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THE ADJOINT REPRESENTATION IN RINGS OF FUNCTIONS

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Abstract. Let G be a connected, simple Lie group of rank n defined over the complex numbers. To a parabolic subgroup P in G of semisimple rank r, one can associate $n-r$ positive integers coming from the theory of hyperplane arrangements (see P. Orlik and L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), 167-189; *Coxeter arrangements*, in Proc. of Symposia in Pure Math., Vol. 40 (1983) Part 2, 269-291). In the case $r=0$, these numbers are just the usual exponents of the Weyl group W of G. These $n-r$ numbers are called coexponents.

Spaltenstein and Lehrer-Shoji have proven the observation of Spaltenstein that the degrees in which the reflection representation of W occurs in a Springer representation associated to P are exactly (twice) the coexponents (see N. Spaltenstein, *On the reflection representation in Springer's theory*, Comment. Math. Helv. **66** (1991), 618-636 and G. I. Lehrer and T. Shoji, *On flag varieties, hyperplane complements and Springer representations of Weyl groups*, J. Austral. Math. Soc. (Series A) **49** (1990), 449-485). On the other hand, Kostant has shown that the degrees in which the adjoint representation of G occurs in the regular functions on the variety of regular nilpotents in $\mathfrak{g} := Lie(G)$ are the usual exponents (see B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327-404). In this paper, we extend Kostant's result to Richardson orbits (or orbit covers) and we get a statement which is dual to Spaltenstein's. We will show that the degrees in which the adjoint representation of G occurs in the regular functions on an orbit cover of a Richardson orbit associated to P are also the coexponents.

1. Introduction

Let G be a connected, simple Lie group of rank n over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. Fix a maximal torus and a Borel subgroup $T \subset B$ with Lie algebras $\mathfrak{t} \subset \mathfrak{b}$. Let Φ be the root system and $W = W(\Phi)$ the Weyl group of Φ. The choice of B determines a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and positive roots Φ^+. Let J be a subset of Π. Denote by Φ_J the root subsystem of Φ spanned by the roots in J. Let W_J be the Weyl group of Φ_J and $\Phi_J^+ \subset \Phi^+$ a set of positive roots of Φ_J.

The natural representation of W on $\mathfrak{t}$ is called the reflection representation and will be denoted by V_{ref}. Let V^J be the subspace of $\mathfrak{t}$ fixed pointwise under the action of W_J. Define M^J to be the complex manifold obtained by removing from V^J all points which lie in the kernel of a root in Φ which is not identically zero on

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V^J. Orlik and Solomon [OS2] have shown that the Poincaré polynomial

$$P(M^J, t) = \sum_{i \geq 0} \dim\, H^i(M^J, \mathbf{C})\ t^i$$

factors as

$$(1 + b_1^J t)(1 + b_2^J t) \cdots (1 + b_{n-r}^J t)$$

where the b^J's are positive integers and r is the cardinality of J. The b^J's are called coexponents. When $r = 0$ the b^J's coincide with the usual exponents of W.

In a previous paper, the first author gave another way to compute the coexponents by counting lattice points in an extended fundamental domain for the affine Weyl group [So]. We recall that result here. Let θ be the highest root in Φ^+. Write $\theta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ and set $c_{-\theta} = 1$. Let $\tilde{\Pi} = \Pi \cup \{-\theta\}$. For $\tilde{J} \subset \tilde{\Pi}$ and $t \in \mathbb{N}$, define $p(\tilde{J}, t)$ to be the number of solutions to the equation

$$\sum c_\alpha x_\alpha = t$$

where the x_α's must be strictly positive integers and the sum is over all $\alpha \in \tilde{\Pi} - \tilde{J}$. Set

$$\chi_J(t) = \sum p(\tilde{J}, t)$$

where this sum is over all subsets $\tilde{J}$ of $\tilde{\Pi}$ that are W-conjugate to J. When t is prime to all c_α's, it turns out that $\chi_J(t)$ is polynomial in t of degree $n - r$ and its roots coincide with the coexponents. The polynomials $\chi_J(t)$ arise geometrically in the context of fixed point varieties of affine flag manifolds (see [So]), where they measure (up to a constant) the Euler characteristic of a family of varieties.

