REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 4, Pages 245–295 (July 31, 2000)
S 1088-4165(00)00105-9

IRREDUCIBLE GENUINE CHARACTERS OF THE METAPLECTIC GROUP: KAZHDAN-LUSZTIG ALGORITHM AND VOGAN DUALITY

DAVID A. RENARD AND PETER E. TRAPA

Abstract. We establish a Kazhdan-Lusztig algorithm to compute characters of irreducible genuine representations of the (nonlinear) metaplectic group with half-integral infinitesimal character. We then prove a character multiplicity duality theorem for representations of $Mp(2n,\mathbb{R})$ at fixed half-integral infinitesimal character. This allows us to extend some of Langlands' ideas to $Mp(2n,\mathbb{R})$.

0. Introduction

Let $G_{\mathbb{R}}$ be a real reductive group. A fundamental problem in the representation theory of $G_{\mathbb{R}}$, originating in the character theory of Harish-Chandra, is to give an explicit description of the character of an irreducible admissible representation of $G_{\mathbb{R}}$. When one fixes an infinitesimal character, the problem becomes a finite one, in the sense that the span of such characters is finite-dimensional. One consequence of the Langlands classification is that this space is also spanned by certain standard characters. Subsequent work of a number of people imply that (at least in principle) the standard characters are readily computable. Hence the problem amounts to explicitly expressing the irreducible characters in terms of the standard ones.

For linear groups (and integral infinitesimal character), the character problem was solved by Vogan in a seminal series of papers ([V1]–[V3], [LV]). A final paper [V4] is devoted to a latent symmetry of the answer, which is encoded in a kind of duality between representations of a real form of a complex group and those of a real form of a reductive subgroup of its Langlands dual. Subsequently this duality was interpreted as a key representation-theoretic ingredient in a geometric reformulation of Langlands' ideas at the real place and, in particular, the Langlands-Shelstad theory of endoscopy for real groups [ABV].

The papers cited above are not enough to handle general nonlinear groups, and for them the character problem (and hence any aspect of a duality theory) remains open. Yet nonlinear groups (especially the metaplectic double cover of the symplectic group) play an important role in the construction of automorphic forms, and it is clearly desirable to extend Langlands' ideas to encompass them as well. The purpose of the present paper is to give a solution of the character problem for the

Received by the editors November 12, 1999 and, in revised form, April 28, 2000.
2000 *Mathematics Subject Classification.* Primary 22E47; Secondary 22E50.
The first author acknowledges the support of NSF grant DMS97-29992 and the Ellentuck Fund of the Institute for Advanced Study.
The second author acknowledges the support of NSF grant DMS97-29995.

metaplectic group at certain kinds of infinitesimal character; in Theorem 7.13 and Corollary 7.11 we give an explicit description of an algorithm to compute the characters of irreducible representations with regular half-integral infinitesimal character (i.e. with an infinitesimal character which coincides with that of a genuine discrete series). We then establish the following duality theorem.

Theorem 0.1. *Fix an infinitesimal character χ so that $Mp(2n,\mathbb{R})$ has a genuine discrete series representation with infinitesimal character χ. Then there is an involution on the set $\mathcal{P}_\chi$ of Langlands parameters of genuine representations with infinitesimal character χ*

$$\gamma \longrightarrow \check{\gamma}$$

so that if $\delta \in \mathcal{P}_\chi$ and

$$[irr(\delta)] = \sum_{\gamma \in \mathcal{P}_\chi} M(\gamma,\delta)[std(\gamma)],$$

then

$$[std(\check{\gamma})] = \sum_{\check{\delta} \in \mathcal{P}_\chi} \epsilon_{\gamma\delta} M(\gamma,\delta)[irr(\check{\delta})];$$

here $\epsilon_{\gamma\delta} = \pm 1$, and is explicitly computable.

Sharper versions of Theorem 0.1 appear in Theorems 6.34 and 8.3 below, and the cases of $Mp(2,\mathbb{R})$ and $Mp(4,\mathbb{R})$ are treated extensively in Section 4 and Example 6.22. Note that Theorem 0.1 is consistent with the long-standing philosophic belief that the metaplectic group is a real form of the Langlands dual of its complexification (which doesn't exist of course); see the introduction of [R] for example.

In future joint work, we will explore the representation theoretic applications of Theorem 0.1 which, as motivation for the present paper, we briefly sketch now. Following the philosophy of [ABV], Theorem 0.1 immediately allows us to define L-packets of representations with a fixed half-integral regular infinitesimal character: $\mathrm{irr}(\gamma)$ and $\mathrm{irr}(\delta)$ are in the same L-packet if and only if the support of the appropriate $\mathcal{D}$-module localization of their duals $\mathrm{irr}(\check{\gamma})$ and $\mathrm{irr}(\check{\delta})$ coincide. Of course Theorem 0.1 begs the questions of general infinitesimal character, but it turns out that the half-integral case is by far the most difficult to understand. In a future paper, we will show how to reduce the case of general infinitesimal character to the half-integral case ultimately to obtain a complete duality theory for the metaplectic group. We can thus extend the intrinsic definition of L-packets to all representations of the metaplectic group. (Previously L-packets for the metaplectic group were defined as theta-lifts of super L-packets of equal-size orthogonal groups [A]. It is an interesting and nontrivial fact that the extrinsic definition coincides with the intrinsic one.) More importantly, we can define a microlocal refinement of L-packets (called geometric packets in the argot of [ABV]) which, except in extremely special cases, are not preserved by theta-lifting. Hence their definition is a genuinely new consequence of the duality. Using them, it should be possible to give a geometric formulation of the results of [R], as well as extend them to nontempered packets.

