcal{X}$ modulo the action of Z.

The calculations needed to understand representation theory (see Section 15) take place entirely in a fixed set $\mathcal{X}(z)$. The sets $\mathcal{X}(z')$ and $\mathcal{X}(z'z\theta(z))$ are canonically identified, so it is safe to think of $\mathcal{X}(z)$ as being defined for $z \in Z_0$. The `Atlas` software takes this approach.

13 Cayley Transforms

Fix basic data (G,γ). We continue to work on the one-sided parameters space $\mathcal{X}$. We begin with some formal constructions.

Fix $x \in \mathcal{X}$ and let $\tau = \phi(x) \in \mathcal{I}_W$. Recall (Section 10.1) τ defines the real, imaginary and complex roots, and x defines a grading gr_x of the imaginary roots. Suppose α is an imaginary non-compact root, i.e. $\tau(\alpha) = \alpha$ and $\mathrm{gr}_x(\alpha) = 1$.

Let G_α be the derived group of $\mathrm{Cent}_G(\ker(\alpha))$, and $H_\alpha \subset G_\alpha$ the one-parameter subgroup corresponding to α. Then G_α is isomorphic to $SL(2)$ or $PSL(2)$ and H_α is a Cartan subgroup of G_α. Choose $\sigma_\alpha \in \mathrm{Norm}_{G_\alpha}(H_\alpha)\backslash H_\alpha$, so $\sigma_\alpha(\alpha) = -\alpha$.

Definition 13.1 *Suppose $x \in \mathcal{X}$ and α is a non-compact imaginary root with respect to θ_x. Choose a representative $\tilde{x} \in \widetilde{\mathcal{X}}$ of x, and define $c^\alpha(x)$ to be the image of $\sigma_\alpha \tilde{x}$ in $\mathcal{X}$.*

Lemma 13.2 *Fix $x \in \mathcal{X}$.*

(1) $c^\alpha(x)$ is well defined, independent of the choice of σ_α and $\tilde{x}$.

(2) $c^\alpha(x)$ is G-conjugate to x, and $c^\alpha(x)^2 = x^2$.

(3) $p(c^\alpha(x)) = s_\alpha p(x) \in \mathcal{I}_W$.

Proof. Fix $\tilde{x} \in \widetilde{\mathcal{X}}$, and let $t = \alpha^\vee(i) \in H_\alpha$. Suppose $h \in H_\alpha$. We have a few elementary identities, essentially in $SL(2)$:

$$\begin{aligned} \sigma_\alpha h \sigma_\alpha^{-1} &= h^{-1}, \quad h\sigma_\alpha h^{-1} = h^2\sigma_\alpha \\ \tilde{x} h \tilde{x}^{-1} &= h \\ tgt^{-1} &= \tilde{x} g \tilde{x}^{-1} \quad (g \in G_\alpha) \\ \tilde{x}\sigma_\alpha\tilde{x}^{-1} &= \sigma_\alpha^{-1} \end{aligned} \tag{13.3}$$

The first two lines follow from $\sigma_\alpha(\alpha^\vee) = -\alpha^\vee$ and $\theta_{\tilde{x}}(\alpha^\vee) = \alpha^\vee$. For the third, $\mathrm{int}(t)$ and $\mathrm{int}(\tilde{x})$ agree on G_α, since they agree on H_α and the $\pm\alpha$ root spaces. The last assertion follows from the third and a calculation in $SL(2)$.

Now $\sigma_\alpha \tilde{x}$ clearly normalizes H, and

$$(\sigma_\alpha\tilde{x})^2 = \sigma_\alpha(\tilde{x}\sigma_\alpha\tilde{x}^{-1})\tilde{x}^2 = \tilde{x}^2 \in Z(G),$$

so $\sigma_\alpha \tilde{x} \in \widetilde{\mathcal{X}}$.

Given a choice of σ_α any other choice is of the form $h^2\sigma_\alpha = h\sigma_\alpha h^{-1}$ for some $h \in H_\alpha$, and

$$(h^2\sigma_\alpha)\tilde{x} = (h\sigma_\alpha h^{-1})\tilde{x} = h\sigma_\alpha(h^{-1}\tilde{x}h)h^{-1} = h(\sigma_\alpha\tilde{x})h^{-1}. \tag{13.4}$$

Therefore the image of $\sigma_\alpha\tilde{x}$ in $\mathcal{X}$ is independent of the choice of σ_α.

We need to show that if $h \in H$ then $\sigma_\alpha\tilde{x}$ and $\sigma_\alpha(h\tilde{x}h^{-1})$ have the same image in $\mathcal{X}$. Write $H = H_\alpha(\ker(\alpha))$. If $h \in H_\alpha$ then $h\tilde{x}h^{-1} = \tilde{x}$ so this is obvious. If $h \in \ker(\alpha)$ $\sigma_\alpha h = h\sigma_\alpha$, and $\sigma_\alpha(h\tilde{x}h^{-1}) = h(\sigma_\alpha\tilde{x})h^{-1}$.