In A_n all the coefficients of θ equal 1. If $1 \leq t \leq n - r$, then $p(\tilde{J}, t) = 0$ for all $\tilde{J}$ that enter into the expression for $\chi_J(t)$. Since all t are prime to the c_α's, we find that $\{1, 2, \ldots, n - r\}$ are the roots of $\chi_J(t)$.

In B_n and C_n similar reasoning shows that $\{1, 3, 5, \ldots, 2(n-r)-1\}$ are the roots of $\chi_J(t)$ since c_α equals 1 or 2.

In D_n there are two cases. If J is not W-conjugate to a set of roots in $A_{n-2} \subset D_n$, then the same reasoning shows that $\{1, 3, \ldots, 2(n-r)-1\}$ are the roots of $\chi_J(t)$. If J is W-conjugate to

$$A_{i_1} \times A_{i_2} \times \cdots \times A_{i_k} \subset A_{n-2},$$

then $\{1, 3, \ldots, 2(n-r)-3\}$ are definitely roots of $\chi_J(t)$. But there is one root missing. It is $n - r + k - 1$ and can be computed by more closely analyzing the expression for $\chi_J(t)$.

In the exceptional groups, case by case computation leads to the determination of the coexponents. Since $\chi_J(t)$ is polynomial when t is prime to the c_α's, we only have to compute it for a few values of t to determine its roots. Complete tables are found in [OS2].

2. Springer representations

Let $P \subset G$ be a parabolic subgroup containing B of semisimple rank r with Lie algebra $\mathfrak{p}$. The choice of P determines a subset J of the simple roots and $b_1, \ldots, b_{n-r}$ are the associated coexponents. Let $P = LU$ be a Levi decomposition with L containing T and with the Lie algebra version $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. Let N be a regular nilpotent element in $\mathfrak{l}$.

The flag manifold $\mathcal{B} := G/B$ can be identified with the set of all Borel subalgebras in $\mathfrak{g}$. Then if $X \in \mathfrak{g}$, the corresponding fixed point subvariety of $\mathcal{B}$ is

$$\mathcal{B}_X = \{\mathfrak{b} \in \mathcal{B} | X \in \mathfrak{b}\}.$$

The Weyl group acts on the cohomology $\mathrm{H}^*(\mathcal{B}_X)$ via Springer representation. It is known that $\mathcal{B}_X$ has no odd cohomology.

Proposition 2.1 (Alvis-Lusztig [AL]). *The Springer representation of W on $\mathrm{H}^*(\mathcal{B}_N)$ is isomorphic to $\mathrm{Ind}_{W_J}^W(1)$.*

By Frobenius reciprocity, it follows that the number of times the reflection representation occurs in $\mathrm{H}^*(\mathcal{B}_N)$ is $n - r$. For two representations V, U of a group, denote by $[V : U]$ the multiplicity of U in V. The next proposition was observed by Spaltenstein and proved by Spaltenstein and Lehrer-Shoji [Sp], [LeS].

Proposition 2.2. *The reflection representation appears in the Springer representation associated to N according to*

$$\sum_{i \geq 0} [\mathrm{H}^{2i}(\mathcal{B}_N) : V_{\mathrm{ref}}]\ q^i = q^{b_1} + q^{b_2} + \cdots + q^{b_{n-r}}.$$

3. Rings of functions

In the previous section, the regular nilpotent in $\mathfrak{l}$ played the central role. Now we take the "dual" nilpotent X which is Richardson in $\mathfrak{u}$, that is, X is in the dense orbit of P on $\mathfrak{u}$. Let $\mathcal{O}_X \subset \mathfrak{g}$ be the orbit of X under the adjoint action of G and $\bar{\mathcal{O}}_X$ its closure in $\mathfrak{g}$. Since two non-associated parabolics can give rise to the same Richardson orbit, we compensate by passing to an orbit cover of $\mathcal{O}_X$. Define $\mathcal{O}_P$ to be G/P^X where P^X is the centralizer of X in P. Since $(G^X)^0 \subset P^X$, $\mathcal{O}_P$ is a finite cover of $\mathcal{O}_X$ of degree $[G^X : P^X]$.