We now return to the present paper in a little more detail. Our computation of characters follows Vogan's treatment of the linear case quite closely, but there are some important new ingredients. The first and most obvious concerns the ubiquitous reduction to rank one subgroups: in the linear case, one need essentially only consider $SU(2)$, $SL(2,\mathbb{R})$, and $GL(2,\mathbb{R})$; but in our setting we also need to include $Mp(2,\mathbb{R})$, the nonlinear double cover of $SL(2,\mathbb{R})$. Said differently, we have to consider a new kind of noncompact and real root (in addition to the usual Type I and Type II roots). This difference, which of course is conceptually very easy, is spelled out carefully in Sections 3.3 and 4. The next ingredient is the Hecht-Miličić-Schmid-Wolf standard module reducibility criterion (extending the linear group criterion of Speh-Vogan); this is recalled in Theorem 1.11. Until this point, we have incorporated no essential features of the metaplectic group in our discussion[1], but we must invoke that setting for the final and most difficult ingredient: a technical vanishing theorem for Lie algebra cohomology (Theorem 1.14)[2].

With these ingredients in place, we are able to retrace the arguments (with appropriate but generally not serious modifications) of [V3]. In order to prove Theorem 0.1, we introduce a closed form of the algorithm to compute irreducible characters. By "closed form" we mean an axiomatic (and then ultimately a recursive) description of the multiplicities of a standard character in an irreducible one. (Actually, as usual, one does not characterize the multiplicities themselves, but rather certain polynomials whose numerical value at 1 is the actual multiplicity.) In doing so, we are able to make the theory of the metaplectic group closely mimic the linear theory, except for one crucial difference. It turns out that the classical role of the representation of the Hecke algebra (of the integral Weyl group) on the Grothendieck group of formal characters is insufficient for our needs. For instance, for $Mp(2,\mathbb{R})$ at infinitesimal character $\frac{1}{2}\rho$, the integral Weyl group is trivial, yet the relevant character formulas are nontrivial. Instead we need to add a new operator which is the analog of a nonintegral wall crossing translation functor. (In contrast to integral wall-crosses, this functor does not preserve blocks; indeed, our algorithm computes characters in one block only by simultaneously computing them in another.) The resulting algebra is *not* a Hecke algebra, though it contains the Hecke algebra of the integral Weyl group as a subalgebra. At any rate, we are able to use the representation of this generalized Hecke algebra, together with a 'Verdier duality' endomorphism, to characterize the basis of irreducible characters of the Grothendieck group. It is this characterization which we use in the proof of Theorem 0.1.

After completing the generalized Hecke algebra formalism, we made a key observation: our formulas bear a striking resemblance to those considered in [MS]. This suggests the possibility of a purely geometric treatment of the results presented here.

We conclude the introduction with a slightly more detailed survey of the contents below. In Section 1, after recalling a few generalities, we describe the shape of Vogan's representation-theoretic algorithm in the linear case. We then state the crucial vanishing theorem in Lie algebra cohomology on which the existence of the algorithm hinges (Theorem 1.13), and our nontrivial extension of it in the metaplectic setting (Theorem 1.14). In other words, the key modification is that

[1]Strictly speaking, this is not quite accurate. For other nonlinear groups, one must consider additional covers (instead of just the double cover) of $SL(2,\mathbb{R})$ in the rank-one reductions mentioned above. But this is very easy.

[2]For general nonlinear groups, a serious problem is finding an appropriate length function satisfying the requirements of this theorem; indeed, it is not clear that one need even exist. For this reason, one may expect that the character problem for general nonlinear groups need not have a "closed form" solution of the kind we describe below.

the ordinary length function replaces the integral length. Once this is established (which is done later in Section 5) the existence of an algorithm to compute irreducible characters relies only on a semisimplicity conjecture; roughly speaking, the semisimplicity of the modules $U_\alpha(X)$. This, in turn, can be deduced in a number of ways. Conceptually, the simplest argument rests on the decomposition theorem, which we explain in Theorem 1.6.

In order to prove Theorem 1.14, we need a few more structure theoretic facts (Section 2) and general representation theoretic preliminaries (Section 3). Section 3.3 pinpoints the fundamental differences between the linear theory and the theory of $Mp(2n,\mathbb{R})$, and develops the technical devices used in the inductive proof of Theorem 1.14. Section 4 deals with $Mp(2,\mathbb{R})$, i.e. the nonlinear base-case of the induction. Finally, we prove Theorem 1.14 in Section 5. The proof proceeds along the lines of Vogan's original and relies on the rather miraculous fact that certain nonintegral wall crossing translation functors always invert the relevant parity condition. (As mentioned above, this fact is a special feature of the $Mp(2n,\mathbb{R})$ setting.)

The remainder of the paper is devoted to establishing Theorem 0.1. As in [V4], we deduce it from a statement about certain Hecke algebra (actually, in our case, generalized Hecke algebra) representations. This is the content of Proposition 8.2. To prove it, one could proceed by suitably modifying the arguments of [V4] to encompass the nonlinear phenomena sketched in Section 3.3. Here we elect to bypass the generalities of graded root systems (developed in [V4]) and instead explicitly define the duality. The hope is that this approach will be more useful to the reader seeking examples rather than abstraction. To proceed in this direction, we need to establish explicit information about the representation theory of $Mp(2n,\mathbb{R})$ at half-integral infinitesimal character.

In a little more detail, in Section 6 we work out the Langlands parameterization in terms of combinatorially defined objects called diagrams. These are introduced out of convenience rather than necessity, but are useful for visualizing certain fundamental computations. Explicit statements relating our diagrammatic parameterization to the Langlands classification or the Beilinson-Bernstein parameterization may be found in Proposition 6.3 and Proposition 6.8. We work out the representation-theoretic constructions of Section 3 (Cayley transforms, cross action, etc.) in terms of simple pictorial descriptions (Lemmas 6.17 and 6.20). From this we obtain a characterization of the blocks of genuine representations with half-integral infinitesimal character (Proposition 6.9). Finally, we define the bijection of Theorem 0.1 in Definition 6.31. The especially informative example of $Mp(4,\mathbb{R})$ is given in Example 6.32.