For the second assertion, we actually show $c^\alpha(x)$ is conjugate to x by an element of G_α. By a calculation in $SL(2)$ is t is easy to see $g(\sigma_\alpha t)g^{-1}=t$ for some $g\in G_\alpha$. Therefore

$$g(\sigma_\alpha\tilde{x})g^{-1}=g(\sigma_\alpha tt^{-1}\tilde{x})g^{-1}=g(\sigma_\alpha t)g^{-1}g(t^{-1}\tilde{x})g^{-1}=tt^{-1}\tilde{x}=\tilde{x}. \tag{13.5}$$

The final assertion is obvious.
qed

We next define inverse Cayley transforms, which have a somewhat different flavor. Suppose $\tilde{x}\in\widetilde{\mathcal{X}}$, and let $\tau=\widetilde{\phi}(\tilde{x})$. Suppose α is a real root with respect to $\theta_{\tilde{x}}$, i.e. $\tau(\alpha)=-\alpha$. Define G_α and H_α as before. Let $m_\alpha=\alpha^\vee(-1)$.

Lemma 13.6 *There exists $\sigma_\alpha\in Norm_{G_\alpha}(H_\alpha)\backslash H_\alpha$ so that $\sigma_\alpha\tilde{x}=g\tilde{x}g^{-1}$ for some $g\in G_\alpha$. The only other element satisfying these conditions is $m_\alpha\sigma_\alpha$.*

Proof. This is similar to the previous case. The involution θ_x restricted to G_α is inner for G_α, and acts by $h\to h^{-1}$ for $h\in H_\alpha$. Therefore we may choose $\tilde{y}\in \mathrm{Norm}_{G_\alpha}(H_\alpha)\backslash H_\alpha$ so that $\tilde{y}g\tilde{y}^{-1}=\tilde{x}g\tilde{x}^{-1}$ for all $g\in G_\alpha$. By a calculation in $SL(2)$ we may choose σ_α so that $g(\sigma_\alpha\tilde{y})g^{-1}=\tilde{y}$ for some $g\in G_\alpha$. Then

$$g(\sigma\tilde{x})g^{-1}=g(\sigma\tilde{y}\tilde{y}^{-1}\tilde{x})g^{-1}=g(\sigma\tilde{y})g^{-1}g(\tilde{y}^{-1}\tilde{x})g^{-1}=\tilde{y}\tilde{y}^{-1}\tilde{x}=\tilde{x} \tag{13.7}$$

We have $\sigma_\alpha\tilde{y}\in H_\alpha$, and $\alpha(\sigma_\alpha\tilde{y})=-1$. Therefore any two such choices differ by m_α.
qed

Definition 13.8 *Suppose $\tilde{x}\in\widetilde{\mathcal{X}}$ and α is a real root with respect to $\theta_{\tilde{x}}$. Let $c_\alpha(\tilde{x})=\{\sigma_\alpha\tilde{x},m_\alpha\sigma_\alpha\tilde{x}\}$.*

If $x\in\mathcal{X}$ choose $\tilde{x}\in\widetilde{\mathcal{X}}$ mapping to x, and define $c_\alpha(x)$ to be the image of $c_\alpha(\tilde{x})$ in $\mathcal{X}$. This is a set with one or two elements.

The analogue of Lemma 13.2 is immediate.

Lemma 13.9 *Suppose $x\in\mathcal{X}$ and α is a a real root with respect to θ_x.*

(1) $c_\alpha(x)$ is well defined, independent of the choice of $\tilde{x}$.

(2) If $c\in c_\alpha(x)$ then c is G-conjugate to x, and $c^2=x^2$.

(3) $p(c_\alpha(x)) = s_\alpha p(x) \in \mathcal{I}_W$.

We deduce some simple formal properties of Cayley transforms. Fix $\tau \in \mathcal{I}_W$. If α imaginary with respect to τ let

$$
\begin{aligned}
\mathcal{X}_\tau(\alpha) &= \{x \in \mathcal{X}_\tau \mid \alpha \text{ is non-compact with respect to } \theta_x\} \\
&= \{x \in \mathcal{X}_\tau \mid \mathrm{gr}_x(\alpha) = 1\} \\
&= \{x \in \mathcal{X}_\tau \mid c^\alpha(x) \text{ is defined}\}
\end{aligned} \tag{13.10}
$$

Lemma 13.11

(1) If $\tau(\alpha) = -\alpha$ then for all $x \in \mathcal{X}_\tau$, $c^\alpha(c_\alpha(x)) = x$,

(2) If $x \in \mathcal{X}_\tau(\alpha)$ then $c_\alpha(c^\alpha(x)) = \{x, m_\alpha x\}$.