Let $R(\mathcal{O}_P)$ be the ring of regular functions on $\mathcal{O}_P$. This ring carries a natural G-action. Its structure as a G-module is given by

Proposition 3.1 (Borho-Kraft [BK]). *The representation of G on $R(\mathcal{O}_P)$ is isomorphic to $\mathrm{Ind}_L^G(1)$.*

Again by Frobenius reciprocity, the number of times the adjoint representation occurs in $R(\mathcal{O}_P)$ is $n - r$.

In fact, there is a grading of $R(\mathcal{O}_P)$ coming from a $\mathbf{C}^*$-action on $\mathcal{O}_P$ (which lifts from a $\mathbf{C}^*$-action on $\mathcal{O}_X$). We can describe the action as follows. Form the G-variety

$$Z = G \times^P \mathfrak{u} := G \times \mathfrak{u} \text{ modulo } (gp, u) \sim (g, Ad(p)u).$$

Let $\lambda \in \mathbf{C}^*$ act on Z by $\lambda \circ (g, u) = (g, \lambda u)$. For $z = (1_G, X) \in Z$, the G-orbit of z is isomorphic to $\mathcal{O}_P$ and it is stable under the $\mathbf{C}^*$-action. This is the desired $\mathbf{C}^*$-action on $\mathcal{O}_P$.

We can get a graded version of the previous proposition by keeping track of the $\mathbf{C}^*$-action. First, we recall Lusztig's q-analog of weight multiplicities [Lu].

Let λ be a character of T and let S be any subset of Φ^+. Define an analog of Kostant's partition function $\mathcal{P}_S(\lambda)$ to be the polynomial in q given by

$$\prod_{\alpha \in S} \frac{1}{(1 - qe^{\alpha})} = \sum_{\lambda} \mathcal{P}_S(\lambda) e^{\lambda}.$$

For the case $S = \Phi^+$, we have the usual q-analog of the partition function, in which case we omit the subscript S. For μ, λ both characters of T, Lusztig's q-analog of weight multiplicity $\mathcal{M}_\lambda^\mu$ is defined to be

$$\mathcal{M}_\lambda^\mu := \sum_{w \in W} \epsilon(w)\mathcal{P}(w(\lambda + \rho) - (\mu + \rho)),$$

where ρ is half the sum of the positive roots and $\epsilon(w)$ is the sign of w. We are also interested in the following polynomials. Define $\mathcal{M}_\lambda(S)$ to be

$$\mathcal{M}_\lambda(S) := \sum_{w \in W} \epsilon(w)\mathcal{P}_S(w(\lambda + \rho) - \rho).$$

We have the identity

$$\mathcal{M}_\lambda(S) = \sum_\mu c(\mu)\mathcal{M}_\lambda^\mu$$

where $c(\mu)$ is the coefficient of e^μ in

$$\prod_{\alpha \in \Phi^+ - S} (1 - qe^\alpha).$$

The following is extracted from McGovern [M] and Borho-Kraft [BK].