Section 7 sets the stage for the proof of Theorem 0.1. As explained above, we need to introduce (what turns out to be) a completely equivalent closed form of the representation-theoretic algorithm for computing irreducible characters. We define the action of the generalized Hecke algebra on the Grothendieck group of formal characters in Definition 7.4, and use the representation of the generalized Hecke algebra to axiomatically define certain canonical elements in the Grothendieck group (Corollary 7.11). Assuming the existence of these elements, we obtain a recursive procedure to compute them. At this point, it is still not clear that the recursion scheme is self-consistent, i.e. it is not clear that the special elements exist. However, we then give a combinatorial verification (Theorem 7.12) that the recursion relations are exactly the same as those implicit in the algorithm given at the end of Section 5. Since these latter relations give the irreducible characters, the former must also, and hence we obtain the closed form we were seeking.

Finally, in Section 8, we use the closed form developed in Section 7 to prove Theorem 0.1. As mentioned above, the key statement here is Proposition 8.2. The proof we sketch is an explicit computation with the combinatorics developed in Section 6.

Acknowledgments

We are grateful to Jeffrey Adams for bringing the conjectural form of Theorem 0.1 to our attention. It is a pleasure to thank David Vogan for many helpful conversations. In particular, he shared his computations for $Mp(4,\mathbb{R})$ with us; without those, we never really could have gotten started.

1. Kazhdan-Lusztig algorithm for Harish-Chandra modules: generalities

1.1. **Notations.** In this section, we recall some material from [V1] and [V2]. Although stated only for linear groups, the definitions and results of [Vgr] also constitute a convenient reference. In this section $G_{\mathbb{R}}$ denotes an arbitrary connected real reductive group in the Harish-Chandra class, and $K_{\mathbb{R}}$ a maximal compact subgroup of $G_{\mathbb{R}}$ with corresponding Cartan involution θ and complexification K. Furthermore, we assume that $\mathrm{rk}(G_{\mathbb{R}}) = \mathrm{rk}(K_{\mathbb{R}})$, i.e. $G_{\mathbb{R}}$ admits discrete series representations, and that all Cartan subgroups of $G_{\mathbb{R}}$ are abelian. We denote real Lie algebras with subscripts $\mathbb{R}$ and delete the subscript for the corresponding complexification. We fix a compact Cartan subgroup $H^a_{\mathbb{R}}$ of $G_{\mathbb{R}}$ contained in $K_{\mathbb{R}}$. The corresponding Cartan subalgebra $\mathfrak{h}^a$ of $\mathfrak{g}$, will be considered as an 'abstract' Cartan subalgebra, and we fix a positive root system Δ^+_a of $\Delta(\mathfrak{g},\mathfrak{h}^a)$. (In [Vgr], Vogan uses the opposite convention by choosing $\mathfrak{h}^a_{\mathbb{R}}$ to be maximally split, but this difference is not serious.) If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and λ is a regular element in $\mathfrak{h}^*$, we will write $\Delta^+(\lambda)$ for a positive root system of $\Delta(\mathfrak{g},\mathfrak{h})$ making λ dominant. (In this context, dominant means that the pairing of the canonical real part of λ with any positive coroot is weakly positive.) We also need

$$R(\lambda) = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) |\, 2\frac{\langle \alpha,\lambda\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}\},$$

the integral roots for λ, and

$$R^+(\lambda) = R(\lambda) \cap \Delta^+(\lambda) \quad \text{and} \quad W(\lambda) = W(R(\lambda)),$$

the positive integral roots and the integral Weyl group. Fix once and for all a Δ^+_a-dominant nonsingular weight λ_a. Let $S \subset W_a := W(\mathfrak{g},\mathfrak{h}^a)$ be the set of reflections with respect to simple roots in Δ^+_a, and $S_{\mathrm{int}} \subset W(\lambda_a)$ the set of reflections with respect to simple roots in $R^+(\lambda_a)$. When the choice of λ_a is clear, we sometimes denote the integral Weyl group by W_{int}.

Let $\mathfrak{h}$ be any Cartan subalgebra of $\mathfrak{g}$, and suppose that $\lambda \in \mathfrak{h}^*$ defines the same infinitesimal character as λ_a; i.e. suppose that there exists an inner automorphism i_λ of $\mathfrak{g}$, sending $(\lambda_a,(\mathfrak{h}^a)^*)$ onto $(\lambda,\mathfrak{h}^*)$. If $\lambda^i \in (\mathfrak{h}^i)^*$, $i = 1,2$ define the same infinitesimal character as λ_a, we set $i_{\lambda^1,\lambda^2} := i_{\lambda^2} \circ (i_{\lambda^1})^{-1}$. The restriction of i_λ to $(\mathfrak{h}_a)^*$ is unique.

Let us denote by $\mathcal{HC}(\mathfrak{g}, K)$ the category of (finite-length) Harish-Chandra modules for $G_\mathbb{R}$. For any infinitesimal character $\chi \in (\mathfrak{h}^a)^*$, $\mathcal{HC}(\mathfrak{g}, K)_\chi$ is the full subcategory of modules having infinitesimal character χ, and we have a well-defined and exact projection functor (see [Vgr, Section 0.3])

$$P_\chi : \mathcal{HC}(\mathfrak{g}, K) \to \mathcal{HC}(\mathfrak{g}, K)_\chi.$$

The Grothendieck groups of these categories are denoted respectively by $\mathbb{K}\mathcal{HC}(\mathfrak{g}, K)$ and $\mathbb{K}\mathcal{HC}(\mathfrak{g}, K)_\chi$. We will write $[X]$ for the image of a module $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, K)$ in $\mathbb{K}\mathcal{HC}(\mathfrak{g}, K)$, and define $P_\chi([X]) = [P_\chi(X)]$. By the exactness of the functor P_χ, this defines a map between Grothendieck groups.