(3) The map $c^\alpha : \mathcal{X}_\tau(\alpha) \to \mathcal{X}_{s_\alpha\tau}$ is surjective, and at most two-to-one

(4) Suppose α is imaginary. The following conditions are equivalent

- *(a) $c^\alpha : \mathcal{X}_\tau(\alpha) \to \mathcal{X}_{s_\alpha\tau}$ is a bijection,*
- *(b) $c_\alpha : \mathcal{X}_{s_\alpha\tau} \to \mathcal{X}_\tau(\alpha)$ is a bijection,*
- *(c) $c_\alpha(x)$ is single valued for all $x \in \mathcal{X}_{s_\alpha\tau}$,*
- *(d) $m_\alpha \in H_2(\tau)^0$ (see* (10.2)(b)*)*
- *(e) $s_\alpha \in W(G^{\theta_x}, H)$ for $x \in \mathcal{X}_\tau(\alpha)$,*
- *(f) $x \equiv m_\alpha x \in \mathcal{X}$ for all $x \in \mathcal{X}_\tau(\alpha)$*

If these conditions fail then $c^\alpha : \mathcal{X}_\tau(\alpha) \to \mathcal{X}_{s_\alpha\tau}$ is two to one, and $c_\alpha(x)$ is double valued for all $x \in \mathcal{X}_{s_\alpha\tau}$.

(5) Suppose α is imaginary with respect to τ. If there exists $h \in H_1(\tau)$ (see (10.2)(a)*) such that $\alpha(h) = -1$ then $\mathcal{X}_\tau$ is the disjoint union of $\mathcal{X}_\tau(\alpha)$ and $h\mathcal{X}_\tau(\alpha)$. Otherwise $\mathcal{X}_\tau(\alpha) = \mathcal{X}_\tau$.*

It is important to understand the effect of Cayley transforms on the grading of the imaginary roots. This is due to Schmid [12]. Also see [15, Definition 5.2 and Lemma 10.9].

Lemma 13.12 *Suppose $\tau \in \mathcal{I}_W$ and $x \in \mathcal{X}_\tau(\alpha)$. Then the imaginary roots for $s_\alpha\tau$ are the roots orthogonal to α, and*

$$
gr_{c^\alpha x}(\beta) = \begin{cases} gr_x(\beta) & \text{if } \alpha+\beta \text{ is not a root} \\ gr_x(\beta)+1 & \text{if } \alpha+\beta \text{ is a root} \end{cases} \tag{13.13}
$$

There is a simple relationship between Cayley transforms and the Weyl group action. Suppose $w \in W$ and $x \in \mathcal{X}_\tau(\alpha)$. Then $x \in X_{w\tau w^{-1}}(w\alpha)$, so

Lemma 13.14

$$w \times c^\alpha(x) = c^{w\alpha}(w \times x) \tag{13.15}$$

With Cayley transforms in hand we can complete the proof of Proposition 11.9.

Proof of Proposition 11.9. Fix $\tau \in \mathcal{I}_W$. Assume there is an imaginary root α for τ. By Lemma 13.11(5) there exists $x \in \mathcal{X}_\tau(\alpha)$, so $c^\alpha(x) \in \mathcal{X}_{s_\alpha\tau}$ is defined. Now suppose β is an imaginary root with respect to $s_\alpha\tau$. By the same argument we may choose $x' \in \mathcal{X}_{s_\alpha\tau}$ so that $x'' = c_\beta(x')$ is defined. Replacing $x \in \mathcal{X}_\tau$ with $c_\alpha(x) \in \mathcal{X}_\tau$ we now have $\mathcal{X}_\tau \ni x \to c^\beta c^\alpha x \in \mathcal{X}_{s_\alpha s_\beta\tau}$. By Lemma 13.2(2) $c^\beta c^\alpha(x)$ is G-conjugate to x.

Continue in this way until we obtain $x \in \mathcal{X}_\tau, x' \in \mathcal{X}_{\tau'}$, where x' is G-conjugate to x, and there are no imaginary roots with respect to τ'. (This corresponds to the most split Cartan subgroup of the quasisplit form of G.) By [5, Proposition 6.24] $\theta_{x'}$ is quasisplit.
qed

14 The Tits group and the algorithmic enumeration of parameters

The combinatorial enumeration of $\mathcal{X}$ is in terms of the *Tits group* $\widetilde{W}$ introduced by Jacques Tits in [14] under the name *extended Coxeter group*.ley

We begin by fixing G and a choice of of splitting data $(H, B, \{X_\alpha\})$ (cf. Section 2). For each simple root α let $\phi_\alpha : SL(2) \to G$ be defined by $\phi_\alpha(\mathrm{diag}(1,-1)) = \alpha^\vee(-1)$ and $d\phi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_\alpha$. Let $\sigma_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (This is consistent with the definition of σ_α in Section 13.)

Definition 14.1 *The Tits group $\widetilde{W}$ is the subgroup of N generated by $\{\sigma_\alpha\}$ for α simple, and the subgroup $H(2)$ of H of elements of order 2.*

Remark 14.2 The definition of $\widetilde{W}$ in [14] uses the subgroup $H_0 \subset H(2)$ generated by the elements $m_\alpha = \alpha^\vee(-1)$. The larger group is more convenient for our purposes, and the difference is unimportant.

