Two remarks on graded nilpotent classes

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1. Let $V = \oplus V_i$ ($i \in \mathbf{Z}$) be a finite-dimensional graded vector space over an algebraically closed field and $E(V)$ a set of linear operators $N: V \to V$ such that $N(V_i) \subset V_{i+1}$ for $i \in \mathbf{Z}$. *Graded nilpotent classes of type* V are defined as the orbits of the natural action of Aut $V = \Pi\mathrm{GL}(V_i)$ on the set $E(V)$. Their properties are analogous to those of ordinary nilpotent classes, that is, orbits of $\mathrm{GL}(V)$ on the set $N(V)$ of all nilpotent operators $V \to V$. The results in this note corroborate this analogy.

It was proved independently in [1] and [3] that there are only finitely many graded nilpotent classes of a given type, and their degeneracies were described. Generally speaking, the closures of classes are singular varieties. Their geometry is studied in [2]. In [3] the author raised the question of computing the Deligne-Goreski-Macpherson cohomology sheaves $\mathcal{H}^i(X)$ in these varieties (see [5], [6]), and made a conjecture about the connection between these sheaves and the representations of the group $\mathrm{GL}(n)$ over a p-adic field. The non-graded analogue of this problem is considered in [6], where the $\mathcal{H}^i(X)$ are calculated for the closures of ordinary nilpotent classes and their connection with the representations of $\mathrm{GL}(n)$ over a finite field is indicated.

Proceeding as in [6], we construct in §2 a compactification of the graded nilpotent classes, which is a Schubert variety. As in [6], this enables us to compute the $\mathcal{H}^i(X)$ for the closures of graded nilpotent classes in terms of the Kazhdan-Lusztig polynomials [4]; in contrast to [6], the resulting polynomials are connected with the ordinary symmetric group.

In §3 we construct a parametrization of an arbitrary graded nilpotent class that enables us to compute very easily the number of its elements over a finite field. The non-graded analogue of this construction enables us to give a simple geometrical proof of Steinberg's well-known result that the number of nilpotent operators in an n-dimensional space over the field $\mathbf{F}_q$ is equal to $q^{n(n-1)}$.

2. We put $n = \dim V$ and $n_i = \dim V_i$ and assume, without loss of generality, that $n_i \neq 0$ only for $1 \leqslant i \leqslant k$. For every operator $N \in E(V)$ and all pairs i, j with $i \leqslant j$, we denote by N_{ij} the compositum $V_i \xrightarrow{N} V_{i+1} \xrightarrow{N} \dots \xrightarrow{N} V_j$ (in particular, N_{ii} is the identity operator $V_i \to V_i$) and we put $r_{ij} = r_{ij}(N) = \mathrm{rk}\, N_{ij}$. We denote by $M(V)$ the set of matrices $M = (m_{ij})$, $1 \leqslant i, j \leqslant k$, having non-negative integral entries such that $m_{ij} = 0$ for $i > j+1$, and with the row- and column-sums $n_1, \dots, n_k$.

Proposition 1 (see [1], [3]). *Graded nilpotent classes of type V are parametrized by the set $M(V)$: to a matrix M there corresponds the class* $X_M = \{N \in E(V): r_{ij}(N) = \sum_{s \leqslant i \leqslant j \leqslant t} m_{st}$ *for* $i \leqslant j\}$.

The numbers m_{ij} are recovered from the r_{ij} by the formulae $m_{ij} = r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1}$ $(i \leqslant j)$, *and* $m_{i,i-1} = r_{i-1,i}$. *The closure of the class of the operator N is the set of those N' for which $r_{ij}(N') \leqslant r_{ij}(N)$ for all $i \leqslant j$.*

We denote by $F(V)$ the variety of flags $0 = U^0 \subset U^1 \subset \dots \subset U^k = V$ such that $\dim U^i/U^{i-1} = n_i$ for $1 \leqslant i \leqslant k$. We put $V^i = V_1 \oplus \dots \oplus V_i$, so that the flag F consisting of the spaces V^i lies in $F(V)$. Let P be the stabilizer of F in $\mathrm{GL}(V)$. The P-orbits on $F(V)$ are called *Schubert cells*, and their closures *Schubert varieties*, of type V. An important example for our purposes is the Schubert variety $G(V) = \{(U^i) \in F(V): U^i \supset V^{i-1}$ for $1 \leqslant i \leqslant k\}$. It is known that the Schubert cells contained in $G(V)$ are parametrized by the set $M(V)$: to the matrix M there corresponds the cell

$O_M = \{(U^i) \in F(V): \dim(U^i \cap V^j)/(U^i \cap V^{j-1} + U^{i-1} \cap V^j) = m_{ij}$ for $1 \leqslant i, j \leqslant k\}$.

