PROBLEMS FOR REAL GROUPS

James Arthur*

At the suggestion (or should I say gentle nagging?) of Bill Casselman, I have tried to compile a set of interesting problems for real groups. I believe that others are also being polled, so I have not made any attempt to represent the field as a whole. Some of the problems are in fact quite idiosyncratic. They all come from real harmonic analysis, and are motivated by global questions in automorphic forms.

I am afraid that the list has been put together quickly, without the reflection it should have had. I am sure that I have overlooked some points, and misstated others. The problems therefore are meant to include supplying corrections, as needed, to what I have written!

Unless otherwise indicated, G will denote a connected, reductive algebraic group over $\mathbb{R}$ in the discussion below.

Contents

1. Endoscopic transfer

It would be very useful to recast the work of Shelstad [She2], [She3], [She4], [She5] explicitly in terms of the general transfer factors defined later by Langlands and Shelstad [LS1]. The setting is an endoscopic embedding

$$\xi' :\ {}^L G' \longrightarrow {}^L G, \tag{1.1}$$

where G' represents an endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G [LS1, (1.2)] for which $\mathcal{G}'$ has been identified with an L-group ${}^L G'$ of G'. Shelstad's work is anchored by two basic results. One is her construction of an endoscopic transfer mapping

$$f \longrightarrow f', \qquad f \in \mathcal{C}(G),$$

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from the Schwartz space $\mathcal{C}(G)$ on $G(\mathbb{R})$ to the stable Schwartz space $\mathcal{S}(G')$ on $G(\mathbb{R})$. The other, which we will leave for §2, is her proof of the resulting family of endoscopic character identities.

We recall that there are three Schwartz spaces attached to G, with surjective mappings

$$\mathcal{C}(G) \longrightarrow \mathcal{I}(G) \longrightarrow \mathcal{S}(G).$$

Besides Harish-Chandra's original (nonabelian) Schwartz space $\mathcal{C}(G)$ [Ha4], we have the invariant Schwartz space

$$\mathcal{I}(G) = I\mathcal{C}(G) = \big\{ f_G : \ f \in \mathcal{C}(G) \big\}$$

of invariant orbital integrals

$$f_G(\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x)\mathrm{d}x,$$

and the stable Schwartz space

$$\mathcal{S}(G) = S\mathcal{C}(G) = \big\{ f^G : \ f \in \mathcal{C}(G) \big\}$$

of stable orbital integrals

$$f^G(\delta) = |D(\delta)|^{\frac{1}{2}} \int_{(G_\delta\backslash G)(\mathbb{R})} f(x^{-1}\delta x)\mathrm{d}x = \sum_{\gamma\to\delta} f_G(\gamma).$$

The space $\mathcal{I}(G)$ consists of functions on the set of strongly regular conjugacy classes γ in $G(\mathbb{R})$, while $\mathcal{S}(G)$ is composed of functions on the set of strongly regular stable conjugacy classes δ. Using the differential equations and boundary conditions of Harish-Chandra [Ha1, Theorem 3], [Ha5, Theorem 9.1], Shelstad characterized $\mathcal{S}(G)$ explicitly as a space of functions of δ [She2]. We note that Shelstad (and Langlands) did not normalize orbital integrals in terms of the Weyl discriminant

$$D(\gamma) = \det\big(1 - \mathrm{Ad}(\gamma)\big)_{\mathfrak{g}/\mathfrak{g}_\gamma},$$

as we have here, but this amounts to a minor notational difference.

Shelstad defined the transfer mapping as a finite linear combination

$$f'(\delta') = \sum_{\gamma} \Delta^{(\mathbb{R})}(\delta', \gamma) f_G(\gamma) \tag{1.2}$$

of invariant orbital integrals on $G(\mathbb{R})$. The coefficients are the somewhat ad hoc transfer factors of [She5] (modified here to accommodate our normalization by the Weyl discriminant). They predated (and anticipated) the systematic transfer factors $\Delta(\delta', \gamma)$ of [LS1]. With the hindsight of [LS2, Theorem 2.6.A], we know that the mapping can be defined equivalently by means of the later transfer factors of [LS1]. In other words,

$$\Delta^{(\mathbb{R})}(\delta', \gamma) = c\Delta(\delta', \gamma),$$

for a nonzero constant c. Since the two transfer factors are defined anyway only up to a multiplicative constant, they are therefore equal. However, the proof of this fact is indirect, and depends on the existence of the mapping $f \to f'$ Shelstad had defined earlier. It would be very instructive to show directly that the mapping defined by (1.2), but with $\Delta(\delta', \gamma)$ in place of $\Delta^{(\mathbb{R})}(\delta', \gamma)$, takes $\mathcal{C}(G)$ to the space $\mathcal{S}(G')$.

The problem is by and large one of exposition, but it is no doubt harder than many questions of original research. A satisfactory solution would probably be very influential. The general transfer factors of [LS1] have still not really been absorbed by mathematicians. A concrete description for real groups of their four subfactors [LS1, (3.2)–(3.5)] would lead to a better understanding of their analogues for p-adic fields. Each of these four factors has a precursor in Shelstad's papers. Shelstad's constructions were driven in turn by certain aspects of the work of Harish-Chandra. These antecedents from Harish-Chandra raised vaguely uncomfortable questions, which in retrospect explain why transfer factors are complicated (and interesting).

One question concerns Harish-Chandra's basic formula for the characters of discrete series. What are the implications of the fact that this formula is given as a sum over the Weyl group of a maximal compact subgroup $K_{\mathbb{R}}$ of $G(\mathbb{R})$, rather than the full Weyl group?

Other questions concern Harish-Chandra's normalization of invariant orbital integrals. He defined G to be *acceptable* if the usual half sum ρ of positive roots $\{\alpha\}$ on the Lie algebra of any maximal torus $T \subset G$ lifts to a character ξ_ρ on $T(\mathbb{C})$. The function

$$\Delta(\gamma) = \xi_\rho(\gamma) \prod_{\alpha>0} \big(1 - \xi_\alpha(\gamma^{-1})\big), \qquad \gamma \in T(\mathbb{R}),$$

is then a refinement of the normalizing factor $|D(\gamma)|^{\frac{1}{2}}$ we used above. In particular, its absolute value equals the nonnegative function $|D(\gamma)|^{\frac{1}{2}}$. Harish-Chandra normalized invariant orbital integrals in this case according to the further refinement

$$F_f(\gamma) = \varepsilon_{\mathbb{R}}(\gamma)\Delta(\gamma) \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x)\mathrm{d}x,$$

in which $\varepsilon_{\mathbb{R}}(\gamma)$ is a locally constant sign function on the set $T_{\mathrm{reg}}(\mathbb{R})$ of regular points in $T(\mathbb{R})$ [Ha1, §22]. This normalization was chosen so that if $T(\mathbb{R})$ is compact, and f is a matrix coefficient of discrete series, then $F_f(\gamma)$ extends from $T_{\mathrm{reg}}(\mathbb{R})$ to a smooth function on $T(\mathbb{R})$.

The transfer factors pertain to relative forms of these questions, as they relate to both G and G'. The term Δ_1 in [LS1, (3.4)] addresses the first point, namely the discrepancy between the Weyl groups of G and $K_{\mathbb{R}}$. The term Δ_{II} in [LS1, (3.3)] addresses the product

$$\varepsilon_{\mathbb{R}}(\gamma) \prod_{\alpha} \big(1 - \xi_\alpha(\gamma^{-1})\big).$$

The reader will observe that the quotient of this function by the factor $|D(\gamma)|^{\frac{1}{2}}$ (which we have built into the basic invariant orbital integrals, and which in [LS1] is the supplementary term Δ_{IV} in (3.6)) is quite simple, especially when G is acceptable. The term Δ_2 from [LS1, (3.5)] deals with the function $\xi_\rho(\gamma)$ if G is acceptable, and accounts more generally for what happens if G (or G') is not acceptable. Finally, the term Δ_{I} from [LS1, (3.2)] is a sign, which is independent of γ, and reflects the fact that the product of the other terms is based on some noncanonical choices. As we have already said, these terms all go back to constructions in Shelstad's papers. For example, the term Δ_2 is closely related to the embeddings (1.1), studied in [S4] and discussed further in [S5, (3.3)]. If I have things straight, the precursor of the term Δ_{I} is the set of signs treated in [S5, (3.5)].

It would be very useful to describe all of this explicitly. The goal might be to illuminate the path that leads from Harish-Chandra to Shelstad to Langlands-Shelstad. A greater appreciation of the role of Harish-Chandra's work in the definition of the transfer factors of [LS1], and hence in the foundations of the theory of endoscopy, would make the theory that much more accessible.

2. Endoscopic character identities

This is a continuation of the proposal of §1. In [She5], Shelstad established an equivalent spectral version of the mapping (1.2). It is given by a linear combination

$$f'(\phi') = \sum_{\pi} \Delta(\phi', \pi) f_G(\pi) \tag{2.1}$$

of irreducible tempered characters

$$f_G(\pi) = \mathrm{tr}\big(\pi(f)\big), \qquad \pi \in \Pi_{\mathrm{temp}}(G),$$

on $G(\mathbb{R})$. The coefficients are spectral transfer factors $\Delta(\phi', \pi)$. They are uniquely determined by the original choice of transfer factors

$$\Delta = \big\{\Delta(\delta', \gamma) = \Delta^{(\mathbb{R})}(\delta', \gamma)\big\},$$

once the linear form $f' \to f'(\phi')$ on $\mathcal{S}(G')$ on the left hand side of (2.1) has been defined. (We recall that Δ is determined up to a scalar multiple.) If (1.2) is taken as the definition of the mapping $f \to f'$, the identity (2.1) is to be regarded as a consequential formula. It expresses f' explicitly as a function on the set of tempered Langlands parameters ϕ' of G'.