Theorem 14.3 (Tits [14])
(1) The kernel of the natural map $\widetilde{W} \to W$ *is* $H(2)$ *(2) The elements* σ_α *satisfy the braid relations.*
(3) There is a canonical lifting of W *to a subgroup of* $\widetilde{W}$*: take a reduced expression* $w = s_{\alpha_1} \dots s_{\alpha_n}$*, and let* $\tilde{w} = \sigma_{\alpha_1} \dots \sigma_{\alpha_n}$*.*

Now fix (G, γ) and an extended group G^Γ for (G, γ) as in Section 5. We fix a distinguished element $\delta \in G^\Gamma$ (cf. Lemma 5.3) and let $\tau = p(\delta) \in \mathcal{I}_W$.

We are going to describe an algorithm for computing $\mathcal{X}(z)$ (cf. 9.9)(o). Assuming $\mathcal{X}(z)$ is non-empty then $\mathcal{X}_\tau(z)$ is non-empty, and we assume we are given $\delta_z \in \mathcal{X}_\tau(z)$. Then $\mathcal{X}_\tau(z)$ is naturally in bijection with $H_2(\tau)/H_2(\tau)^0$ (cf. Proposition 10.1).

The algorithm goes roughly as follows.

We will maintain a first-in-first-out a list of triples

$$(\tau, x_\tau, B_\tau) \tag{14.4}$$

where $\tau \in \mathcal{I}_W$, x_τ is an element of $\widetilde{W}\delta_z$ mapping to τ, and B_τ is a subset B_τ of $T_\tau(2)$. The set B_τ is chosen to be a basis of

$$H^{-\tau}/(H^{-\tau})^0 \simeq T_\tau(2)/T_\tau \cap A_\tau \tag{14.5}$$

as a $\mathbb{Z}/2\mathbb{Z}$ vector space (cf. Remark 10.3). We initialize this list with (τ, δ_z, B_τ) where B_τ is a basis of $T_\tau(2)/T_\tau \cap A_\tau$.

Whenever we put a triple (τ, x_τ, B_τ) on the list, we add the corresponding elements $x \in \mathcal{X}_\tau$, obtained by acting on x_τ with $T_\tau(2)/T_\tau \cap A_\tau$, to a store.

If the list is non-empty, take the first element (τ, x_τ, B_τ). Try conjugating τ by s_α for each simple root α. If $s_\alpha \tau s_\alpha$ is not on the list, add $s_\alpha(\tau, x_\alpha, B_\tau)s_\alpha$ to the list.

Next, for each simple imaginary root α, see if $\tau' = s_\alpha\tau$ is not on our list, and if there are is an element $h \in H(2)$ so that $hx_\tau \in \mathcal{X}_\tau(\alpha)$ i.e. so that α is non-compact imaginary with respect to hx_τ. If so, choose such an h (this is either half or all of $T_\tau(2)$) and let $x_{\tau'} = \sigma_\alpha h x_\tau$. Now $T_{\tau'}(2)/T_{\tau'} \cap A_{\tau'}$ is the quotient of $T_\tau(2)/T_\tau \cap A_\tau$ by $\{1, m_\alpha\}$. Use this to compute a basis $B_{\tau'}$ of $T_{\tau'}(2)/T_{\tau'} \cap A_{\tau'}$. Add $(\tau', x_{\tau'}, B_{\tau'})$ to the list, and the corresponding elements of $X_{\tau'}$ to the store.

Continue until the list is empty, at which point the store will contain a list of the elements of $\mathcal{X}(z)$.

15 The Parameter Space $\mathcal{Z}$

We can now describe the parameter space for representations of strong real forms of G.

Fix basic data (G, γ), and let $(chG, \gamma^\vee)$ be the dual data (cf. Section 4). Let

$$\begin{aligned} \mathcal{X} &= \mathcal{X}(G, \gamma) \\ \mathcal{Y} &= \mathcal{X}(G^\vee, \gamma^\vee). \end{aligned} \tag{15.1}$$

Definition 15.2 *Define $\mathcal{Z}$ to be a subset of $\mathcal{X} \times \mathcal{Y}$ as follows:*

$$\mathcal{Z} = \{(x, y) \in \mathcal{X} \times \mathcal{Y} \,|\, \theta_x^t = -\theta_y\}. \tag{15.3}$$

Corollary 7.16(1) now becomes:

Theorem 15.4 *There is a natural bijection between $\mathcal{Z}$ and the set of translation families of irreducible representations of strong real forms of G with regular integral infinitesimal character.*

We give several alternative formulations, restating those of Corollary 7.16.

Corollary 15.5 *(1)*
Fix a set $\Lambda \subset P_{reg}$ of representatives of $P/X^*(H)$. Then there is a natural bijection between $\mathcal{Z}$ and the union, over $\lambda \in \Lambda$, of irreducible representations of strong real forms of G, with infinitesimal character λ.
(2) Suppose G is semisimple and simply connected. Then there is a natural bijection between $\mathcal{Z}$ and the irreducible representations of strong involutions of G with infinitesimal character ρ.
(3) Suppose G is adjoint, and fix a set $\Lambda \subset P_{reg}$ of representatives of P/R. Then there is a natural bijection between $\mathcal{Z}$ and the irreducible representations of real forms of G, with infinitesimal character in Λ.