1.2. **Pseudocharacters and the Langlands classification.** We recall the definition of a pseudocharacter ([Vgr]). A pseudocharacter of $G_\mathbb{R}$ is a pair $(H_\mathbb{R}, \gamma)$ where $H_\mathbb{R}$ is a θ-stable Cartan subgroup of $G_\mathbb{R}$, and $\gamma = (\Gamma, \overline{\gamma})$, consists of an irreducible representation Γ of $H_\mathbb{R}$ and an element $\gamma \in \mathfrak{h}^*$, with certain compatibility conditions. We write the Cartan decomposition

$$\mathfrak{h}_\mathbb{R} = \mathfrak{t}_\mathbb{R} \oplus \mathfrak{a}_\mathbb{R},$$

and let $M_\mathbb{R} \exp \mathfrak{a}_\mathbb{R}$ be the Langlands decomposition of the centralizer $\mathrm{Cent}(G_\mathbb{R}, \mathfrak{a}_\mathbb{R})$. We require:

(i) $\overline{\gamma}_{|\mathfrak{t}_\mathbb{R}}$ is purely imaginary and regular with respect to $\Delta(\mathfrak{m}, \mathfrak{h})$.

(ii) Let $\Delta^+(\mathfrak{m}, \mathfrak{h})$ be the positive root system making $\overline{\gamma}_{|\mathfrak{t}}$ dominant. We impose

$$d\Gamma = \overline{\gamma} + \rho_I - 2\rho_c, \quad \text{where } \rho_I = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{m}, \mathfrak{h})} \alpha \ , \quad \rho_c = \frac{1}{2} \sum_{\substack{\alpha \in \Delta^+(\mathfrak{m}, \mathfrak{h}) \\ \alpha \text{ compact}}} \alpha \ . \tag{1.1}$$

We write $(\widehat{H_\mathbb{R}})'$ for the set of pseudocharacters having $H_\mathbb{R}$ as their first component. The pair $\gamma = (\Gamma, \overline{\gamma})$ defines in a natural way a discrete series representation $\delta(\gamma)$ of $M_\mathbb{R}$ (specified by the requirement that its lowest K-type have highest weight Γ) and a character $e^\nu = \Gamma_{|\exp \mathfrak{a}_\mathbb{R}}$ of $\exp \mathfrak{a}_\mathbb{R}$.

An important invariant of a pseudocharacter is its length, which we now define.

Definition 1.1. Let $(H_\mathbb{R}, \gamma)$ be a pseudocharacter of $G_\mathbb{R}$. The length of γ is

$$l(\gamma) = \frac{1}{2} |\{\alpha \in \Delta^+(\overline{\gamma}) \mid \theta(\alpha) \notin \Delta^+(\overline{\gamma})\}| + \frac{1}{2} \dim \mathfrak{a}.$$

We are interested in irreducible admissible Harish-Chandra modules of $G_\mathbb{R}$ with our fixed infinitesimal character $\lambda_a \in (\mathfrak{h}^a)^*$. We write $(\widehat{H_\mathbb{R}})'_{\lambda_a}$ for the subset of $(\widehat{H_\mathbb{R}})'$ of pseudocharacters γ such that $\overline{\gamma}$ and λ_a are conjugate by an inner automorphism of $\mathfrak{g}$ (i.e. they define the same infinitesimal character); in this case, we say that γ is a λ_a-pseudocharacter.

Definition 1.2. Let $\gamma \in (\widehat{H_\mathbb{R}})'_{\lambda_a}$ and recall that λ_a was assumed to be nonsingular. Choose a cuspidal parabolic subgroup $P_\mathbb{R} = M_\mathbb{R} A_\mathbb{R} N_\mathbb{R}$ associated to $H_\mathbb{R}$ in such a way that if $\nu = \overline{\gamma}_{|\mathfrak{a}_\mathbb{R}}$, then

$$\mathrm{Re}\langle \alpha, \nu \rangle \leq 0$$

for every root α of $\mathfrak{a}$ in $\mathfrak{n}$. The Langlands standard module with parameter γ is defined by

$$X(\gamma) = \mathrm{Ind}_{P_\mathbb{R}}^{G_\mathbb{R}} (\delta(\gamma) \otimes e^\nu \otimes 1).$$

Theorem 1.3 (Langlands [La], see [V2])**.** *Let the notations be as in the definition above, and recall that λ_a is regular.*

(a) *The standard module $X(\gamma)$ admits a unique irreducible submodule $\overline{X}(\gamma)$.*

(b) *Let X be an irreducible Harish-Chandra module of $G_{\mathbb{R}}$ with regular infinitesimal character λ_a. Then there exist a θ-stable Cartan subgroup $H_{\mathbb{R}}$ of $G_{\mathbb{R}}$ and a pseudocharacter $\gamma \in (\widehat{H}_{\mathbb{R}})'_{\lambda_a}$ such that X is equivalent to $\overline{X}(\gamma)$.*

(c) *Let $\gamma^i \in (\widehat{H}^i_{\mathbb{R}})'_{\lambda_a}$, $i = 1, 2$ be two pseudocharacters. The irreducible modules $\overline{X}(\gamma^i)$, $i = 1, 2$ are equivalent if and only if $(H^1_{\mathbb{R}}, \gamma^1)$ and $(H^2_{\mathbb{R}}, \gamma^2)$ are conjugate under $K_{\mathbb{R}}$.*

(d) *Write $cl(H_{\mathbb{R}}, \gamma)$ for the $K_{\mathbb{R}}$-conjugacy class of pseudocharacters containing $(H_{\mathbb{R}}, \gamma)$, and $\mathcal{P}_{\lambda_a}$ for the (finite) set of $K_{\mathbb{R}}$-conjugacy classes of λ_a pseudocharacters. By (c), $\mathcal{P}_{\lambda_a}$ parameterizes the irreducible Harish-Chandra modules with infinitesimal character λ_a. The following two sets are bases of the Grothendieck group $\mathbb{K}\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$:*

$$\{\ [\overline{X}(\gamma)]\ \}_{cl(\gamma)\in\mathcal{P}_{\lambda_a}} \quad \textit{and} \quad \{\ [X(\gamma)]\ \}_{cl(\gamma)\in\mathcal{P}_{\lambda_a}}. \tag{1.2}$$

Remark. We slightly abuse notation by writing γ for the pseudocharacter $(H_{\mathbb{R}}, \gamma)$, and often γ for $\mathrm{cl}(\gamma)$. The context will make clear what we mean.