Recall that a tempered Langlands parameter for G is an L-homomorphism

$$\phi : \ W_{\mathbb{R}} \longrightarrow {}^L G, \qquad \phi \in \Phi_{\mathrm{temp}}(G),$$

taken up to $\widehat{G}$-conjugacy, whose image in $\widehat{G}$ is relatively compact. We assume implicitly that ϕ is *relevant* to G, in the sense that if its image is contained in a parabolic subgroup ${}^L P \supset {}^L G$, then ${}^L P$ is dual to a $\mathbb{Q}$-rational parabolic subgroup $P \subset G$. It then gives rise to the L-packet Π_ϕ that was an integral part of Langlands' earlier classification [L1] of representations of real groups. Recall that Π_ϕ is a finite subset of representations in $\Pi_{\mathrm{temp}}(G)$ whose constituents have the same local L-functions and ε-factors, and that $\Pi_{\mathrm{temp}}(G)$ is a disjoint union over ϕ of the subsets Π_ϕ. Shelstad observed that for any ϕ, the distribution

$$f^G(\phi) = \sum_{\pi \in \Pi_\phi} f_G(\pi), \qquad f \in \mathcal{C}(G), \tag{2.2}$$

is stable, in the sense that it depends only on the image f^G of f in $\mathcal{S}(G)$. Applied to G' instead of G, this gives meaning to the left hand side of (2.1).

Shelstad established striking properties of the spectral transfer factors $\Delta(\phi', \pi)$ in (2.1), which had been conjectured earlier by Langlands [She1]. The problem we propose here is, again, to establish them explicitly in terms of the transfer factors of [LS1].

In describing Shelstad's spectral results, we assume implicitly that the given pair (G', ϕ') is relevant to G, in the sense that the composite Langlands parameter

$$\phi = \xi' \circ \phi' : \ W_{\mathbb{R}} \longrightarrow {}^L G$$

is relevant to G. Given ϕ, one forms the centralizer

$$S_\phi = \mathrm{Cent}\big(\widehat{G}, \phi(W_{\mathbb{R}})\big)$$

of the image of ϕ in $\widehat{G}$, and its group

$$\mathcal{S}_\phi = S_\phi / S_\phi^0 = \pi_0(S_\phi)$$

of connected components. The semisimple element s' that is part of the endoscopic datum represented by G' belongs to S_ϕ. We thus have a mapping

$$(G',\phi') \longrightarrow (\phi, s').$$

Conversely, for any $\phi \in \Phi_{\rm temp}(G)$ and any semisimple element $s' \in S_\phi$, (ϕ, s') is the image of the unique pair (G',ϕ'). The mapping is therefore invertible. (We have assumed for simplicity that every endoscopic datum G' for G has an endoscopic embedding (1.1). This is not true for arbitrary G, but is easily accounted for [LS1, (4.4)].)

Shelstad's spectral results may be summarized as the existence of nonvanishing normalizing functions

$$\rho(\Delta, s'), \qquad \phi \in \Phi_{\rm temp}(G),\ s' \in S_\phi,$$

where (G',ϕ') maps to (ϕ, s'), and Δ is a Langlands-Shelstad transfer factor for G', with the following two properties.

(i) The quotient

$$\langle s, \pi\rangle = \rho(\Delta, s')^{-1}\Delta(\phi', \pi), \qquad s' \in S_\phi,\ \pi \in \Pi_{\rm temp}(G),$$

depends only on the image s of s' in $\mathcal{S}_\phi$, and vanishes unless π lies in the subset Π_ϕ of $\Pi_{\rm temp}(G)$.

(ii) For any $\pi \in \Pi_\phi$, the function

$$s \longrightarrow \langle s, \pi\rangle, \qquad s \in \mathcal{S}_\phi,$$

is a character on $\mathcal{S}_\phi$.

It is not hard to see that the group $\mathcal{S}_\phi$ is abelian. Shelstad showed that the quotient

$$\overline{\mathcal{S}}_\phi = \mathcal{S}_\phi/\pi_0\big(Z(\widehat{G})^\Gamma\big),$$

where $Z(\widehat{G})^\Gamma$ is the centralizer of ${}^LG$ in $\widehat{G}$, is actually a 2-group. She also arranged matters (in the choice of the function $\rho(\Delta, s')$) so that the characters in (ii) were trivial $\pi_0\big(Z(\widehat{G})^\Gamma\big)$. Since the mapping $\pi \rightarrow \langle\cdot, \pi\rangle$ is injective by construction, an irreducible representation $\pi \in \Pi_{\rm temp}(G)$ can thus be identified with a parameter $\phi \in \Phi_{\rm temp}(G)$, together with a character on a 2-group. A slightly different way to say things is that Shelstad's spectral results impose an endoscopic interpretation on the tempered representations in the Langlands classification. This of course is very important for the theory of automorphic forms.

The problem, once again, is to try to reorganize the proofs of Shelstad's spectral results. As they stand now, they are quite difficult to extract from their source in [S5, §4–5]. An exposition would include the straightforward stabilization

$$(zf)' = z'f', \qquad z \in \mathcal{Z}(G),$$

of Harish-Chandra's differential equations for invariant orbital integrals, as well as Shelstad's more difficult stabilization of the boundary conditions of [Ha5, Theorem 9.1].

3. Orthogonality relations

Elliptic tempered characters satisfy orthogonality relations. For example, the characters of discrete series form an orthonormal set on the (regular) elliptic set $G_{\rm ell}(\mathbb{R})$ in $G(\mathbb{R})$. We assume that G is *cuspidal*, in the sense that $G_{\rm ell}(\mathbb{R})$ is nonempty. This is to say that $G(\mathbb{R})$ has a maximal torus $T_{\rm ell}(\mathbb{R})$ that is compact modulo the split part of the center $A_G(\mathbb{R})$ of $G(\mathbb{R})$. In general, suppose that $\Theta = \Theta_\pi$ and $\Theta' = \Theta_{\pi'}$ are two irreducible tempered characters with the same central character on $A_G(\mathbb{R})$. One forms their elliptic inner product

$$(\Theta, \Theta')_{\mathrm{ell}} = \int_{G_{\mathrm{ell}}(\mathbb{R})/A_G(\mathbb{R})} \Theta(x)\,\overline{\Theta'(x)}\,\mathrm{d}x, \tag{3.1}$$

in which $\mathrm{d}x$ is the normalized invariant measure. That is,

$$(\Theta, \Theta')_{\mathrm{ell}} = \big|W\big(G(\mathbb{R}), T_{\mathrm{ell}}(\mathbb{R})\big)\big|^{-1} \int_{T_{\mathrm{ell}}(\mathbb{R})/A_G(\mathbb{R})} |D(\gamma)|\Theta(\gamma)\,\overline{\Theta'(\gamma)}\,\mathrm{d}\gamma,$$

where $W\big(G(\mathbb{R}), T_{\mathrm{reg}}(\mathbb{R})\big)$ is the Weyl group of $\big(G(\mathbb{R}), T_{\mathrm{ell}}(\mathbb{R})\big)$, and $\mathrm{d}\gamma$ is the normalized Haar measure on the compact abelian group $T_{\mathrm{ell}}(\mathbb{R})/A_G(\mathbb{R})$.

If π and π' belong to the discrete series, Harish-Chandra established the relations

$$(\Theta, \Theta') = \begin{cases} 1, & \text{if } \pi = \pi' \\ 0, & \text{otherwise,} \end{cases}$$

in the course of his monumental classification [Ha4]. More general orthogonality relations apply to irreducible constituents of induced tempered representations. They can be described elegantly in terms of the finite groups $\mathcal{S}_\phi$. To adopt a broader perspective, let us take S to be *any* complex reductive group, and

$$\mathcal{S} = \pi_0(S) = S/S^0$$

to be its finite group of connected components. The example we have in mind here is of course the case that S equals the group S_ϕ, so that $\mathcal{S}$ equals $\mathcal{S}_\phi$.

Given S, we define $\mathcal{S}^1$ to be the subgroup of connected components in $\mathcal{S}$ that have representatives that commute with the identity component. The quotient

$$R = \mathcal{S}/\mathcal{S}^1$$

then acts faithfully by outer automorphisms on S^0. We also obtain an action of R on any maximal torus T in S^0 by fixing a Borel subgroup B of S^0 that contains T, and choosing representatives of classes in R that stabilize the pair (B, T). This in turn gives an action of R on the real vector space

$$\mathfrak{a}_T = \mathrm{Hom}\big(X(T)_{\mathbb{R}}, \mathbb{R}\big).$$

The function

$$d(r) = \det(1 - r)_{\mathfrak{a}_T}, \qquad r \in R,$$

on R is independent of the pair (B, T), as is the subset

$$R_{\mathrm{reg}} = \{r \in R : \ d(r) \neq 0\}$$

of R.

If ξ is an irreducible character on the group $\mathcal{S}^1$, let R_ξ be the subgroup of elements in R that stabilize ξ. There is no a priori reason why ξ should extend to an irreducible character on the preimage of R_ξ in $\mathcal{S}$. The obstruction will be a class in $H^2(R_\xi, \mathbb{C}^*)$. It has never been determined, so far as I know, whether this cocycle always splits, at least in the case that S is the centralizer in $\widehat{G}$ of some L-subgroup of ${}^LG$. I pose this as a problem, even though it does not look like it concerns real groups. Indeed, if $S = S_\phi$, the group $\mathcal{S}_\phi$ is abelian, and the answer is obvious. However, the problem does seem to be relevant to the nontempered packets for $G(\mathbb{R})$. We will return to it briefly later.