16 Examples

16.1 Example: Defining G and an inner class of real forms

(1) Here is $SL(2)$. There is only one inner class.

```
empty: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
```

(2) Here is $SO(3) = PGL(2) = PSL(2)$, again there is only one inner class.

```
real: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
ad
enter inner class(es): s
main: components
(weak) real forms are:
```

Let's check the component group:

```
main: components
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
component group is (Z/2)^1
```

Yes, this is $PGL(2)$ and not $SL(2)$.

(3) Here is $GL(2)$, which is $\mathbb{C}^\times \times SL(2, \mathbb{C})/\{\pm(1, 1)\}$, there is only one inner class.

```
main: type
Lie type: T1.A1
elements of finite order in the center of the simply connected group:
Q/Z.Z/2
```

```
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): ss
main: components
(weak) real forms are:
0: gl(1,R).su(2)
1: gl(1,R).sl(2,R)
enter your choice: 1
component group is (Z/2)^1
real:
```

(3) There are two inner classes in type A_2. First the split one:

```
real: type
Lie type: A2
elements of finite order in the center of the simply connected group:
Z/3
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
main: realform
there is a unique real form: sl(3,R)
real:
```

Here is the other (compact) inner class:

```
real: type
Lie type: A2
elements of finite order in the center of the simply connected group:
Z/3
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): c
main: realform
(weak) real forms are:
```

```
0: su(3)
1: su(2,1)
enter your choice:
```

16.2 Example 4: Real forms of $SO(2n)$ with n even

In the usual terminology there is a single equivalence class real forms of $G = SO(2n)$ denoted $SO^*(2n)$ (with maximal compact subgroup $U(n)$). This corresponds to a single $\mathrm{Aut}(G(\mathbb{C}))$ conjugacy class of anti-holomorphic involutions of $G(\mathbb{C})$. However this conjugacy class is the union of two G conjugacy classes; what this means is that there are two subgroups $SO(2n, \mathbb{C})$ which are conjugate via the outer automorphism of $SO(2n, \mathbb{C})$ (and are therefore isomorphic real Lie groups) which are not conjugate by $SO(2n, \mathbb{C})$.

In terms of Cartan involutions there is a single $\mathrm{Aut}(G)$ conjugacy $\mathcal{C}$ class of involutions in $\mathrm{Aut}(G)$ such that $G^\theta \simeq GL(n)$ ($\theta \in \mathcal{C}$); $\mathcal{C}$ is the union of two G conjugacy classes.

What this means in practice is that $SO(2n)$ (with n even) has two weak real forms labelled $SO^*(2n)[0, 1]$ and $SO^*(2n)[1, 0]$ in the software.

```
real: type
Lie type: D4 ad c
main: realform
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
```

If $G = Spin(2n), SO(2n)$ or $PSO(2n)$ then the two real forms corresponding to the two version of $SO^*(2n)$ are isomorphic. However there is another isogeny in which these two groups become different. This is a very subtle example:

```
main: type
Lie type: D4
elements of finite order in the center of the simply connected group:
Z/2.Z/2
```

```
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

enter inner class(es): c
main: realform
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 2
real: components
group is connected
real: realform
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 3
real: components
component group is (Z/2)^1
```

In the case of $SO(8)$ something more dramatic happens. Consider the adjoint group $PSO(8)$. There are three real forms: $PSO(6,2), PSO^(8)[0,1]$ and $PSO^*(8)[1,0]$. These are interchanged by outer automorphisms of G, and are isomorphic as real groups, and are considered one equivalence class in the usual sense. The same statements are true in $Spin(8)$. However in $SO(8)$, $SO(6,2)$ is disconnected, and the two versions of $SO^*(8)$ are connected, and isomorphic.

```
main: type
Lie type: D4
elements of finite order in the center of the simply connected group:
Z/2.Z/2
```

```
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): c
main: realform
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 1
real: components
component group is (Z/2)^1
real: realform
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 2
real: components
group is connected
real: realform
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 3
real: components
group is connected
real:
```

See the information on `realform` under the `help` command.

16.3 Example 3: involutions in the compact inner class

Ordinary (weak) involutions of G are the same as those of the adjoint group, so suppose G is adjoint. Also assume $\gamma = 1$, so we are considering involutions in the inner class of the compact group. Then

$$X_1 = \{x \in H \mid x^2 = 1\} \simeq P^\vee/2P^\vee. \tag{16.1}$$

involutions of G are parametrized by $\mathcal{X}_1/W$.

For example take $G = PSO(2n)$. Then $P^\vee = \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n$. For representatives of $\mathcal{X}_1$ we can take

$$\{(a_1, \dots, a_{n-1}, 0) \mid a_i = 0, 1\} \cup \{\frac{1}{2}(b_1, \dots, b_{n-1}, 1) \mid b_i = \pm 1\} \tag{16.2}$$

For representatives of $\mathcal{X}_1/W$ we take

$$\begin{aligned} x_k &= (\overbrace{1, \dots, 1}^{k}, 0, \dots, 0) \quad (0 \le k \le n/2) \\ x'_\pm &= \frac{1}{2}(1, \dots, 1, \pm 1) \end{aligned} \tag{16.3}$$

The first line corresponds to the real forms $PSO(2k, 2n-2k)$. In addition $PSO(2n)$ has two real forms, which are isomorphic as Lie groups, denote $PSO^*(2n)$. See the next example.