Since characters of modules induced from discrete series are fairly well-understood (see for example [H] or [GKM]), a way to compute the character of an irreducible module $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$ is to express $[X]$ in the second basis of (1.2). That is, we look for an expression

$$[\overline{X}(\delta)] = \sum_{\gamma\in\mathcal{P}_{\lambda_a}} M(\gamma, \delta)[X(\gamma)], \tag{1.3}$$

where the $M(\gamma, \delta)$'s are nonnegative integers. The inverse and equivalent problem of decomposing a standard module in terms of its irreducible constituents consists of finding nonnegative integers $m(\gamma, \delta)$ such that

$$[X(\delta)] = \sum_{\gamma\in\mathcal{P}_{\lambda_a}} m(\gamma, \delta)[\overline{X}(\gamma)]. \tag{1.4}$$

In [V2] (building on [V1]), Vogan gave an explicit conjectural algorithm computing the integers $m(\gamma, \delta)$ in the case where $G_{\mathbb{R}}$ is linear. In the case that λ_a is integral, the conjectural part amounted to the complete reducibility of certain modules $U_\alpha(\overline{X})$ (see Theorem 1.4 below), where α is a simple root. In [V3] Vogan used the results of [LV] to establish the requisite semisimplicity. It has been known from various experts in the field how to drop the integrality assumption; see [ABV, Chapter 17], for instance. In any case, the hypothesis of linearity makes its key appearance in [V2, Proposition 7.2]. We will explain this in more detail in Section 1.5 below, but first we need to recall a few more definitions.

1.3. **Translation functors.** The weight lattice of a Cartan subgroup $H_{\mathbb{R}}$ of $G_{\mathbb{R}}$ is the set of weights of $H_{\mathbb{R}}$ in finite dimensional representations of $G_{\mathbb{R}}$. The weights of a finite dimensional representation of $G_{\mathbb{R}}$ in $H^a_{\mathbb{R}}$ can be identified with their differential in $(\mathfrak{h}^a)^*$ since $H^a_{\mathbb{R}}$ is connected. (Recall that $H^a_{\mathbb{R}}$ is a Cartan subgroup of the connected compact group $K_{\mathbb{R}}$.)

Let μ be an extremal weight in $(\mathfrak{h}^a)^*$ of a finite dimensional representation F_μ of $G_{\mathbb{R}}$. The translation functor $\psi_\chi^{\chi+\mu} : \mathcal{HC}(\mathfrak{g}, K)_\chi \longrightarrow \mathcal{HC}(\mathfrak{g}, K)_{\chi+\mu}$ is defined by

$$\psi_\chi^{\chi+\mu}(X) = P_{\chi+\mu}(F_\mu \otimes X).$$

We now define translation functors that push-to and push-off walls defined by simple integral roots. Since such a root need not be simple for Δ_a^+, we need to perform translations into a chamber for which our simple integral root is actually simple. The details are as follows. Let α be a simple root in $R^+(\lambda_a)$. Since α is not necessary simple in Δ_a^+, we choose a positive root system Ψ_α of $\Delta(\mathfrak{g}, \mathfrak{h}^a)$ containing $R^+(\lambda^a)$ such that α is simple for Ψ_α. Next we choose an integral weight μ_α^1 such that $\lambda_a + \mu_\alpha^1$ is strictly dominant for Ψ_α. (If α was already simple for Δ_a^+, we take $\mu_\alpha^1 = 0$.) Now let

$$m = 2\frac{\langle \alpha, \lambda_a + \mu_\alpha^1 \rangle}{\langle \alpha, \alpha \rangle}$$

and define μ_α^2 to be m times the fundamental weight corresponding to α. It follows that if $\beta \in R^+(\lambda_a)$,

$$\langle \beta, \lambda_a + \mu_\alpha^1 - \mu_\alpha^2 \rangle \geq 0$$

with equality if and only if $\beta = \alpha$. Let F_α^i denote the finite dimensional module of highest weight μ_α^i with respect to Ψ_α. We set

$$\psi_\alpha^1 := \psi_\lambda^{\lambda+\mu_\alpha^1},\ \psi_\alpha^2 := \psi_{\lambda+\mu_\alpha^1}^{\lambda+\mu_\alpha^1-\mu_\alpha^2},\ \phi_\alpha^2 := \psi_{\lambda+\mu_\alpha^1-\mu_\alpha^2}^{\lambda+\mu_\alpha^1},\ \phi_\alpha^1 := \psi_{\lambda+\mu_\alpha^1}^{\lambda},$$
$$\psi_\alpha = \psi_\alpha^2 \circ \psi_\alpha^1,\ \phi_\alpha = \phi_\alpha^1 \circ \phi_\alpha^2.$$

The functors ψ_α and ϕ_α are adjoint (e.g. [V1], Lemma 3.4).

Note that in the case that α is a simple integral root that is not simple for Δ_a^+, the composition $\phi_\alpha\psi_\alpha$ can be envisioned as the conjugation of a simple wall-cross by a nonintegral wall-cross. We will return to this in Remark 7.6 below.

The τ-invariant of $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, K)$ is the set of simple roots in $R^+(\lambda_a)$ such that $\psi_\alpha(X) = 0$ and is denoted by $\tau^a(X)$. Sometimes it will be convenient to transport this to another Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Suppose we have fixed $\lambda \in \mathfrak{h}^*$ such that λ is inner conjugate to λ_a in $\mathfrak{g}$. The τ-invariant of X with respect to $(\mathfrak{h}, \lambda)$ is the subset $\tau(X) = i_\lambda(\tau^a(X))$ of simple roots in $\mathbb{R}^+(\lambda)$. Although not explicit in the notation, the choice of $(\mathfrak{h}, \lambda)$ is usually clear from the context.