Suppose now that $S = S_\phi$, and consider the associated short exact sequence

$$1 \longrightarrow \mathcal{S}_\phi^1 \longrightarrow \mathcal{S}_\phi \longrightarrow R_\phi \longrightarrow 1. \tag{3.2}$$

The subgroup $\mathcal{S}_\phi^1$ of $\mathcal{S}_\phi$ is isomorphic to $\mathcal{S}_{\phi_M}$, where M is a Levi subgroup of G, and

$$\phi_M : \ W_\mathbb{R} \longrightarrow {}^L M$$

is a Langlands parameter for M whose image in ${}^L G$ equals ϕ, and whose L-packet π_{ϕ_M} consists of representations in the relative discrete series of $M(\mathbb{R})$. Suppose $\sigma \in \Pi_{\phi_M}$ corresponds to the character ξ on the group $\mathcal{S}_{\phi_M} \cong \mathcal{S}_\phi^1$. Since $\mathcal{S}_\phi$ is abelian, R_ξ equals the full group R_ϕ, and ξ extends to a character θ on $\mathcal{S}_\phi$. The set of such θ is a torsor under the action of the group of characters in R_ϕ. It corresponds to a subset $\Pi_{\phi,\sigma}$ of Π_ϕ, composed of the irreducible constituents of the induced representation $\mathcal{I}_P^G(\sigma)$, where P belongs to the set $\mathcal{P}(M)$ of parabolic subgroups of G with Levi component M. The group $R_\sigma = R_\phi$ is known as the R-group of σ. We have described it here in its spectral form, rather than the dual form [KnSt, §13] defined originally in terms of Plancherel densities. (See [KnZ].)

The general orthogonality relations apply to an arbitrary pair of irreducible tempered characters $\Theta = \Theta_\pi$ and $\Theta' = \Theta_{\pi'}$, where π and π' correspond to characters θ and θ' on respective groups $\mathcal{S}_\phi$ and $\mathcal{S}_{\phi'}$. We form an elliptic inner product $(\theta, \theta')_{\mathrm{ell}}$ by defining it to be 0 unless π and π' belongs to a set $\Pi_{\phi,\sigma}$ as above, in which case we define

$$(\theta, \theta')_{\mathrm{ell}} = |R_\phi|^{-1} \sum_{r \in R_{\phi,\mathrm{reg}}} |d(r)| \theta(r)\, \overline{\theta'(r)}. \tag{3.3}$$

Since θ and θ' here both restrict to the same character ξ on the subgroup $\mathcal{S}_\phi^1$ of $\mathcal{S}_\phi$, the summand in (3.3) is a well defined function on $R_\phi = \mathcal{S}_\phi / \mathcal{S}_\phi^1$. The general orthogonality relations are then given by the identity

$$(\Theta, \Theta')_{\mathrm{ell}} = (\theta, \theta')_{\mathrm{ell}}. \tag{3.4}$$

(See [A6, Corollary 6.3].) Notice the parallel structure in the definitions (3.1) and (3.3) of the two sides of the identity. In particular, $R_{\phi,\mathrm{reg}}$ is an analogue of the elliptic set $G_{\mathrm{ell}}(\mathbb{R})$, and $d(r)$ is an analogue of the Weyl discriminant $D(\gamma)$.

I pose a second question of deciding whether the identity (3.4) might have any role in the proof of Shelstad's spectral results. For example, is the identity easy to derive directly from what is known of the characters Θ_π? (The proof of (3.4) by the local trace formula is indirect, and applies uniformly to all local fields of characteristic 0.) The R-groups R_ϕ are an essential part of Shelstad's arguments. The question of whether the orthogonality relations have anything further to contribute would not be hard to resolve. The answer might well be negative. Nonetheless, it is useful to be armed with concrete questions upon entering territory as unfamiliar as the work of Shelstad might be to some. In any case, I have tried to raise the question in a form that might also be posed for nontempered representations discussed in §8 and §9.

4. Weighted orbital integrals

Weighted orbital integrals are generalizations of invariant orbital integrals. They are integrals

$$J_M(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(\mathbb{R}) \backslash G(\mathbb{R})} f(x^{-1}\gamma x) v_M(x) \mathrm{d}x, \qquad f \in \mathcal{C}(G), \tag{4.1}$$

over the $G(\mathbb{R})$-conjugacy class of a G-regular class γ in $M(\mathbb{R})$, with respect to a noninvariant measure $v_M(x)\mathrm{d}x$. The weight function $v_M(x)$ is the volume of a certain convex hull, which depends on x, and is trivial in case $M = G$. Weighted orbital integrals are terms in the noninvariant trace formula that are attributable to the boundary. Their invariant refinements

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{\substack{L \supset M \\ L \neq M}} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big) \tag{4.2}$$

represent corresponding terms in the invariant trace formula. If G is quasisplit, these objects in turn have stable refinements

$$S_M(\delta, f) = I_M(\delta, f) - \sum_{\substack{G' \in \mathcal{E}_M(G) \\ G' \neq G}} \iota_M(G, G') \widehat{S}_{\tilde{M}}^{\tilde{G}'}(\delta, f'), \tag{4.3}$$

where δ is a G-regular, *stable* conjugacy class in $M(\mathbb{R})$, and

$$I_M(\delta, f) = \sum_{\gamma \to \delta} I_M(\gamma, f).$$

They become corresponding terms in the stable trace formula.

We refer the reader to [A7, §1] and [A8, §1] for discussion of the inductive definitions (4.2) and (4.3). Keep in mind that although $I_M(\gamma, f)$ is an invariant distribution, in the sense that it is invariant under conjugation by $G(\mathbb{R})$, it is by no means equal to an invariant orbital integral. Similarly, $S_M(\delta, f)$ is not a stable orbital integral, even though it is a stable distribution.

The problem is to compute the Fourier transform of any of the three kinds of tempered distributions. In each case, the problem is to compute the distribution explicitly as a linear form on the relevant dual space. For example, since $J_M(\gamma, f)$ is a noninvariant distribution, its Fourier transform is a continuous linear form on the Schwartz space $\mathcal{C}(\widehat{G})$ of matrix valued functions on $\Pi_{\mathrm{temp}}(G)$. Since $I_M(\gamma, f)$ is invariant, its Fourier transform can be treated as a continuous linear form on $\mathcal{I}(G)$, regarded now as a Schwartz space of functions on $\Pi_{\mathrm{temp}}(G)$. Since $S_M(\delta, f)$ is stable, its Fourier transform becomes a continuous linear form on $\mathcal{S}(G)$, regarded as a Schwartz space on the set $\Phi_{\mathrm{temp}}(G)$ of tempered Langlands parameters. (It is on this understanding that the notation $\widehat{I}$ and $\widehat{S}$ in (4.2) and (4.3) is based.)

The problem was solved for $G = SL(2)$ in [AHS], and for G of real rank 1 in [Ho]. In general, there are two sources of difficulty. The first is analytic. One tries to characterize the Fourier transform uniquely in terms of the analytic properties it satisfies. The second is combinatorial. This entails imposing some sort of order on the complicated functions that make up the Fourier transform. W. Hoffmann has made considerable progress on both fronts.

The equation (4.3) is actually part of the solution of a similar (though simpler) problem. It represents an inductive definition of the terms in a general identity, which was stated and proved in [A8, Theorem 1.1], and which amounts to a stabilization of the invariant distributions $I_M(\gamma, f)$. The analytic properties used in the proof are the differential equations

$$I_M(\gamma, zf) = \sum_{L \supset M} \partial_M^L(\gamma, z_L) \widehat{I}_L(\gamma, f), \qquad z \in \mathcal{Z}(G),$$

satisfied by $I_M(\gamma, f)$ as a function of γ, the boundary conditions satisfied by $I_M(\gamma, f)$ as γ approaches a singular hypersurface, and an asymptotic formula [A7] for $I_M(\gamma, f)$ as both γ and the support of f approach infinity. (Much of the paper [A8] was devoted to the stabilization of these properties.) The three properties might also suffice to characterize the relevant Fourier transforms. However, they do not seem to help with combinatorial questions.

The problem of computing Fourier transforms of weighted orbital integrals goes back to Selberg, or at least to the study of his work by Langlands in the 1960's. The complicated nature of these objects, and the lack of a clear application, has been discouraging. However, it does seem to me that a solution could now be very useful. It might allow us to investigate local aspects of Langlands' proposal [L3] for using the trace formula to study the principle of functoriality. For example, Langlands has used Hoffmann's solution of the problem for $G = GL(2)$ to investigate relationships among some of the terms that arise for this group [L2].

5. Intertwining operators and residues

We can agree that the spectral analogues of *invariant* orbital integrals are irreducible characters

$$f_G(\pi) = \mathrm{tr}\big(\pi(f)\big) = \int_{G(\mathbb{R})} f(x)\Theta_\pi(x)\mathrm{d}x, \qquad f \in \mathcal{C}(G).$$

Weighted orbital integrals have their own spectral analogues, known as weighted characters. These objects are distributions obtained by taking a "noninvariant trace" of operators $\mathcal{I}_P(\sigma, f)$, for representations $\sigma \in \Pi_{\mathrm{temp}}(M)$. In other words, they are defined by a trace

$$J_M(\sigma, f) = \mathrm{tr}\big(\mathcal{R}_P(\sigma, P)\mathcal{I}_P(\sigma, f)\big), \qquad f \in \mathcal{C}(G), \tag{5.1}$$

of the product of $\mathcal{I}_P(\sigma, f)$ with a natural non-scalar operator

$$\mathcal{R}_P(\sigma, P): \ \mathcal{H}_{\mathcal{P}}(\sigma) \longrightarrow \mathcal{H}_{\mathcal{P}}(\sigma),$$

on the space $\mathcal{H}_P(\sigma)$ on which $\mathcal{I}_P(\sigma, f)$ acts. This operator-valued weight factor is built out of the normalized intertwining operators [A2]

$$R_{Q|P}(\sigma_\lambda): \ \mathcal{H}_P(\sigma) \longrightarrow \mathcal{H}_Q(\sigma), \qquad P, Q \in \mathcal{P}(M), \tag{5.2}$$

between the induced representations $\mathcal{I}_P(\sigma_\lambda)$ and $\mathcal{I}_Q(\sigma_\lambda)$. It is defined as (a multiple of) a limit

$$\mathcal{R}_M(\sigma, P) = \lim_{\lambda \to 0} \sum_{Q \in \mathcal{P}(M)} R_{Q|P}(\sigma)^{-1} R_{Q|P}(\sigma_\lambda)\Big(\prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee)\Big)^{-1}, \tag{5.3}$$

which reduces to the logarithmic derivative

$$\lim_{\lambda \to 0} \Big(R_{\bar{P}|P}(\sigma)^{-1} \frac{d}{d\lambda} R_{\bar{P}|P}(\sigma_\lambda)\Big),$$

in case M is maximal in G.