Proposition 3.2. *Let V_λ be a representation of G of highest weight λ. Let $S = \Phi^+ - \Phi_J^+$ (the positive roots of G which are not positive roots of L). Then $\sum_{i \geq 0}[R^i(\mathcal{O}_P) : V_\lambda]\ q^i = \mathcal{M}_\lambda(S)$.*

Proof. Consider again the smooth variety $Z = G \times^P \mathfrak{u}$, which is the cotangent bundle of G/P. We have a natural, surjective map $\phi : Z \to \bar{\mathcal{O}}_X$ given by $\phi(g, u) = Ad(g)u$. Consider the Stein factorization

$$Z \xrightarrow{\phi_1} Y \xrightarrow{\phi_2} \bar{\mathcal{O}}_X$$

so that Y is affine with coordinate ring $R(Y) = R(Z)$. The action of G on Y has an open, dense orbit which identifies naturally with $\mathcal{O}_P$. The complement of $\mathcal{O}_P$ in Y has codimension at least 2 since the complement of $\mathcal{O}_X$ in $\bar{\mathcal{O}}_X$ has codimension at least 2. Hence, $R(\mathcal{O}_P) \simeq R(Y) \simeq R(Z)$ since Y is normal. This reduces the problem to computing $R(Z)$.

Let ω_Z be the canonical line bundle on Z. The map ϕ_1 is a proper, surjective, birational map between the smooth variety Z and the variety Y. This is a setting for the vanishing theorem of Grauert-Riemenschneider [GR] which implies that $R^i(\phi_1)_*\omega_Z = 0$ for $i > 0$. But the canonical divisor of Z is trivial because Z is the cotangent bundle of G/P. Consequently if $\mathbf{O}_Z$ is the structure sheaf of Z, then $H^i(Z, \mathbf{O}_Z) = 0$ for $i > 0$ since Y is affine.

Consider the map $\pi : Z \to G/P$. This map is affine with fiber $\mathfrak{u}$ which implies

$$H^i(Z, \mathbf{O}_Z) \simeq H^i(G/P, \pi_*(\mathbf{O}_Z)) \simeq H^i(G/P, G \times^P R(\mathfrak{u})).$$

The left hand side vanishes for $i > 0$ so the functions of degree j on Z can be computed as the Euler characteristic of the vector bundle $G \times^P R^j(\mathfrak{u})$ over G/P. By using a suitable filtration of this vector bundle, we can use the Bott-Borel-Weil theorem to get the desired formula.

Remark 3.3. The case $r = 0$ is due to Hesselink [He2] and D. Peterson.

Corollary 3.4. *Let J, J' be subsets of Π which are conjugate under W. Let $S = \Phi^+ - \Phi^+_J$ and $S' = \Phi^+ - \Phi^+_{J'}$. Then*

$$\mathcal{M}_\lambda(S) = \mathcal{M}_\lambda(S').$$

Proof. Since J, J' are conjugate under W, the Levi subgroups they determine are conjugate under G. In other words, the parabolic subgroups of G that they determine are associated. But if P, P' are associated parabolics, then $\mathcal{O}_P$ and $\mathcal{O}_{P'}$ are isomorphic G-varieties (with isomorphic $\mathbf{C}^*$-actions) [LuS].

Our main result is

Theorem 3.5. [1] *The adjoint representation V_θ of G appears in the functions on $\mathcal{O}_P$ according to*

$$\sum_{i\geq 0} [R^i(\mathcal{O}_P) : V_\theta]\ q^i = q^{b_1} + q^{b_2} + \cdots + q^{b_{n-r}}$$

(θ is the highest root of Φ^+).

Remark 3.6. When $r = 0$ Proposition 3.1 and Theorem 3.5 are due to Kostant [Ko].

4. Proof of the theorem

The proof is case by case. In the classical groups we will use geometric arguments so that we get slightly stronger results about the occurrence of the adjoint representation in $R(\bar{\mathcal{O}}_X)$. In type D_n, however, in the case when J is conjugate to a set of simple roots in A_{n-2}, we use a different argument.