Theorem 1.4. *Let $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$ be an irreducible module, and let α be a simple root in $R^+(\lambda_a)$ not in $\tau^a(X)$. Then $\phi_\alpha\psi_\alpha(X)$ has X as its unique irreducible submodule and irreducible quotient, and the following sequence*

$$0 \to X \to \phi_\alpha\psi_\alpha(X) \to X \to 0,$$

defined by the adjointness of the two functors ψ_α and ϕ_α, is a chain complex. Define $U_\alpha(X)$ to be its cohomology. Then the module $U_\alpha(X)$ has finite composition series, and $\alpha \in \tau(U_\alpha(X))$.

Proof. See Section 7.3 of [Vgr].

Conjecture 1.5. *The modules $U_\alpha(X)$ are semisimple.*

As we said above, this conjecture is proved by Vogan for linear groups at integral infinitesimal character in [V3]. For nonintegral infinitesimal character (but still for linear groups), a sketch of the proof can be found in [ABV, Chapter 17]. For a nonlinear group, the exact status of the conjecture is unclear, at least to the authors of the present paper. Fortunately, we will need only a special case which we now state.

Theorem 1.6. *Let $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$ be an irreducible module. Suppose $\alpha \in R^+(\lambda_a)$ is simple, $\alpha \notin \tau^a(X)$, and α is also simple as a root in Δ_a^+. Then $U_\alpha(X)$ is a semisimple module in $\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$.*

Sketch. The Beilinson-Bernstein localization theory gives an equivalence of the category between $\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$ and $\mathcal{M}(\mathcal{D}_{-\lambda_a}, K)$, the category of Harish-Chandra sheaves on the flag manifold. The functor U_α has a geometric counterpart U_α^g from $\mathcal{M}(\mathcal{D}_{-\lambda_a}, K)$ into itself. If $\mathcal{F}$ is an irreducible Harish-Chandra sheaf in $\mathcal{M}(\mathcal{D}_{-\lambda_a}, K)$, the semisimplicity of $U_\alpha^g(\mathcal{F})$ is a consequence of the Riemann-Hilbert correspondence and the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [BBD]. For more details and a similar argument for Verma modules, see [Mi2]. □

We will also need to define translation functors across nonintegral walls.

Theorem 1.7 ([Vgr], Proposition 7.3.3)**.** *Let $\alpha \in \Delta_a^+$ be a simple nonintegral root, and choose a weight $\mu_\alpha \in (\mathfrak{h}^a)^*$ such that $\lambda_a + \mu_\alpha$ is dominant and regular for $s_\alpha \cdot \Delta_a^+$. If $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, K)$, define*

$$\psi_\alpha(X) := \psi_{\lambda_a}^{\lambda_a+\mu_\alpha}(X), \quad \phi_\alpha(X) := \psi_{\lambda_a+\mu_\alpha}^{\lambda_a}(X).$$

The functor ψ_α realizes an equivalence of the categories between $\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$ and $\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a+\mu}$; its inverse is ϕ_α.

The notation ψ_α across a nonintegral wall depends (in an inessential way) on the choice of μ. Nonetheless, we will find it convenient to adhere to the following convention.

Convention 1.8. *If $s_\alpha \cdot \lambda_a - \lambda_a = -2\frac{\langle \alpha, \lambda_a \rangle}{\langle \alpha, \alpha \rangle}\alpha$ in the theorem above is an integral weight, then we choose μ_α to be this weight.*

1.4. **Cohomology of Harish-Chandra modules.** Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. Up to $K_\mathbb{R}$-conjugacy, our fixed (compact) Cartan subalgebra $\mathfrak{h}^a$ is contained in $\mathfrak{l}$, since $\mathfrak{l}$ is the centralizer of an elliptic element of $\mathfrak{g}_\mathbb{R}$. If $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, K)$, the cohomology groups $H^i(\mathfrak{u}, X)$ are $(\mathfrak{l}, L \cap K)$ modules. Set

$$\rho_\mathfrak{u} = \frac{1}{2} \sum_{\beta \in \Delta(\mathfrak{h}^a, \mathfrak{u})} \beta.$$

We introduce translation functors for modules in $\mathcal{HC}(\mathfrak{l}, L \cap K)$ similar to the ones in the previous section. Because of the following result there is a ρ-shift in the definitions.

Theorem 1.9 ([V2], Theorem 4.1)**.** *Let $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$. Then $Y = H^i(\mathfrak{u}, X)$ decomposes as*

$$Y = \bigoplus_{w \in W(\mathfrak{h}^a, \mathfrak{l}) \backslash W(\mathfrak{h}^a, \mathfrak{g})} P_{w\lambda_a - \rho_\mathfrak{u}}(Y).$$

For Harish-Chandra modules, one needs an appropriate analog of a highest weight of a Verma module. Roughly speaking, the next definition is related to such an analog.

Definition 1.10 ([V2], Definitions 6.8, 6.9)**.** Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ containing a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ defined over $\mathbb{R}$. Let $H_{\mathbb{R}} = T_{\mathbb{R}} \exp \mathfrak{a}_{\mathbb{R}}$ be the Cartan decomposition of $H_{\mathbb{R}}$. Suppose $\gamma \in (\widehat{H}_{\mathbb{R}})'$. Let $\gamma_{\mathfrak{q}}$ be the regular pseudocharacter of $H_{\mathbb{R}}$ with respect to $\mathfrak{l}$ specified by

(a) $\gamma_{\mathfrak{q}} = (\Gamma_{\mathfrak{q}}, \overline{\gamma}_{\mathfrak{q}})$;
(b) $\overline{\gamma}_{\mathfrak{q}} = \overline{\gamma} - \rho_{\mathfrak{u}}$;
(c) $(\Gamma_{\mathfrak{q}})_{|T_{\mathbb{R}}} = \Gamma_{|T_{\mathbb{R}}} \otimes V$ where V is a one-dimensional character in the root lattice of $T_{\mathbb{R}}$.