Recall that σ_λ is a twist of the representation σ by a point λ in a complex vector space

$$\mathfrak{a}^*_{M,\mathbb{C}} = X(M)_\mathbb{R} \otimes \mathbb{C}^*.$$

The normalized intertwining operator (5.2) is a meromorphic function of λ, whose restriction to the space $i\mathfrak{a}^*_M$ is analytic. It follows that weight factor (5.3) can be continued to a mermorphic function $\mathcal{R}_M(\sigma_\lambda, P)$, which is also analytic on $i\mathfrak{a}^*_M$. If f is compactly supported, $\mathcal{I}_P(\sigma_\lambda, f)$ is an entire function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. In fact, we may as well take f from the Hecke algebra $\mathcal{H}(G)$ on $G(\mathbb{R})$, relative to the maximal compact subgroup $K_\mathbb{R}$ which is already implicit in the construction above. Then the weighted character also extends to a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, which is again analytic on $i\mathfrak{a}^*_M$. Can one say anything about its residues at singular points?

The linear form $J_M(\sigma_\lambda, f)$ on $\mathcal{H}(G)$ is not invariant. It turns out in fact that the failure of the weighted character (5.1) to be invariant is parallel (in a precise quantitative sense) to the failure also of the weighted orbital integral (4.1) to be invariant. It is this property that is behind the definition of the invariant distribution (4.2). Indeed, the argument

$$\phi_L(f), \qquad f \in \mathcal{C}(G),$$

on the right hand side of this formula is the function in $\mathcal{I}(L)$ defined by

$$\phi_L(f):\ \pi_L \longrightarrow J_L(\pi_L, f),$$

for irreducible tempered representations π_L of $L(\mathbb{R})$. However, it is the meromorphic function $J_M(\sigma_\lambda, f)$, defined for $f \in \mathcal{H}(G)$, that is our focus here. One sees from the quantitative description of its failure to be invariant that its multi-residues in λ are actually are invariant linear forms in f. What are they?

To form a multi-residue one takes a residue datum, consisting of the flag

$$\mathfrak{a}_M = \mathfrak{a}_{M_0} \supset a_{M_1} \supset a_{M_r} = \mathfrak{a}_G$$

attached to a maximal chain of Levi subgroups

$$M = M_0 \subset M_1 \subset \cdots \subset M_r = G,$$

a unit vector E_i in the orthogonal complement of $\mathfrak{a}^*_{M_i}$ in $\mathfrak{a}^*_{M_{i-1}}$ for each i, and a point Λ_Ω in $\mathfrak{a}^*_{M,\mathbb{C}}$. The associated iterated residue

$$\operatorname*{Res}_{\Omega}\big(J_M(\sigma, f)\big) = \operatorname*{Res}_{\Omega,\Lambda\to\Lambda_\Omega}\big(J_M(\sigma_\Lambda, f)\big), \qquad f \in \mathcal{H}(G),$$

is invariant in f. More generally, the iterated residue

$$\operatorname*{Res}_{\Omega,X}\big(J_M(\sigma, f)\big) = \operatorname*{Res}_{\Omega,\Lambda\to\Lambda_\Omega}\big(J_M(\sigma_\lambda, f)e^{-\Lambda(X)}\big), \qquad f \in \mathcal{H}(G), \tag{5.4}$$

is an invariant linear form in f that depends on a point $X \in \mathfrak{a}_M$ [A2, Lemma 8.1]. Can one describe it as a linear combination of characters? Does the answer require derivatives at $\Lambda = \Lambda_\Omega$ of the function $\mathcal{I}_P(\sigma_\Lambda, f)$?

There is a formula that imposes some further structure on these questions by relating them to elliptic tempered characters. It applies to *cuspidal* functions $f \in \mathcal{H}(G)$. We are assuming here that G is cuspidal, as in §3. For simplicity, assume also that A_G is trivial, and that f satisfies the stronger condition that its image in $\mathcal{I}(G)$ is supported on the discrete series characters of $G(\mathbb{R})$. Suppose in addition that M is cuspidal, and that $\gamma \in M_{\mathrm{ell}}(\mathbb{R})$ is in general position. There has been much study of the normalized discrete series characters

$$\Phi(\pi, \gamma) = \Phi_G(\pi, \gamma) = |D(\gamma)|^{\frac{1}{2}}\Theta(\pi, \gamma)$$

over the years, beginning with the first [Ha3] of the two papers of Harish-Chandra. The formula in question gives a relation

$$\sum_{\pi} \Phi_G(\pi^\vee, \gamma)\,\mathrm{tr}\big(\pi(f)\big) = \sum_{\sigma}\sum_{\Omega}(-1)^{\dim(A_M)}\,\Phi_M(\sigma^\vee, \gamma)\operatorname*{Res}_{\Omega,X}\big(J_M(\sigma, f)\big) \tag{5.5}$$

between these objects and the residues (5.4), in which

$$X = H_M(\gamma)$$

is the image of γ in the vector space $\mathfrak{a}_M$.

The formula (5.5) is a special case of [A6, (9.4)]. (See [A6, (9.1) and Remark (1) on p. 135].) The first two sums are over discrete series representations π of $G(\mathbb{R})$ and σ of $M(\mathbb{R})/A_M(\mathbb{R})$, with contragredients $\pi^\vee$ and $\sigma^\vee$. The third sum is over the finite set of residue data Ω associated with the residue scheme of the real Paley-Wiener theorem or the spectral decomposition of Eisenstein series. Namely, it is the sum of residues encountered in deforming the contour of an integral

$$\int J_M(\sigma_\Lambda, f)e^{-\Lambda(X)}\mathrm{d}\Lambda$$

from $\mu(X)+i\mathfrak{a}_M^*$ to $\varepsilon+i\mathfrak{a}_M^*$, where $\mu(X)$ is a large point in general position in the chamber $(\mathfrak{a}_P^*)^+$ for which X lies in $\mathfrak{a}_P^+$, and ε is a *small* point in general position in $\mathfrak{a}_M^*$. What is the meaning of the right hand side of (5.5)? It seems to keep track of constituents of induced representations $\mathcal{I}_P(\sigma_\Lambda)$ that are discrete series, or at least that match discrete series on cuspidal functions f. What is its relation to Osborne's conjecture, which is the real analogue of Casselman's p-adic embedding theorem [C], and has been proved by Hecht and Schmid [HeSc]?

6. Twisted groups

All of the problems discussed so far can be posed more generally for twisted groups. To do so, we need to inflate G to a triplet (G, θ, ω), where θ is an automorphism of G over $\mathbb{R}$, and ω is a character on $G(\mathbb{R})$. This is the setting of Kottwitz and Shelstad [KS], who construct transfer factors that generalize those of [LS1]. We shall include a few remarks here, leaving to the reader the exercise of formulating more precisely the problems of §1–§5 for twisted groups.

The notation is easier to reconcile with that of previous sections if we write G^0 in place of G. We can then use the symbol G for the variety

$$G = G^0 \rtimes \theta$$

over $\mathbb{R}$, equipped with the obvious two-sided G^0-action

$$x_1(x \rtimes \theta)x_2 = (x_1 x\theta(x_2)) \rtimes \theta.$$

In fact, following our convention for endoscopic data, we may as well let G also represent the triplet

$$(G^0, \theta, \omega).$$

A point $\gamma \in G(\mathbb{R})$ may be called *strongly G-regular* if its G^0-centralizer

$$G^0_\gamma = \{y \in G^0 : y^{-1}\gamma y = \gamma\}$$

is a torus, with the property that $G^0_\gamma(\mathbb{R})$ lies in the kernel of ω. It gives rise to a (twisted) invariant orbital integral

$$f_G(\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{G^0_\gamma(\mathbb{R})\backslash G^0(\mathbb{R})} f(x^{-1}\gamma x)\omega(x)\,\mathrm{d}x, \qquad f \in \mathcal{C}(G).$$

(Our understanding here is that a tempered distribution D in $G(\mathbb{R})$ will be called *invariant* if

$$D(f^y) = D(f)\omega(y), \qquad f \in \mathcal{C}(G),\ y \in G^0(\mathbb{R}),$$

where $f^y(x) = f(yxy^{-1})$.)