The number of elements in the W-orbit of these elements are

$$\begin{aligned} x_k &: \binom{n}{k} \quad k \le \frac{n-1}{2} \\ x_{n/2} &: \frac{1}{2}\binom{n}{n/2} \\ x_\pm &: 2^{n-2} \end{aligned} \tag{16.4}$$

Note that

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{k} + \frac{1}{2}\binom{n}{n/2} + 2 \times 2^{n-2} = 2^n \tag{16.5}$$

```
real: type
Lie type: D4 ad c
```

```
main: strongreal
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 4
cartan class (one of 0,1,2,3,4,5,6): 0
Name an output file (hit return for stdout):

real form #4: [0,1,2] (3)
real form #0: [3] (1)
real form #3: [4,5,6,7] (4)
real form #2: [8,9,10,11] (4)
real form #1: [12,13,14,15] (4)
```

In this case the fiber $\mathcal{X}_1$ has order $16 = 3 + 1 + 4 + 4 + 4$. See the next example for an example of $\mathcal{X}_1$ in a non-adjoint group.

16.4 Example: Strong real forms

Here are the strong real forms of $SL(2)$:

```
real: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
main: strongreal
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
```

```
cartan class (one of 0,1): 0
Name an output file (hit return for stdout):

there are 2 real form classes:

class #0:
real form #1: [0,1] (2)

class #1:
real form #0: [0] (1)
real form #0: [1] (1)
```

This says there are two strong real forms corresponding to (weak) real form 0, i.e. $SU(2)$. There is one strong real form corresonding to (weak) real form 1, i.e. $SL(2,\mathbb{R})$.

Here are the strong real forms of $Spin(8)$

```
real: type
Lie type: D4 sc c
main: realform
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 4
real: strongreal
cartan class (one of 0,1,2,3,4,5,6):
sorry, value must be one of 0,1,2,3,4,5,6
try again (? to abort): 0
Name an output file (hit return for stdout):

there are 4 real form classes:

class #0:
real form #4: [0,1,2,4,5,6,8,9,10,12,13,14] (12)
real form #0: [3] (1)
```

```
real form #0: [7] (1)
real form #0: [11] (1)
real form #0: [15] (1)

class #1:
real form #2: [0,1,2,7,8,9,10,15] (8)
real form #2: [3,4,5,6,11,12,13,14] (8)

class #2:
real form #3: [0,2,3,4,6,7,9,13] (8)
real form #3: [1,5,8,10,11,12,14,15] (8)

class #3:
real form #1: [0,2,3,5,9,12,14,15] (8)
real form #1: [1,4,6,7,8,10,11,13] (8)
```

This says there are 5 real forms. The number of corresponding strong real forms is:

real form	#strong real forms	$\lvert\mathcal{O}_x\rvert$
$Spin(8)$	4	1
$Spin(6,2)$	2	8
$Spin(4,4)$	1	12
$Spin^*(8)[0,1]$	2	8
$Spin^*(8)[1,0]$	2	8

The last column in the table gives the cardinality of the W-orbit of x, in the fiber $\mathcal{X}_1$. The cardinality of this fiber is $|Z|2^4 = 4 \times 16 = 64$, note that $4 \times 1 + 2 \times 8 + 1 \times 12 + 2 \times 8 + 2 \times 8 = 64$.

If $\gamma = 1$ then $G^\Gamma = G \times \mathbb{Z}/2\mathbb{Z}$, and we may safely drop δ from the notation.

16.5 Example 1: $SL(2)$

Let $G = SL(2)$, so then $G^\vee = PGL(2)$. In this case $\text{Out}(G) = 1$ so there is only one inner class of involutions, and we drop δ from the notation.

Write $H = \{\text{diag}(z, \frac{1}{z}) \mid z \in \mathbb{C}^*\}$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Write $W = \{1, s\}$ and let $t = \text{diag}(i, -i)$. In this case

$$\begin{aligned} \mathcal{I} &= \{x \in G \mid x^2 = \pm I\} \\ \widetilde{\mathcal{X}} &= \{h \in H \mid h^2 = \pm I\} \cup Hw \\ &= \{\pm I, \pm t\} \cup Hw \\ \mathcal{X} &= \{\pm I, \pm t, w\} \\ \mathcal{X}_1 &= \{\pm I, \pm t\}, \quad \mathcal{X}_s = \{w\} \end{aligned} \tag{16.6}$$

Of course we mean the elements on the right are representatives of the elements of $\mathcal{X}$.

Proposition 11.16 says

$$\{\text{strong involutions of } G\} \leftrightarrow \{I, -I, t\} \tag{16.7}$$

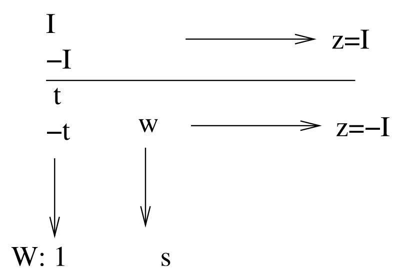
It is easy to see directly that each element of $\mathcal{X}$ is conjugate to precisely one of these. The group K_x is equal to G if $x = \pm I$, or H otherwise. The first two cases correspond to the (weak) involution $SU(2)$, and the third to $SL(2, \mathbb{R}) \simeq SU(1, 1)$. It is helpful to think of these groups as $SU(2, 0), SU(1, 1)$ and $SU(0, 2)$.