In this context, set

$$l_{\mathfrak{q}}(\gamma) = \frac{1}{2}|\{\alpha \in \Delta(\mathfrak{h},\mathfrak{u}) \,|\, \theta(\alpha) \neq \alpha\}| + |\{\alpha \in \Delta(\mathfrak{h},\mathfrak{u}) \,|\, \alpha \text{ compact and } \langle \alpha, \overline{\gamma}\rangle < 0 \text{ or } \alpha \text{ imaginary noncompact and } \langle \alpha, \overline{\gamma}\rangle > 0\}|.$$

Finally, we will need translation functors for $\mathfrak{l}$. Define for α simple in Δ_a^+

$$\psi_\alpha^{\mathfrak{l}} = \psi_{\lambda-\rho_{\mathfrak{u}}}^{\lambda-\rho_{\mathfrak{u}}+\mu_\alpha}, \ \phi_\alpha^{\mathfrak{l}} = \psi_{\lambda-\rho_{\mathfrak{u}}+\mu_\alpha}^{\lambda-\rho_{\mathfrak{u}}}$$

where μ_α is chosen as in the previous section.

1.5. **Kazhdan-Lusztig algorithm: first remarks.** The Kazhdan-Lusztig algorithm for Harish-Chandra modules, for linear groups, is described in [V2] or [Vgr]. It proceeds by induction on the length, $l(\gamma)$, of a pseudocharacter $(H_{\mathbb{R}}, \gamma)$ and computes

(a) the composition series of $X(\gamma)$;
(b) the cohomology groups $H^i(\mathfrak{u}, \overline{X}(\gamma))$ as a $(\mathfrak{l}, L \cap K)$-module, for each θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$;
(c) for each simple root $\alpha \in R^+(\overline{\gamma})$ such that $\alpha \notin \tau(\overline{X}(\gamma))$, the composition series of $U_\alpha(\overline{X}(\gamma))$.

When $X(\gamma)$ is irreducible, we say that γ is minimal. For linear groups, [Vgr, Theorem 8.6.4] gives a necessary and sufficient condition for a pseudocharacter γ to be minimal. For nonlinear groups we need a generalization of this result, found in [Mi1, Theorem 2.1].

Theorem 1.11. *Suppose $\gamma \in (\widehat{H}_{\mathbb{R}})'_{\lambda_a}$ is a regular pseudocharacter. Then γ is minimal if and only if the following conditions are satisfied:*

(a) *For any complex root $\alpha \in R^+(\overline{\gamma})$:*
  (i) *$\theta(\alpha) \in R^+(\overline{\gamma})$; or*
  (ii) *$\theta(\alpha) \notin R^+(\overline{\gamma})$ and α is not minimal in $\{\alpha, -\theta(\alpha)\}$ with respect to the standard ordering of R_α^+; here R_α is the smallest θ-stable root system containing α and $-\theta(\alpha)$ and $R_\alpha^+ = R_\alpha \cap R^+(\overline{\gamma})$.*

(b) *No real root in $\Delta^+(\overline{\gamma})$ satisfies the parity condition.*

For a minimal pseudocharacter γ, step (a) in the algorithm is trivial by definition. Part (b) is Theorem 6.13 of [V2] which computes the cohomology of standard irreducible modules. Part (c) is obtained by observing that the only constituents of $U_\alpha(\overline{X}(\gamma))$ are the 'special constituents' of [V1, Theorem 4.12]. For non-minimal γ, it is possible to find a pseudocharacter γ' of length $l(\gamma') = l(\gamma) - 1$ obtained from γ either by Cayley transform with respect to a simple real root satisfying the parity conditions, or coherent continuation across a simple complex wall. Steps (a), (b), and (c) for γ are computable from the corresponding ones for γ' and other pseudocharacters of smaller length—this will briefly be recalled at the end of Section 5. For linear groups, what is needed is an important intermediate result, namely a vanishing theorem in cohomology ([V2]). To state it we first need a definition.

Definition 1.12. Suppose $G_{\mathbb{R}}$ is linear and $(H_{\mathbb{R}}, \gamma)$ is a regular pseudocharacter of $G_{\mathbb{R}}$. Put

$$l^I(\gamma) = \frac{1}{2}|\{\alpha \in R^+(\overline{\gamma}) \mid \theta(\alpha) \notin R^+(\overline{\gamma})\}| + \frac{1}{2}\dim \mathfrak{a} - c_0,$$

where c_0 is a constant independent of γ chosen such that $l^I(\gamma)$ is an integer for all pseudocharacters γ.

Remark. If $G_{\mathbb{R}}$ is not linear, the definition still makes sense, but it may be false that there exists c_0 with the property that $l^I(\gamma)$ is an integer for all γ, as the example of $Mp(2,\mathbb{R})$ and modules with infinitesimal character $\frac{1}{2}\rho$ shows (see Section 4).

Now we come to the vanishing theorem alluded to above. The hypothesis of linearity is crucial here, essentially because in the nonlinear case it is not obvious what the correct analog of l^I should be, or even if such an analog exists.

Theorem 1.13 ([V2], Theorem 7.2)**.** *Suppose that $G_{\mathbb{R}}$ is linear and let $\gamma^i \in (\widehat{H}^i_{\mathbb{R}})'$, $i = 1,2$ be two regular pseudocharacters of $G_{\mathbb{R}}$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}^2$. Then*

(a) *$H^i(\mathfrak{u}, \overline{X}(\gamma^1))$ contains $\overline{X}^L(\gamma^2_{\mathfrak{q}})$ as a composition factor only if $(l^I(\gamma^1) - l^I(\gamma^2)) - (l_{\mathfrak{q}}(\gamma^2) - i)$ is even.*

(b) *If $\overline{X}(\gamma^1)$ and $\overline{X}(\gamma^2)$ are distinct, $l(\gamma^1) \geq l(\gamma^2)$, and*

$$\mathrm{Ext}^1_{(\mathfrak{g},K)}(\overline{X}(\gamma^1), \overline{X}(\gamma^2)) \neq 0,$$

then $(l^I(\gamma^1) - l^I(\gamma^2))$ is odd.