For the spectral analogue, we let $\Pi_{\mathrm{temp}}(G)$ denote the set of representations $\pi \in \Pi_{\mathrm{temp}}(G^0)$ such that $\pi \circ \theta$ is equivalent to $\omega \otimes \pi$. These representations extend to $G(\mathbb{R})$. To be more precise, we introduce a set $\widetilde{\Pi}_{\mathrm{temp}}(G)$, consisting of unitary equivalence classes of (continuous) mappings $\tilde{\pi}$ from $G(\mathbb{R})$ to the space of unitary operators on a Hilbert space V such that

$$\tilde{\pi}(x_1 x x_2) = \pi(x_1)\tilde{\pi}(x)\pi(x_2)\omega(x_2), \qquad x_1, x_2 \in G^0(\mathbb{R}), \tag{6.1}$$

for a representation $\pi \in \Pi_{\mathrm{temp}}(G)$ on V. Then $\widetilde{\Pi}_{\mathrm{temp}}(G)$ is a principal $U(1)$-bundle over $\Pi_{\mathrm{temp}}(G)$, relative to the mapping $\tilde{\pi} \to \pi$, and the obvious action of $U(1)$ on $\widetilde{\Pi}_{\mathrm{temp}}(G)$. The (twisted) character of $\tilde{\pi} \in \widetilde{\Pi}_{\mathrm{temp}}(G)$ is the tempered invariant distribution

$$f_G(\tilde{\pi}) = \mathrm{tr}\big(\tilde{\pi}(f)\big) = \mathrm{tr}\Big(\int_{G(\mathbb{R})} f(x)\tilde{\pi}(x)\mathrm{d}x\Big), \qquad f \in \mathcal{C}(G),$$

on $G(\mathbb{R})$. For any f, the function $f_G(\tilde{\pi})$ can be regarded as a section of the bundle $\widetilde{\Pi}_{\mathrm{temp}}(G)$.

Following [KS, (2.1)], we choose an automorphism $\hat{\theta}$ of $\hat{G}^0$ that is dual to θ, and that preserves a Γ-splitting. We also choose a 1-cocycle a_ω from $W_{\mathbb{R}}$ to $Z(\hat{G})$ that is the Langlands dual of the character ω on $G(\mathbb{R})$. We can then form the L-automorphism

$${}^L\theta = {}^L\theta_\omega : g \times w \to \hat{\theta}(g)a_\omega(w)^{-1} \times w, \qquad g \times w \in {}^LG^0,$$

of the L-group ${}^LG^0$ of G^0. This in turn gives rise to the dual set

$$\hat{G} = \hat{G}_\omega = \hat{G}^0 \rtimes {}^L\theta_\omega.$$

The L-group ${}^LG^0$ acts by conjugation on $\hat{G}$.

Suppose that ϕ is a tempered Langlands parameter for G^0. We can then form the finite group

$$\mathcal{S}^0_\phi = \mathcal{S}_\phi(G^0) = \pi_0(S_\phi(G^0)),$$

relative to G^0 as in §2, and the finite set

$$\mathcal{S}_\phi = \mathcal{S}_\phi(G) = \pi_0(S_\phi(G)),$$

where

$$S_\phi(G) = \mathrm{Cent}(\hat{G}, \phi(W_{\mathbb{R}}))$$

is defined relative to G. We let $\Phi_{\mathrm{temp}}(G)$ denote the set of parameters ϕ such that $\mathcal{S}_\phi(G)$ is nonempty. For any such ϕ, the finite group $\mathcal{S}^0_\phi$ acts simply transitively on both the left and the right of $\mathcal{S}_\phi$.

There is no need to generalize stable orbital integrals and stable characters to twisted groups, since these objects are needed only for the twisted endoscopic groups G' of G, which are again quasisplit and connected. This time G' represents a larger twisted endoscopic datum $(G', \mathcal{G}', s', \xi')$, defined for G as in [KS, §2.1]. In particular, s' is a semisimple element in $\hat{G}$, $\mathcal{G}'$ is a split extension of $W_{\mathbb{R}}$ by $\widehat{G}'$ and ξ' is an L-embedding of $\mathcal{G}'$ into ${}^LG^0$, whose image centralizes s'. Given an L-embedding

$$\xi' : {}^LG' \longrightarrow {}^LG^0,$$

obtained from an identification of $\mathcal{G}'$ with ${}^LG'$, one defines the twisted form of the original transfer mapping by the natural analogue

$$f'(\delta') = \sum_\gamma \Delta(\delta', \gamma) f_G(\gamma)$$

of (1.2). Thus δ' is a stable conjugacy class in $G'(\mathbb{R})$ in general position, γ ranges over strongly G-regular $G^0(\mathbb{R})$-conjugacy classes in $G(\mathbb{R})$, and $\Delta(\delta',\gamma)$ is a twisted transfer factor of [KS]. As in §1, the problem is to show that f' belongs to $\mathcal{S}(G')$. It was solved by Renard [R], at least in the case that ω is trivial.

Renard's results are based on the twisted transfer factors $\Delta(\delta',\gamma)$, which generalize the ordinary transfer factors of [LS1]. In this sense, they represent an answer to some of the inquiries of §1. However, they also depend to some extent on constructions from [She5]. It would again be very useful to establish the twisted transfer mapping in more concrete terms, relating it if possible to the work of Harish-Chandra.

The corresponding twisted character identities should be similar in form to the identities of §2. For a given pair (G',ϕ'), there will be an expansion

$$f'(\phi') = \sum_{\pi\in\Pi_{\mathrm{temp}}(G)} \Delta(\phi',\tilde{\pi}) f_G(\tilde{\pi}),$$

for coefficients $\Delta(\phi',\widetilde{\pi})$ which are determined by the choice of twisted transfer factor Δ that defines f', and which satisfy

$$\Delta(\phi',u\tilde{\pi}) = \Delta(\phi',\tilde{\pi})u^{-1}, \qquad u\in U(1),\ \tilde{\pi}\in\tilde{\Pi}_{\mathrm{temp}}(G).$$

There ought then to be a function $\rho(\Delta,s')$ with analogues of the properties (i) and (ii) of §2. In particular, the function

$$\langle s,\tilde{\pi}\rangle = \rho(\Delta,s')^{-1}\Delta(\phi',\tilde{\pi}), \qquad s\in\mathcal{S}_\phi,$$

on $\mathcal{S}_\phi$ attached to any $\tilde{\pi}\in\widetilde{\Pi}_\phi$ should be an extension of the corresponding function $\langle\cdot,\pi\rangle$ on $\mathcal{S}^0_\phi$, in the sense that

$$\langle s_1 s s_2,\tilde{\pi}\rangle = \langle s_1,\pi\rangle\langle s,\tilde{\pi}\rangle\langle s_2,\pi\rangle, \qquad s_1,s_2\in\mathcal{S}^0_\phi.$$

(We write Π^0_ϕ, Π_ϕ and $\tilde{\Pi}_\phi$ for the subsets of $\Pi_{\mathrm{temp}}(G^0)$, $\Pi_{\mathrm{temp}}(G)$ and $\tilde{\Pi}_{\mathrm{temp}}(G)$ respectively attached to ϕ.) The problem of establishing such identities appears to be completely open.

The twisted analogues of other problems require the notion of a Levi subset of G. A *parabolic subset* of G is a nonempty subset P that equals the normalizer in G of a parabolic subgroup P^0 of G^0 over F. The correspondence $P\to P^0$ is an injection from the set of such P to the set of P^0 such that $\theta(P^0)$ is conjugate to P^0. A *Levi subset* of G is a rational Levi component M of a parabolic subset P, which is to say, the normalizer in P of a Levi component M^0 of P^0 over F. For any such M, one forms the finite set $\mathcal{P}(M)\subset\mathcal{P}(M^0)$ and the real vector space $\mathfrak{a}_M\subset\mathfrak{a}_{M^0}$. One can then formulate twisted versions of weighted orbital integrals and weighted characters, and corresponding analogues of the problems of §4 and §5.

7. Trace identities for intertwining operators

To simplify the remaining discussion, we assume again that G is a connected reductive group. The normalized intertwining operators $R_{Q|P}(\sigma)$ appear as local ingredients of several terms in the global trace formula. The most sensitive of these concerns the case that

$$Q = w^{-1}Pw, \qquad w\in W(M),$$

and

$$\sigma\cong w^{-1}\sigma w, \qquad \sigma\in\Pi_{\mathrm{temp}}(M),$$

where w is represented by an element in $G(\mathbb{R})$ that normalizes M. In this case, one uses $R_{Q|P}(\sigma)$ to construct a self-intertwining operator

$$R_P(\sigma_w) = A(\sigma_w)\circ R_{Q|P}(\sigma) \qquad (7.1)$$

of the induced representation $\mathcal{I}_P(\sigma)$. The relevant local object is the trace

$$\operatorname{tr}\big(R_P(\sigma_w)\mathcal{I}_P(\sigma, f)\big). \tag{7.2}$$

The problem is to interpret this trace in terms of the endoscopic character identities of Shelstad.

There are really two questions. The first is to formulate a precise conjectural identity for (7.2) in terms of the spectral transfer factors $\Delta(\phi', \pi)$ and the characters $\langle s, \pi\rangle$. This is already quite subtle. Such a formula was stated in [A5, §7], by allowing representations with Whittaker models to serve as base points. The conjectural formula is pretty complicated, partly because it was stated in much greater generality. The theory of Whittaker models is well understood for real groups [V1]. Following the discussion of Sections 1, 2 and 6, one might try to formulate the conjectural identity as clearly and simply as possible in the special case of tempered distributions of real groups we are dealing with here.

The second problem would be to prove the identity! Shahidi has done so in the special case that the original inducing representation $\sigma \in \Pi_{\mathrm{temp}}(M)$ has a Whittaker model [Sha]. The more general situation seems to be considerably harder. If G is a classical group, it is likely that such identities can be established by global means, which at present rely on the fundamental lemma. Perhaps one could approach the problem locally through the theory of minimal K-types [V2].