We know that the number of times V_θ appears in $R(\mathcal{O}_P)$ is $n-r$. Also the following inclusions hold:

$$R(\bar{\mathcal{O}}_X) \subset R(\mathcal{O}_X) \subset R(\mathcal{O}_P).$$

The first inclusion is an isomorphism if and only if $\bar{\mathcal{O}}_X$ is a normal variety. Both inclusions are always isomorphisms in type A_n, but neither holds in general [KP1], [KP2]. When possible, our strategy will be to find $n-r$ copies of the adjoint representation in the symmetric algebra $S(\mathfrak{g}^*)$ which are non-zero when restricted to $R(\bar{\mathcal{O}}_X)$ and which occur in the correct degrees. We will use Hesselink's results about Richardson elements in classical groups [He1].

Type $\mathbf{A_n}$. We have $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbf{C})$, the set of $n+1 \times n+1$ matrices of trace zero. Let $\phi_m : \mathfrak{g} \to \mathfrak{g}$ be the G-equivariant map defined by $\phi_m(X) = X^m$. Let $x_{i,j}$ be the linear function on $\mathfrak{g}$ which returns the value of the i,j-component of a matrix in $\mathfrak{g}$. The composition $x_{i,j} \circ \phi_m$ is then a function on $\mathfrak{g}$ of degree m. In fact, the linear span of the functions $\{x_{i,j} \circ \phi_m | 1 \leq i, j \leq n+1\}$ determine a copy of the adjoint representation of G in $S(\mathfrak{g}^*)$ in degree m. Denote this representation by V_θ^m.

Given a parabolic subgroup P of semisimple rank r, its Richardson element X will satisfy $X^{n-r+1} = 0$ and $X^{n-r} \neq 0$ [He1]. Hence for $1 \leq m \leq n-r$ the adjoint representation V_θ^m in $S(\mathfrak{g}^*)$ is non-zero when restricted to $R(\bar{\mathcal{O}}_X)$. The proof follows since we have located $n-r$ copies of the adjoint representation in $R(\bar{\mathcal{O}}_X)$ (which in this case equals $R(\mathcal{O}_P)$).

[1] *Note added in proof*: A. Broer has informed us that he has also proved this theorem using methods different from ours.

Type $\mathbf{C_n}$. Let

$$M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathfrak{sl}_{2n}(\mathbf{C})$$

and $\mathfrak{g} = \{X \in \mathfrak{sl}_{2n}(\mathbf{C}) | XM + MX^t = 0\}$. The map ϕ_m preserves $\mathfrak{g}$ if m is odd. Hence if m is odd, the representation V_θ^m when restricted from functions on $\mathfrak{sl}_{2n}(\mathbf{C})$ to functions on $\mathfrak{g}$ is isomorphic to the adjoint representation of G.

Given a parabolic subgroup P of semisimple rank r, its Richardson element X will satisfy $X^{2(n-r)+1} = 0$ and $X^{2(n-r)-1} \neq 0$ [He1]. Hence for $m = 1, 3, \ldots, 2(n-r)-1$, V_θ^m will be non-zero when restricted to $R(\bar{\mathcal{O}}_X)$. We have located the required $n-r$ copies of the adjoint representation in $R(\bar{\mathcal{O}}_X)$. Note in this case, $R(\mathcal{O}_P)$ can be larger than $R(\bar{\mathcal{O}}_X)$ (on account of normality failing or the fact that $P^X \neq G^X$) but this is not detectable with the adjoint representation.

Orthogonal cases. Let $\mathfrak{g}$ be the set of skew-symmetric matrices in $\mathfrak{sl}_l(\mathbf{C})$. As above, when m is odd the map ϕ_m preserves $\mathfrak{g}$ and so the representation V_θ^m when restricted to functions on $\mathfrak{g}$ is isomorphic to the adjoint representation of G.

Type $\mathbf{B_n}$. If P is a parabolic subgroup of semisimple rank r, its Richardson element X will satisfy $X^{2(n-r)+1} = 0$ and $X^{2(n-r)-1} \neq 0$ [He1]. Hence for $m = 1, 3, \ldots, 2(n-r)-1$, V_θ^m will be non-zero when restricted to $R(\bar{\mathcal{O}}_X)$. We have located the required $n-r$ copies of the adjoint representation in $R(\bar{\mathcal{O}}_X)$. As in type C_n, $R(\mathcal{O}_P)$ can be larger than $R(\bar{\mathcal{O}}_X)$.