Then $\mathcal{X}/W = \{I, -I, t, w\}$. Proposition 11.9 says this corresponds to: the compact Cartan subgroup of $SU(2, 0)$, $SU(0, 2)$, $SL(2, \mathbb{R})$, and the split Cartan subgroup of $SL(2, \mathbb{R})$, respectively.

We now consider Proposition 10.1. We have

$$\begin{aligned} H_1(1) &= \{\pm I, \pm t\} \\ H_2(1) &= \{\pm I\} \quad H_2(1)^0 = I \\ H_1(1)/H_2(1)^0 &= \{\pm I, \pm t\} = \mathcal{X}_1 \\ H_2(1)/H_2(1)^0 &\leftrightarrow \{\pm I\} = \mathcal{X}_1(I) \\ &\leftrightarrow \{\pm t\} = \mathcal{X}_1(-I) \end{aligned} \tag{16.8}$$

On the other hand if s is the non-trivial element of W, $H_1(s) = 1$, and $\mathcal{X}_s$ is a singleton. We have the following picture:



We next consider the orbit picture of Section 8. The space G/B is isomorphic to the two-sphere. The group $K_I = G$ has a single orbit on G/B, which we denote $\mathcal{O}_{2,0}$. Similarly write $\mathcal{O}_{0,2}$ for the single orbit of K_{-I}. The group K_t has three orbits, labelled $\mathcal{O}_\pm$ (which are points) and $\mathcal{O}_*$ (which is open and of dimension 1).

The bijection of Proposition 8.4 (using Proposition 9.4) is

$\mathcal{X}$	orbit
I	$\mathcal{O}_{2,0}$
-I	$\mathcal{O}_{0,2}$
t	$\mathcal{O}_+$
t	$\mathcal{O}_-$
w	$\mathcal{O}_*$

16.6 Example 2: $PGL(2)$

Recall $PGL(2) \simeq SO(3)$, and it is easier to work with the latter realization, with respect to the form $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We can take $H = \{\mathrm{diag}(z, \frac{1}{z}, 1 \,|\, z \in \mathbb{C}^\times\}$. The center is trivial, and it is easy to see

$$\mathcal{X} = \{I, \mathrm{diag}(-1,-1,1), w\} \tag{16.9}$$

where w is the matrix above with -1 in place of 1 in the lower right hand corner. The latter two elements are conjugate by G, and the corresponding involutions are $SO(3,0)$ and $SO(2,1) \simeq PGL(2,\mathbb{R})$.

For $x = I$ we write $\mathcal{O}'_{3,0}$ for the corresponding orbit (which is a point). If $x = \mathrm{diag}(-1,-1,1)$ there are two orbits, the closed orbit $\mathcal{O}'_+$ and the open orbit $\mathcal{O}'_*$. a The representation theory of $SL(2)$ and $PGL(2)$ may be summarized in the following table. We write $\mathbb{C}$ for the trivial representation, $DS_\pm$ for the two discrete series representations of $SL(2,\mathbb{R})$, and PS_{odd} for the (irreducible) non-spherical principal series representation of $SL(2,\mathbb{R})$. All these representations have infinitesimal character ρ.

On the other hand $PGL(2,\mathbb{R})$ has the trivial, sgn, and discrete series representations at infinitesimal character ρ. It also has two irreducible principal series representations $PS_{\mathbb{C}}$ and PS_{sgn} at infinitesimal character 2ρ, distinguished by their lowest K-types $\mathbb{C}$ and sgn of $O(2)$.

Table of representations of $SL(2)$ and $PGL(2)$

Orbit	x	x^2	θ_x	G_x	λ	rep	Orbit	y	y^2	θ_y	G_y	λ	rep
$\mathcal{O}_{2,0}$	I	I	1	$SU(2,0)$	ρ	$\mathbb{C}$	$\mathcal{O}'_*$	w	I	-1	$SO(2,1)$	2ρ	$PS_{\mathbb{C}}$
$\mathcal{O}_{0,2}$	-I	I	1	$SU(0,2)$	ρ	$\mathbb{C}$	$\mathcal{O}'_*$	w	I	-1	$SO(2,1)$	2ρ	PS_{sgn}
$\mathcal{O}_+$	t	-I	1	$SU(1,1)$	ρ	DS_+	$\mathcal{O}'_*$	w	I	-1	$SO(2,1)$	ρ	$\mathbb{C}$
$\mathcal{O}_-$	-t	-I	1	$SU(1,1)$	ρ	DS_-	$\mathcal{O}'_*$	w	I	-1	$SO(2,1)$	ρ	sgn
$\mathcal{O}_*$	w	-I	1	$SU(1,1)$	ρ	$\mathbb{C}$	$\mathcal{O}'_+$	t	I	-1	$SO(2,1)$	ρ	DS
$\mathcal{O}_*$	w	I	1	$SU(1,1)$	ρ	PS_{odd}	$\mathcal{O}'_{3,0}$	I	I	1	$SO(3)$	ρ	$\mathbb{C}$