It might be helpful to add a few remarks about the first question, by way of introduction to the conjectural identity of [A5, §7]. We write M_w for both the $\mathbb{R}$-rational subvariety Mw of G, and the triplet $\big(M, \mathrm{Int}(w), 1\big)$, following the convention of §6. The operator

$$A(\sigma_w):\ \mathcal{H}_Q(\sigma) = \mathcal{H}_{w^{-1}Pw}(\sigma) \longrightarrow \mathcal{H}_P(\sigma)$$

in (7.1) is defined by

$$\big(A(\sigma_w)\phi_1\big)(x) = \sigma_w(w)\phi_1(w^{-1}x), \qquad \phi_1 \in \mathcal{H}_Q(\sigma),$$

where σ_w is an extension of the representation σ of $M(\mathbb{R})$ to the group generated by $M_w(\mathbb{R})$. This last object is an essential ingredient. Since its restriction to $M_w(\mathbb{R})$ satisfies (6.1), in the special case here, σ_w can be regarded as an element in $\tilde{\Pi}_{\mathrm{temp}}(M_w)$. Let ϕ_w be the Langlands parameter for M such that σ lies in the packet Π_{ϕ_w}. Then ϕ_w is an element in the subset $\Phi_{\mathrm{temp}}(M_w)$ of $\Phi_{\mathrm{temp}}(M)$. As such, it gives rise to the $\mathcal{S}^0_{\phi_w}$-torsor $\mathcal{S}_{\phi_w}$ described in §6. Since σ_w belongs to the packet $\tilde{\Pi}_{\sigma_w}$, it should yield a function

$$\langle s_w, \sigma_w\rangle = \rho(\Delta_w, s'_w)\Delta_w(\phi'_w, \sigma_w), \qquad s_w \in \mathcal{S}_{\phi_w}.$$

The extension of σ_w of σ is not unique. If it is replaced by its product $\xi\sigma_w$ with a character ξ of cyclic subgroup of $W(M)$ generated by w, the expression (7.2) is multiplied by the complex number $u = \xi(w)$. On the other hand,

$$\Delta_w(\phi'_w, \xi\sigma_w) = \Delta_w(\phi'_w, \sigma_w)u^{-1}.$$

It follows that the product

$$\langle s_w, \sigma_w\rangle \operatorname{tr}\big(R_P(\sigma_w), \mathcal{I}_P(\sigma, f)\big) \tag{7.3}$$

of (7.2) with $\langle s_w, \sigma_w\rangle$ is independent of the extension σ_w. The product is also independent of the transfer factor Δ_w. However, it does depend on the choice of function $\rho(\Delta_w, s'_w)$, which in turn is determined only up to multiplication by a function $\rho(s_w)$ on $\mathcal{S}_{\phi_w}$ such that

$$\rho(s_1 s_w s_2) = \rho^0(s_1)\rho(s_w)\rho^0(s_2), \qquad s_1, s_2 \in \mathcal{S}^0_{\phi_w},$$

for some character ρ^0 on $\mathcal{S}^0_{\phi_w}$.

Let $\phi \in \Phi_{\mathrm{temp}}(G)$ be the Langlands parameter for G induced from ϕ_w. The short exact sequence (3.2) actually sits in a larger commutative diagram [A5, (7.1)], which consists of four exact sequences. The set $\mathcal{S}_{\phi_w} = \mathcal{S}^0_{\phi_w} \rtimes w$ is in bijection with the $\mathcal{S}^1_\phi$-coset $\mathcal{S}^1_{\phi,w} = \mathcal{S}^1_\phi w$ of w in the group $\mathfrak{N}_\phi$ at the center of this diagram. (We are following the notation of [A5, (7.1)] here, but with ϕ in place of the more general parameter ψ.) Let

$$s_w \longrightarrow s, \qquad\qquad s_w \in \mathcal{S}_{\phi_w},$$

be the projection mapping from this subset of $\mathfrak{N}_\phi$ onto the group $\mathcal{S}_\phi$. The conjectural identity of [A5] amounts to the assertion that (7.3) equals

$$c(s_w) \sum_{\pi \in \Pi_{\phi,\sigma}} \langle s, \pi \rangle f_G(\pi), \tag{7.4}$$

for a constant $c(s_w)$ [A5, Conjecture 7.1]. The constant becomes explicit (and independent of s_w) with a judicious choice of functions $\rho(\Delta_w, s_w)$ and $\rho(\Delta, s)$ that is ultimately based on Whittaker models. In order to formulate the identity in case G is not quasisplit, one would also want to verify that the expression [A5, (7.9)] is a transfer factor for G, a hypothesis that was put forward before the twisted transfer factors appeared in [KS].

8. Construction of A-packets

The spectral questions we have discussed to this point apply only to tempered representations. General nontempered representations do not behave in the same way. However, there are some nontempered representations that inherit much of the structure of tempered representations. They are the representations that are thought to occur in discrete spectra of spaces of automorphic forms. Such representations should of course be unitary. I do not know whether it is expected that, conversely, unitary representations should all have structure in common with tempered representations.

The structure in question arises from the endoscopic transfer of characters. In particular, the relevant nontempered representations occur in packets Π_ψ. These packets generalize tempered L-packets. However, they are quite different from general nontempered L-packets, which are incompatible with endoscopic transfer. They are parametrized by mappings

$$\psi :\ W_{\mathbb{R}} \times SL(2, \mathbb{C}) \longrightarrow {}^L G$$

for which the projection onto $\widehat{G}$ of $\psi(W_{\mathbb{R}})$ is relatively compact. We take such mappings up to $\widehat{G}$-conjugacy, and denote the resulting family by $\Psi(G)$. The packets Π_ψ were constructed by geometric means in [ABV], and were shown there to satisfy the conjectured endoscopic properties.

As originally envisaged [A4], the representations in a packet Π_ψ were conjectured to be irreducible. This would have provided a well defined construction of the packets in terms of harmonic analysis, specifically a series of conjectural character identities. However, in their study of the characters of unitary representations with $(\mathfrak{g}, K)$-cohomology [AJ], Adams and Johnson showed that the constituents of a packet need not be irreducible. (See also [A5, §5].) This was reflected in the expanded account [A5] of the conjectures, without however being accompanied by a corresponding means for determining the packets Π_ψ uniquely. The geometric methods by which the packets were eventually defined in [ABV] are remarkable, and will probably be an important part of future progress. Nonetheless, it would be interesting to have an alternative way to characterize the packets that are based purely on harmonic analysis. I pose this as a problem, without having a sense of whether any such thing is possible in general.

The basic problem is to define a stable distribution

$$f \longrightarrow f^G(\psi), \qquad f \in \mathcal{H}(G), \tag{8.1}$$

in case G is quasisplit and ψ is any parameter in $\Psi(G)$. If ψ is trivial on the second factor $SL(2,\mathbb{C})$, it reduces to a parameter $\phi \in \Phi_{\mathrm{temp}}(G)$. The stable distribution was defined in this case by the sum (2.2), taken over representations in the packet Π_ϕ. The point here is that the packet Π_ϕ had already been defined in Langlands' original classification [L1]. There is no such a priori construction of a general nontempered packet Π_ψ.

Suppose that the distributions (8.1) have been defined, in some fashion, whenever G is quasisplit. Let me recall how the conjectures, stated in [A5] and proved in [ABV], then lead to the general packets Π_ψ.

For an arbitrary G and ψ, we first form the centralizer

$$S_\psi = \mathrm{Cent}\big(\widehat{G}, \psi(W_{\mathbb{R}} \times SL(2,\mathbb{C})\big),$$

and its group $\mathcal{S}_\psi$ of connected components, as in the special case of §2. This gives rise to a bijective correspondence

$$(G', \psi') \longrightarrow (\psi, s'),$$

again as in §2. The distribution

$$f \longrightarrow f'(\psi'), \qquad f \in \mathcal{H}(G),$$

is then defined by hypothesis, and depends implicitly on a choice of transfer factor Δ for (G, G'). Following (2.1), we decompose it as a linear combination

$$f'(\psi') = \sum_\pi \Delta(\psi', \pi) f_G(\pi) \tag{8.2}$$

of irreducible (nontempered) characters π, with coefficients $\Delta(\psi', \pi)$.

The first assertion is that there is a function $\rho(\Delta, s')$ such that for any π, the quotient

$$q(s, \pi) = \rho(\Delta, s')^{-1} \Delta(\psi', \pi)$$

depends only on the image s of s' in $\mathcal{S}_\psi$. At this point there is a new wrinkle. It comes from the central element

$$s_\psi = \psi\left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

in S_ψ, which we identify with its image in $\mathcal{S}_\psi$. The second assertion is that for any π, the function

$$s \longrightarrow q(s_\psi^{-1} s, \pi) = q(s_\psi s, \pi), \qquad s \in \mathcal{S}_\psi,$$

is a character (possibly 0) on the group (possibly nonabelian) $\mathcal{S}_\psi$. We decompose it as a linear combination

$$q(s_\psi^{-1} s, \pi) = \sum_\xi \xi(s) n_{\xi,\pi}$$

of irreducible characters ξ of $\mathcal{S}_\psi$, with nonnegative integral coefficients $n_{\xi,\pi}$. For any ξ such that $\eta_{\xi,\pi} \neq 0$ for some π, we set

$$\tau = \tau_\xi = \bigoplus_\pi n_{\xi,\pi} \pi.$$

We then define the packet Π_ψ to be the disjoint union over ξ of the representations (possibly reducible) τ_ξ. Any representation $\tau = \tau_\xi$ in Π_ψ thus comes with an irreducible character

$$s \longrightarrow \langle s, \tau \rangle = \xi(s)$$

on $\mathcal{S}_\psi$. The decomposition (8.2) then takes the form

$$f'(\psi') = \rho(\Delta, s') \sum_{\tau\in\Pi_\psi} \langle s_\psi s, \tau\rangle f_G(\tau). \tag{8.3}$$

With its explicit dependence on the transfer factors Δ, the statement of (8.3) differs slightly from that of its counterpart in [ABV] (or rather the special case in [ABV] that applies to functions supported on only one of the several groups G that make up an "extended group".) In the spirit of the questions posed in §1, one could try to compare the two formulations directly. Notice that the function $\rho(\Delta, s')$ is forced on us here, as it was in §2, since the transfer factor Δ attached to s' is determined only up to a scalar multiple. This function ought to be determined up to multiplication by a linear character in s, a change that would be reflected in a corresponding translation of the image of the injective mapping

$$\tau \longrightarrow \langle\cdot, \tau\rangle, \qquad \tau \in \Pi_\psi.$$

In [ABV], the mapping was normalized by relating it to a certain representation with a Whittaker model.