Type $\mathbf{D_n}$. Let P be a parabolic subgroup of semisimple rank r such that J is not conjugate to a set of simple roots of A_{n-2}. Then its Richardson element X satisfies the same conditions as in type B_n [He1] and we get the desired occurrences of the adjoint representation in $R(\bar{\mathcal{O}}_X)$.

Now assume P is a parabolic subgroup of semisimple rank r such that $\Phi_J \cong A_{i_1} \times \cdots \times A_{i_k} \subset A_{n-2}$. If $r > k$, then there are only $n-r-1$ occurrences of the adjoint representation in $R(\bar{\mathcal{O}}_X)$. The missing coexponent detects either the failure of normality of $\bar{\mathcal{O}}_X$ or the fact that $P^X \neq G^X$. So we give a combinatorial proof using Proposition 3.2 to reduce to the case $r = k$ and then use a normality result of Broer [Br].

Suppose $\Phi_J \cong A_{i_1} \times \cdots \times A_{i_k} \subset A_{n-2}$ and some $i_j > 1$. We show in Lemma 4.1 that the polynomial $\mathcal{M}_\theta(\Phi^+ - \Phi_J^+)$ stays the same when we pass from D_n to D_{n-1} and i_j decreases by one. Noting that the coexponents coincide in these two situations allows us to repeat this process until $i_1 = i_2 = \cdots = i_k = 1$ (or equivalently, $r = k$).

Let $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n$ be the simple roots (in standard coordinates) of a root system Φ_n of type D_n. Then $\alpha_2, \ldots, \alpha_n$ span a root system Φ_{n-1} of type D_{n-1}. We may assume $\alpha_1, \alpha_2 \in J$ by Corollary 3.4. Set $S = \Phi_n^+ - \Phi_J^+$. Let $J' = J - \{\alpha_1\}$ and $S' = \Phi_{n-1}^+ - \Phi_{J'}^+$. Define a map σ from the real span of the α_i to the subspace spanned by the α_i for $i \geq 2$ via

$$\sigma(a_1, a_2, \ldots, a_n) = (0, a_1, a_2 + a_3, a_4, \ldots, a_n).$$

Denote by S_0 the roots of S which have zero coefficient on e_3. Then σ restricts to a bijection between S_0 and S' and carries the highest root $\theta = e_1 + e_2$ of Φ_n to the highest root $\sigma(\theta)$ of Φ_{n-1}. Let $s_i \in W(\Phi_n)$ be the reflection by the simple root α_i.

Consider the parabolic subgroups

$$W_1 = \langle s_1, s_4, s_5, \ldots, s_n \rangle \subset W(\Phi_n),$$
$$W_2 = \langle s_2, s_4, s_5, \ldots, s_n \rangle \subset W(\Phi_{n-1}) \subset W(\Phi_n).$$

Denote the map from W_1 to W_2 taking s_1 to s_2 and s_j to s_j for $j \geq 4$ likewise by σ; then σ extends to an isomorphism from W_1 to W_2. For any weight γ, set $\gamma_w = w(\gamma + \rho) - \rho$.