16.7 Example 7: Discrete Series

Suppose G is semisimple and $\gamma = 1$. Then we can take $\delta = 1$ and $\theta_1 = 1$, and this inner class of groups contains a compact Cartan subgroup. Then

$$X_1 = \{x \in H \mid x^2 \in Z(G)\} \tag{16.10}$$

and $|X_1| = 2^n|Z(G)|$ where $n = \text{rank}(G)$. In particular $\mathcal{X}_1$ is a two-group if G is adjoint.

Let $P^\vee$ be the co-weight lattice, and X_* the co-character lattice. Then the map $P^\vee \ni \gamma^\vee \to \exp(\pi i \gamma^\vee) \in X_1$ induces an isomorphism

$$X_1 \simeq P^\vee/2X_*. \tag{16.11}$$

Let $G^\vee$ be the dual group, with extended group $G^{\vee\Gamma}$. Then $\mathcal{I}_W(G^\vee)$ contains an element w_0 satisfying $w_0(h) = h^{-1}$ for all $h \in H^\vee$. It is easy to see that $X^\vee_{w_0} = \{y_0\}$ is a singleton. If $x \in X_1$ then $(x, y_0) \in \mathcal{Z}$, and we see

$$X_1 \leftrightarrow \{(x, \pi)\} \tag{16.12}$$

where x is a strong involution of G and π is a discrete series representation of this strong involution with infinitesimal character ρ. In this bijection a W-orbit on $\mathcal{X}_1$ corresponds to the discrete series of a fixed strong involution.

The parameters

$$\{(x, y_0) \mid x \in H, x^2 \in Z(G)\} \tag{16.13}$$

correspond to discrete series representation of strong involutions of G, with infinitesimal character ρ. These are dual to the principal series representation of the split involution of $G^\vee$. There are 2^n of these at each of $|Z(G)|$ infinitesimal characters for $G^\vee$.

In particular if G is adjoint there are 2^n discrete series representations of involutions of G, with trivial infinitesimal character ρ. These are dual to the 2^n principal series representation of the split involution of $G^\vee$, also at trivial infinitesimal character.

16.8 Example 4: Discrete Series of $SO(5)$

Let $G = SO(5)$. Then $\gamma = 1$, and the preceding example applies. We can choose a quadratic form so that $H = \{t(z_1, z_2) \mid z_i \in \mathbb{C}^\times\}$ where $t(z_1, z_2) = \{\text{diag}(z_1, z_2, z_1^{-1}, z_2^{-1}, 1)\}$ Then

$$X_1 = \{t(\epsilon_1, \epsilon_2) \mid \epsilon_i = \pm 1\} \tag{16.14}$$

There are three (strong) involutions:

$$t(1,1) \leftrightarrow SO(5,0) \\ t(-1,-1) \leftrightarrow SO(4,1) \\ t(1,-1) \leftrightarrow SO(3,2) \tag{16.15}$$

The dual group is $Sp(4)$. The long element of the Weyl group of $Sp(4)$ is $w_0 = -1$, and

$$X^\vee_{w_0} = \{y_0\}. \tag{16.16}$$

Then

(16.17)

$$\begin{aligned} (t(1,1), y_0) &\leftrightarrow \text{ trivial representation of } SO(5) \\ (t(-1,-1), y_0) &\leftrightarrow \text{ unique discrete seres representation of } SO(4,1) \\ (t(1,-1), y_0) &\leftrightarrow \text{ holomorphic discrete series representation of } SO(3,2) \\ (t(-1,1), y_0) &\leftrightarrow \text{ anti-holomorphic discrete series representation of } SO(3,2) \end{aligned}$$

These four representations are dual to the four principal series representations of $Sp(4,\mathbb{R})$ at infinitesimal character ρ.

16.9 Example: Representations of E_8

```
main: type
Lie type: E8 sc s
main: blocksizes
       0       0       1
       0    3150   73410
       1   73410  453060
```

These are the blocks for real forms of E_8. The split group has a self-dual block of size $453,060$. It also has an irreducible principal series representation; this is a block of size 1, dual to the trivial representation of the compact form. The split group also has a block of size $73,410$ dual to the quaternionic real form of E_8, with K of type $A1 \times E_7$.

These blocks on the split form of E_7, together with their duals, give every block, except for a self-dual block of size $3,150$ on the quaternionic real form.

The total number of representations (for example at infinitesimal character ρ) is $603,320$.

The number of unitary representations with infinitesimal character ρ is

$$\begin{aligned} &1 \text{ compact form}\\ &1575 \text{ the quaternionic form}\\ &2157 \text{ split form}\\ &3,733 \text{ total}\end{aligned} \tag{16.18}$$

(this has also been computed by Scott Crofts). This is approximately .62% of the representations.

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