From the perspective of harmonic analysis, the problem is thus to characterize stable distributions (8.1). For many classical groups, a candidate for (8.1) can be obtained through endoscopic transfer from $GL(N)$.

Suppose for example that G is a quasisplit orthogonal or symplectic group. There is then a triplet

$$\widetilde{G} = (\widetilde{G}^0, \widetilde{\theta}, 1),$$

where $\widetilde{G}^0 = GL(N)$ and $\widetilde{\theta}(x) = {}^t x^{-1}$, for which G represents a twisted endoscopic datum. In particular, there is a canonical L-embedding

$$\xi :\ {}^L G \longrightarrow {}^L\widetilde{G}^0.$$

For any ψ, the mapping

$$w \longrightarrow \xi\left(\psi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right)\right), \qquad w \in W_{\mathbb{R}},$$

is a Langlands parameter (typically nontempered) for $\widetilde{G}^0 = GL(N)$. Its L-packet consists of course of one element π_ψ^0, a unitary Langlands quotient sometimes called a Speh representation. Using the theory of Whittaker models, one can define a canonical extension π_ψ of π_ψ^0 to the group generated by $\widetilde{G}(\mathbb{R})$. This granted, we set

$$\widetilde{f}^G(\psi) = \widetilde{f}_{\widetilde{G}}(\pi_\psi) = \mathrm{tr}\big(\pi_\psi(\widetilde{f})\big), \qquad \widetilde{f} = \mathcal{H}(\widetilde{G}). \tag{8.4}$$

Of course, for this to make sense, one has to show that the right hand side depends only on the image $\widetilde{f}^G$ of $\widetilde{f}$ in the stable Hecke algebra

$$S\mathcal{H}(G) = \big\{f^G :\ f \in \mathcal{H}(G)\big\}$$

on $G(\mathbb{R})$ under twisted endoscopic transfer. If G is of the form $SO(2n+1)$ or $Sp(2n)$, the image of $\mathcal{H}(\widetilde{G})$ under the mapping $\widetilde{f} \to \widetilde{f}^G$ is the entire space $S\mathcal{H}(G)$. In this case, (8.4) serves to define the distribution (8.1). If G is of the form $SO(2n)$, the image is the subspace of functions in $S\mathcal{H}(G)$ that are fixed by the nontrivial outer automorphism θ of $G(\mathbb{R})$ (an automorphism induced from the nontrivial component in $O(2n)$). In this case, (8.4) specifies only the symmetrized distribution

$$\frac{1}{2}\big(f^G(\psi) + f^G(\theta\psi)\big).$$

The linear form (8.4) can be studied by global means. A global comparison of trace formulas will likely yield stable distributions and character formulas (8.3), with the caveat above that implies weaker assertions in case $G = SO(2n)$. Such methods lead also to results for p-adic groups, but for the moment are conditional on the fundamental lemma (in its most general form that applies to twisted weighted orbital integrals).

Let me pose the problem of comparing (8.4) with [ABV]. In the case of classical groups, does the definition (8.4) match the geometric construction of [ABV]? In general, the nontempered character identity (8.3) should have a twisted analogue, along the lines of the twisted generalization in §6 of the discussion of §2. The formula (8.4) amounts to a very special case of this. An extension of the general results of [ABV] to twisted groups would undoubtedly provide an answer to the question.

9. Properties of A-packets

There are many other questions one can pose for A-packets Π_ψ. The most obvious concern the structure of the representations in a given packet. When are these representations irreducible? When do they have tempered constituents? When does a packet Π_ψ contain elliptic representations? I do not know whether such questions are amenable to the geometric methods of [ABV]. It is not even clear the extent to which explicit answers might exist. Be this as it may, the case of classical groups is particularly interesting. Any new information in this case is likely to have immediate applications to spectra of automorphic forms.

As a matter of fact, most of the questions posed for tempered representations in §1–§7 have natural analogues for the nontempered packets Π_ψ. For example, the conjectural trace identity, described for tempered representations in §7, was originally stated for A-packets in [A5, §7]. Once again, it is likely that for classical groups the identity can be established by global methods that rely on the fundamental lemma. Any local insights would of course be very interesting.

Consider the orthogonality relations of §3. Might they have some analogue that applies to representations in a packet Π_ψ? This is a sharper form of the question of which representations in Π_ψ are elliptic. I have no idea whether it has any kind of reasonable answer. The first step would be to look at examples — say the unitary representations with $(\mathfrak{g}, K)$-cohomology studied by Adams and Johnson [AJ]. The characters of these representations are quite transparent. It ought to be possible to compute their elliptic inner products. In so doing, can one discern any pattern? If an answer does emerge, will it have any bearing on whether the representations in Π_ψ are irreducible?

In the tempered case, the stable characters (2.2) satisfy their own orthogonality relations. These formulas are simpler, for the reason that a stable character is elliptic if and only if the corresponding packet Π_ϕ is composed of discrete series. They give rise to a stabilization of the orthogonality relations for representations $\pi \in \Pi_\phi$. Is there anything similar for the stable characters (8.1)? In the case of classical groups, can one relate such things to the formula (8.4)? This would entail establishing twisted orthogonality relations for characters of Speh representations.

In another direction, consider the identity (5.5). This formula relates residues of intertwining operators with values of tempered characters on noncompact tori. Does it have any analogue for characters of representations $\tau \in \Pi_\psi$? The residue scheme that defines the right hand side (5.5) is given by a deformation of a contour $\mu(X) + i\mathfrak{a}_M^*$, where $\mu(X)$ is a large point in the chamber $(\mathfrak{a}_P^*)^+$ such that X lies in $\mathfrak{a}_P^+$. If there is any nontempered analogue of (5.5), it will have to involve deformation of other contours. These would presumably be of the form $\mu(c) + i\mathfrak{a}_M^*$, where $\mu(c)$ is a large point in some other chamber $c = c(\tau, X)$ in $\mathfrak{a}_M^X$ that depends on τ and X.

The questions I have tossed about in this section are quite scattered. They need to be better focused before we can see what merit (if any) they have. They do at least have a common foundation in harmonic analysis. For this reason, we can hope that any answers for the real groups under discussion here might also apply to p-adic groups. As I have suggested, the case of classical groups is worthy of special consideration.

10. Functorial transfer

The problems we have discussed up until now are all related in one way or another to endoscopic transfer. For example, the question of Fourier transforms from §4 is probably most natural for the stable distributions $S_M(\delta, f)$, even though it was originally posed for the basic weighted orbital integrals $J_M(\gamma, f)$. I would like to end by bringing up another open ended question. This one applies to a completely different kind of transfer.

The starting point for endoscopic transfer was the endoscopic embedding (1.1). Suppose now that

$$\rho : \ {}^LG' \longrightarrow {}^LG \tag{10.1}$$

is an arbitrary embedding. We assume only that G and G' are quasisplit groups over $\mathbb{R}$, and that ρ is an L-embedding of their L-groups. This is the local setting for Langlands' principle of functoriality, which applies to reductive groups over a global field. In [L3], Langlands proposed a tentative strategy for attacking the general global conjecture. It is highly speculative. However, it is also of great interest for what it offers, the possibility of being able to extend functoriality beyond the limited number of cases that are related to endoscopy.

Since [L3] is ultimately predicated on a comparison of trace formulas, it implicitly includes a transfer of functions. Recall that the stable Schwartz space $\mathcal{S}(G')$ can be identified with the natural Schwartz space on the set $\Phi_{\mathrm{temp}}(G')$ of tempered Langlands parameters for G'. Given the general L-embedding (10.1), and also a function $f \in \mathcal{C}(G)$, we define a function f^ρ on $\Phi_{\mathrm{temp}}(G')$ by setting

$$f^\rho(\phi') = f^G(\rho \circ \phi'), \qquad \phi' \in \Phi_{\mathrm{temp}}(G'). \tag{10.2}$$

It follows from the definitions imply that $f \to f^\rho$ is a continuous linear mapping from $\mathcal{C}(G)$ to $\mathcal{S}(G')$. Since f^ρ depends only on the image f^G of f in $\mathcal{S}(G)$, we in fact obtain a mapping $f^G \to f^\rho$ from $\mathcal{S}(G)$ to $\mathcal{S}(G')$. The hope is that it will some day be part of a comparison of stable trace formulas.

The trouble is that the mapping does not have a simple geometric characterization. Unlike its endoscopic companion $f \to f'$, it does not have a simple expression in terms of invariant orbital integrals. Since any comparison of trace formulas would be focused primarily on the geometric terms, the matter is serious. On the other hand, it is not the stable trace formula for G that one would hope to compare with its counterpart for G'. It is rather a hypothetical formula, attached to a finite dimensional representation r of ${}^LG$ (or perhaps several r), and derived from the stable trace formula of G. The relevant point here is that the geometric terms in the latter would depend on the stable orbital integrals of f only obliquely. The situation is murky, to say the least. Might one be able to guess at some aspect of the structure of the hypothetical r-trace formula for G by studying the mapping $f^G \to f^\rho$ in terms of stable orbital integrals? Do matters become any simpler if one takes a linear combination of mappings over several related groups G'?