Lemma 4.1. $\mathcal{M}_\theta(S) = \mathcal{M}_{\sigma(\theta)}(S')$. *The former computation is in D_n and the latter is in D_{n-1}.*

Proof. In fact we show that σ sets up a bijection between the terms contributing to $\mathcal{M}_\theta(S)$ and $\mathcal{M}_{\sigma(\theta)}(S')$. By definition,

$$\mathcal{M}_\theta(S) = \sum_{w \in W(\Phi_n)} \epsilon(w)\ \mathcal{P}_S(\theta_w) \tag{4.1}$$

and

$$\mathcal{M}_{\sigma(\theta)}(S') = \sum_{w \in W(\Phi_{n-1})} \epsilon(w)\ \mathcal{P}_{S'}(\sigma(\theta)_w). \tag{4.2}$$

In the notation introduced above, we have

$$\theta_w = w(n, n-1, n-3, n-4, \ldots, 1, 0) - (n-1, n-2, n-3, n-4, \ldots, 1, 0).$$

Since $\alpha_1, \alpha_2, \alpha_1 + \alpha_2 \notin S$ (by assumption), all roots in S have non-negative coefficient on e_1, e_2, and e_3. Consequently if $\mathcal{P}_S(\theta_w) \neq 0$, then the first three coordinates of θ_w must be non-negative. This implies that $w \in W_1$. For such a w, the third coordinate of θ_w is zero and hence all terms in an expression contributing to $\mathcal{P}_S(\theta_w)$ must actually belong to S_0. In other words, for $w \in W_1$,

$$\mathcal{P}_S(\theta_w) = \mathcal{P}_{S_0}(\theta_w).$$

On the other hand, if $\mathcal{P}_{S'}(\sigma(\theta)_w) \neq 0$, then $w \in W_2$. So the sum in equation (4.1) is really over W_1 and the sum in equation (4.2) is really over $W_2 = \sigma(W_1)$. Furthermore, $\sigma(\theta_w) = \sigma(\theta)_{\sigma(w)}$ for $w \in W_1$. To complete the proof of the lemma, we have to show that

$$\mathcal{P}_{S_0}(\theta_w) = \mathcal{P}_{S'}(\sigma(\theta_w))$$

whenever $w \in W_1$. But this is clear since σ is a bijection between S_0 and S'. The lemma is proved.

We can complete the proof of the theorem. When $\Phi_J \cong A_1 \times A_1 \times \cdots \times A_1$ (k times) $\subset A_{n-2}$, the Richardson element X has Jordan normal form in $\mathfrak{sl}_{2n}(\mathbf{C})$ corresponding to the partition $[2(n-k)-1, 2k+1]$ and also $\mathcal{O}_X = \mathcal{O}_P$. Furthermore, Broer [Br] has shown that for these nilpotents $\bar{\mathcal{O}}_X$ is a normal variety, whence $R(\bar{\mathcal{O}}_X) = R(\mathcal{O}_X)$. So we need to locate the degrees in which the $n-k$ copies of V_θ occur in $R(\bar{\mathcal{O}}_X)$. All occurrences of V_θ in $R(\bar{\mathcal{O}}_X)$ are just restrictions of occurrences of V_θ in the functions on the nilpotent cone $\bar{\mathcal{O}}_{\text{reg}} \subset \mathfrak{g}$. By Kostant [Ko] V_θ occurs in $R(\bar{\mathcal{O}}_{\text{reg}})$ in degrees $1, 3, \ldots, 2n-3, n-1$, where all but one of the occurrences comes from the restriction of V_θ^m from functions on $\mathfrak{sl}_{2n}(\mathbf{C})$ to $R(\bar{\mathcal{O}}_{\text{reg}})$. Since $X^{2(n-k)-1} = 0$, we see that for $m = 2(n-k)-1, 2(n-k)+1, \ldots, 2n-3$, the restriction of V_θ^m to $R(\bar{\mathcal{O}}_X)$ is zero. But there are $n-k$ copies of V_θ in $R(\bar{\mathcal{O}}_X)$ and so they must occur in degrees $1, 3, \ldots, 2(n-k)-3, n-1$ which is the desired result.

Exceptional Groups. In the exceptional cases, we used a computer to calculate the polynomials $\mathcal{M}_\theta(\Phi^+ - \Phi_J^+)$ for $J \subset \Pi$ verifying the theorem in those cases. By Corollary 3.4 we only needed to do the compution for one subset J in each W-conjugacy class of subsets of Π.

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