We define a map $E(V) \to F(V)$ that associates with $N \in E(V)$ the flag (U^i), where $U^i = \{(v_1, \dots, v_k) \in V_1 \oplus \dots \oplus V_k: v_{j+1} = N(v_j)$ for $j \geqslant i\}$.

Theorem 1. *This map is an isomorphism between $E(V)$ and an open subvariety of $G(V)$ for which $X_M = O_M \cap E(V)$ for all $M \in M(V)$.*

We split the set $\{1, 2, \dots, n\}$ in running order into blocks $B_1, \dots, B_k$, where Card $B_i = n_i$. For every $M = (m_{ij}) \in M(V)$ we put $S^M = \{w \in S_n: \mathrm{Card}(w(B_i) \cap B_j) = m_{ij}$ for $1 \leqslant i, j \leqslant k\}$. It is known that S_M has a unique element $w(M)$ of maximal length.

Corollary 1 (see [6]). *The sheaves* $\mathcal{H}^i(\overline{X}_M)$ *are* 0 *for odd* i, *and for* $x \in X_{M'} \subset \overline{X}_M$ *we have* $\sum_{i\geqslant 0} q^i \dim \mathcal{H}^{2i}(\overline{X}_M)_x = P_{w(M'),\, w(M)}(q)$, *where the right-hand side is the Kazhdan-Lusztig polynomial* ([4], [5]).

3. Let $M = (m_{ij}) \in M(V)$, $N \in X_M$, and $r_{ij} = r_{ij}(N)$. For all i, j with $i \leqslant j$, let $I_{ij} = \operatorname{Im} N_{ij}$, so that $\dim I_{ij} = r_{ij}$. For $1 \leqslant j \leqslant k$ we denote by F_j the flag ($I_{1j} \subset I_{2j} \subset \ldots \subset I_{jj} = V_j$) in the space V_j. It is clear that the map $N \mapsto (F_1, \ldots, F_k)$ of X_M into the product of flag varieties of this type is a fibration. Let us describe a typical fibre, that is, the set of operators N with fixed I_{ij}. We choose a complement C_{ij} to $I_{i-1,j}$ in I_{ij}. The operator N is determined by its restrictions to all the C_{ij}, which can be chosen independently of one another, as the sum of an arbitrary operator $C_{ij} \to I_{i-1,j+1}$ and an arbitrary surjective operator $C_{ij} \to C_{i,j+1}$. This leads to the following result.

Proposition 2. *The number of elements of* X_M *over* $\mathbf{F}_q$ *is* $q^d \prod_i \Phi_q(n_i) / \prod_{i\leqslant j} \Phi_q(m_{ij})$ *where* $\Phi_q(r) = (q-1)(q^2-1) \ldots (q^r-1)$, *and*

$$d = \sum_{i<j} r_{ij} m_{ij} + \sum_i \left[n_i r_{i-1,\, i+1} - \binom{r_{i,\, i+1}+1}{2} \right].$$

We now consider the non-graded analogue of our construction. The nilpotent classes in $N(V)$ are parametrized by the collections $\lambda = (m_i)$ of non-negative integers with $\sum (i+1) m_i = n$: the class X_λ consists of operators with m_i Jordan blocks of order $(i+1)$. (The numbers m_{ij} above have a similar meaning.) For $N \in X_\lambda$ and $0 \leqslant i \leqslant n$ we put $I_i = \operatorname{Im} N^i$ and $r_i = \dim I_i$. By assigning to N the flag (I_i) we see that X_λ is a fibration over the variety of flags of this type. As above, this readily yields the well-known formula for the number of elements of X_λ over $\mathbf{F}_q$ (see [7]).

In conclusion, we show how to break up $N(V)$ into pieces from which we can build an affine space of dimension $n(n-1)$. (The assumption that such a decomposition exists is made in [7].) We consider an $(n-1)$-dimensional vector space U with a preferred complete flag $0 = U^0 \subset U^1 \subset \ldots \subset U^{n-1} = U$. To any nilpotent class X_λ we assign the variety

$$Y_\lambda = \{A \in \operatorname{Hom}(U, V)\colon \dim A(U^{r_i - 1}) = r_{i+1} \text{ for } 0 \leqslant i \leqslant n-1\}.$$

It is easy to see that the Y_λ are pairwise disjoint and that their union is $\operatorname{Hom}(U, V)$. By assigning to any $A \in Y_\lambda$ the flag $(A(U^{r_i - 1}))$ we see that Y_λ is a fibration over the same variety as X_λ, and it is easy to see that their fibres are isomorphic. From this it is clear that X_λ breaks into pieces from which we can build Y_λ. By taking these decompositions for all X_λ, we obtain the required decomposition of $N(V)$.

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Received by the Board of Governors 4 April 1983