One can of course consider special cases. For example, if G and G' are tori, the mapping $f^G \to f^\rho$ has a simple geometric formulation. This observation can be applied to the general case if f is restricted to be a cuspidal function. The image f^ρ then vanishes unless there are elliptic maximal tori $T_{\mathrm{ell}} \subset G$ and $T'_{\mathrm{ell}} \subset {}^LG$, with admissible L-embeddings ${}^LT'_{\mathrm{ell}} \subset {}^LG'$ and ${}^LT_{\mathrm{ell}} \subset {}^LG$ [LS1, (2.6)] such that $\rho({}^LT'_{\mathrm{ell}})$ is contained in ${}^LT_{\mathrm{ell}}$. In this case, the problem reduces to its analogue for the groups T'_{ell} and T_{ell}. The admissible embeddings of ${}^LT'_{\mathrm{ell}}$ and ${}^LT_{\mathrm{ell}}$ are of course an essential part of the answer. They play the same role as they did for endoscopic transfer [She4], [LS1, (3.5)], though it is more transparent here.

We could also take minimal Levi subgroups $M \subset G$ and $M' \subset G'$, since these groups are again maximal tori. Suppose that there are admissible embeddings ${}^LM' \subset {}^LG'$ and ${}^LM \subset {}^LG$ such that $\rho({}^LM')$ is contained in ${}^LM$. If f is any function in $\mathcal{C}(G)$, the restriction

$$f^\rho(\delta'), \qquad\qquad \delta' \in M'(\mathbb{R}),$$

of f^ρ to the stable conjugacy classes in $G'(\mathbb{R})$ that meet $M'(\mathbb{R})$ then has a simple formulation in terms of the associated restriction

$$f^G(\gamma), \qquad\qquad \gamma \in M(\mathbb{R}),$$

of f^G. It is of course given by the obvious reduction of the problem to the tori M and M'.

I mention the last example for the relation it bears to the stable distributions $S_M(\delta, f)$. These objects are among the most interesting geometric terms in the stable trace formula. I am assuming now that M and M' are as above, so in particular, M is a minimal Levi subgroup of G. It seems to me that it would be useful to try to relate the function

$$S_M(\delta, f), \qquad\qquad \delta \in M(\mathbb{R}) \cap G_{\mathrm{reg}}(\mathbb{R}),$$

with its analogue

$$\widehat{S}_{M'}(\delta', f^\rho), \qquad\qquad \delta' \in M'(\mathbb{R}) \cap G'_{\mathrm{reg}}(\mathbb{R}),$$

for G', which is obtained by functorial transfer of f. One would first transform $S_M(\delta, f)$ to a function

$$S_M(\delta', f), \qquad\qquad \delta' \in M'(\mathbb{R}) \cap G'_{\mathrm{reg}}(\mathbb{R}),$$

of δ' by the simple prescription above for M and M'. There would then be two functions of δ' one could try to compare. It is at this point that an explicit formula for the Fourier transform of $S_M(\delta, f)$ would be needed.

If there are any simple relations to be found, they will probably show up in a linear combination of functions $S_{M'}(\delta', f^\rho)$. Assume that we are given only the torus M', together with an L-embedding ${}^LM' \to {}^LM$. It is conceivable that the endoscopic relations (4.3) could offer guidance. One might look for a family $\mathcal{F}_{M'}(G)$ of functorial embeddings (10.1), with M' being a minimal Levi subgroup of G', such that

$$S_M(\delta', f) = \sum_{\rho \in \mathcal{F}_{M'}(G)} \iota_{M'}(G, \rho) \widehat{S}_{M'}(\delta', f^\rho), \tag{10.3}$$

for coefficients $\iota_{M'}(G, \rho)$. This is at best only a natural guess. My point is simply that there seem to be a number of experiments that can be performed with the distributions $S_M(\delta, f)$.

Suppose for example that $M' = M$. In this case, we could take $\mathcal{F}_{M'}(G)$ to be set $\mathcal{E}_M(G)$ that indexes the sum in (4.3), or perhaps some related set of endoscopic data for G. The functorial embeddings ρ would then coincide with endoscopic embeddings ξ'. This of course does not mean that the functorial transfer mappings $f \to f^\rho$ are the same as their endoscopic companions $f \to f'$. What are the implications in this case for a possible identity (10.3)? If f is restricted to be a cuspidal function, $S_M(\delta, f)$ has a simple expression as a linear combination of stabilized discrete series characters, evaluated at the point $\delta \in M(\mathbb{R})$. The experiment then becomes quite accessible. However, it still seems to offer us the possibility of new insights.

References

[ABV] J. Adams, D. Barbasch, and D. Vogan, *The Langlands Classification and Irreducible Characters for Real Reductive Groups*, Progr. Math. **104**, Birkhauser, Boston, 1992.

[AJ] J. Adams and J. Johnson, *Endoscopic groups and packets of non-tempered representations*, Compositio Math. **64** (1987), 271–309.

[A1] J. Arthur, *Some problems suggested by the trace formula*, in *Lie Group Representations II*, Springer Lecture Notes, Vol. 1041, 1984, 1–49.

[A2] ———, *Intertwining operators and residues I. Weighted characters*, J. Funct. Anal. **84** (1989), 19–84.

[A3] ———, *Intertwining operators and residues II. Invariant distributions,* Compositio Math. **70** (1989), 51–99.

[A4] ———, *On some problems suggested by the trace formula*, in *Lie Group Representations II*, Lecture Notes in Math., vol. 1041, Springer-Verlag, 1983, 1–49.

[A5] ———, *Unipotent automorphic representations: conjectures*, Astérisque **171–172** (1989), 13–71.

[A6] ———, *On elliptic tempered characters*, Acta. Math. **171** (1993), 73–138.

[A7] ———, *An asymptotic formula for real groups*, to appear in J. reine angew. Math.

[A8] ———, *Parabolic transfer for real groups,* preprint.

[AHS] J. Arthur, R. Herb and P. Sally, *The Fourier transform of weighted orbital integrals on $SL(2,\mathbb{R})$,* Contemp. Math. **53** (1986), 17–37.

[C] W. Casselman, *Characters and Jacquet modules*, Math. Ann. **230** (1977), 101–105.

[Ha1] Harish-Chandra, *A formula for semisimple Lie groups,* Amer. J. Math. **79** (1979), 733–760.

[Ha2] ———, *Invariant eigendistributions on a semisimple Lie group,* Trans. Amer. Math. Soc. **119** (1965), 457–508.

[Ha3] ———, *Discrete series for semisimple Lie groups. I,* Acta Math. **113** (1965), 241–318.

[Ha4] ———, *Discrete series for semisimple Lie groups. II,* Acta Math. **116** (1966), 1–111.

[Ha5] ———, *Harmonic analysis on real reductive groups. I,* J. Funct. Analysis **19** (1975), 104–204.

[HeSc] H. Hecht and W. Schmid, *Characters, asymptotics, and n-homology of Harish-Chandra modules*, Acta Math. **151** (1983), 49–151.

[Ho] W. Hoffmann, *The Fourier transforms of weighted orbital integrals on semisimple groups of real rank one,* J. reine angew. Math. **489** (1997), 53–97.

[KnSt] A. Knapp and E. Stein, *Intertwining operators for semisimple groups* II, Invent. Math. **60** (1980), 9–84.

[KnZ] A. Knapp and G. Zuckerman, *Normalizing factors and L-groups*, in *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math. vol. 33, Part 1, Amer. Math. Soc., 1979, 93–106.

[KoSh] R. Kottwitz and D. Shelstad, *Foundations of Twisted Endoscopy*, Astérisque, vol. 255.

[L1] R. Langlands, *On the classification of irreducible representations of real algebraic groups*, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, AMS Mathematical Surveys and Monographs, vol. 31, 1989, 101–170.

[L2] ———, *Beyond endoscopy*, in *Contributions to Automorphic Forms, Geometry, and Number Theory*, Johns Hopkins University Press, 2004, 611–698.

[L3] ———, *Un nouveau point de repère dans la théorie des formes automorphes,* to appear in Canad. Math. Bull.

[LS1] R. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), 219–271.

[LS2] ———, *Descent for transfer factors, The Grothendieck Festschift*, Vol. II, Birkhauser, Boston, 1990, 485–563.

[R] R. Reynard, *Twisted endoscopy for real groups,* Journal of the Inst. of Math. Jussieu **2** (2003), 529–566.

[Sha] F. Shahidi, *Local coefficients as Artin factors for real groups,* Duke Math. J. **52** (1985), 973–1007.

[She1] D. Shelstad, *Notes on L-indistinguishability (based on a lecture by R.P. Langlands)*, in *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math. vol. 33, Part 2, Amer. Math. Soc., 1979, 193–204.

[She2] ———, *Characters and inner forms of a quasi-split group over* $\mathbb{R}$, Compositio Math. **39** (1979), 11–45.

[She3] ———, *Orbital integrals and a family of groups attached to a real reductive group*, Ann. Sci. École Norm. Sup. **12** (1979), 1–31.

[She4] ———, *Embeddings of L-groups*, Canad. J. Math. **33** (1981), 513–558.

[She5] ———, *L-indistinguishability for real groups*, Math. Ann. **259** (1982), 385–430.

[V1] D. Vogan, *Gelfand-Kirillov dimension for Harish-Chandra modules,* Invent. Math. **48** (1978), 75–98.

[V2] ———, *The algebraic structure of the representations of semi-simple Lie groups I,* Ann. Math. **109** (1979), 1–60.