(c) *$H^i(\mathfrak{u}, \overline{X}(\gamma^1))$ is completely reducible as an $\mathfrak{l}$ module.*

(d) *Suppose that $\alpha \in \Delta^+(\overline{\gamma}^2)$ is a simple integral root not in $\tau(\overline{X}(\gamma^1))$, with $n = 2\frac{\langle \alpha, \overline{\gamma}^2\rangle}{\langle \alpha, \alpha\rangle}$.*

(d1) *If $\alpha \in \Delta(\mathfrak{h}^2, \mathfrak{u})$, then the multiplicity of $\overline{X}^L(\gamma^2_{\mathfrak{q}})$ in $H^i(\mathfrak{u}, U_\alpha(\overline{X}(\gamma^1))$ is its multiplicity in $H^{i+1}(\mathfrak{u}, \overline{X}(\gamma^1))$ plus the multiplicity of $\overline{X}^L(\gamma^2_{\mathfrak{q}} - n\alpha)$ in $H^i(\mathfrak{u}, \overline{X}(\gamma^1))$.*

(d2) *If $\alpha \in \Delta(\mathfrak{h}^2, \mathfrak{l})$, then the multiplicity of $\overline{X}^L(\gamma^2_{\mathfrak{q}})$ in $H^i(\mathfrak{u}, U_\alpha(\overline{X}(\gamma^1))$ is zero unless $\psi^{\mathfrak{l}}_\alpha(\overline{X}^L(\gamma^2_{\mathfrak{q}})) = 0$ (i.e. α lies in the τ-invariant with respect to $\mathfrak{l}$) and in that case it is the multiplicity of $\overline{X}^L(\gamma^2_{\mathfrak{q}})$ in*

$$\phi^{\mathfrak{l}}_\alpha \psi^{\mathfrak{l}}_\alpha (H^i(\mathfrak{u}, \overline{X}(\gamma^1)) \oplus H^{i-1}(\mathfrak{u}, \overline{X}(\gamma^1)) \oplus H^{i+1}(\mathfrak{u}, \overline{X}(\gamma^1)).$$

(d3) *If $-\alpha \in \Delta(\mathfrak{h}^2, \mathfrak{u})$, then the multiplicity of $\overline{X}^L(\gamma^2_{\mathfrak{q}})$ in $H^i(\mathfrak{u}, U_\alpha(\overline{X}(\gamma^1))$ is its multiplicity in $H^{i-1}(\mathfrak{u}, \overline{X}(\gamma^1))$ plus the multiplicity of $\overline{X}^L(\gamma^2_{\mathfrak{q}} - n\alpha)$ in $H^i(\mathfrak{u}, \overline{X}(\gamma^1))$.*

(e) *In the setting of (d), $\overline{X}(\gamma^3)$ occurs in $U_\alpha(\overline{X}(\gamma^1))$ only if $(l^I(\gamma^1) - l^I(\gamma^3))$ is odd.*

Our main result in this section is an analog of this theorem when $G_{\mathbb{R}} = Mp(2n,\mathbb{R})$, and λ_a is the infinitesimal character of a genuine discrete series. What we need is an appropriate length function to replace l^I. It turns out that the length (Definition 1.1) is what we want. (Since the proof of Theorem 1.14 proceeds by induction on the dimension of Levi factors in $Mp(2n,\mathbb{R})$, we need to slightly enlarge the class of groups under consideration.)

Theorem 1.14. *For representations having the same infinitesimal character as a genuine discrete series representation, Theorem 1.13 is valid for any group in the class $\mathcal{L}(Mp)$ (defined in Section 3.1 below) with l^I replaced by l.*

We will prove this theorem in Section 3, after introducing some material about the metaplectic group. Once this result is in place, we can proceed as in the linear case and arrive at the following conclusion, which is one of our main results.

Corollary 1.15. *There is an effective algorithm for computing composition series of genuine standard Harish-Chandra modules for $Mp(2n,\mathbb{R})$ whose infinitesimal character coincides with that of a genuine discrete series representation.*

For completeness, we will describe steps (a), (b), (c) of the Kazhdan-Lusztig algorithm for $Mp(2n,\mathbb{R})$ in Section 3. This closely follows Section 9.6 of [Vgr], but we will give the few extra arguments needed in our setting.

2. Notation and structure theory

In this section we recall the structural facts that we need for our study of the representation theory of $Mp(2n,\mathbb{R})$.

2.1. $Sp(2n,\mathbb{R})$**.** To make the pictures in Section 6 a little cleaner, we choose a different realization of the symplectic group than is usually considered, and define $Sp(2n,\mathbb{R})$ to be the set of real $2n$-dimensional matrices preserving the form defined by

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}.$$

In the foregoing, we will let $G_{\mathbb{R}}$ denote $Sp(2n,\mathbb{R})$ and will write $\mathfrak{g}_{\mathbb{R}}$ for its Lie algebra. With our choice of J, we have that $\mathfrak{g}_{\mathbb{R}}$ consists of all $2n$-dimensional real matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{R}) \text{ such that } A = -D^{atr}, B = B^{atr}, \text{ and } C = C^{atr};$$

here X^{atr} denotes the *antitranspose* of X, i.e. the flip of X about its antidiagonal.

We write $\mathfrak{g}$ for the complexification of $\mathfrak{g}_{\mathbb{R}}$ and let θ denote the Cartan involution of negative transpose.

2.2. **Cartan subalgebras in $\mathfrak{g}_{\mathbb{R}}$.** Given $x_k, y_k \in \mathbb{R}$, define the 2-by-2 matrices $M_{ij}^{(k)}$ as follows:

$$\begin{pmatrix} M_{11}^{(k)} & M_{12}^{(k)} \\ -M_{12}^{(k)} & -M_{11}^{(k)} \end{pmatrix} = \begin{pmatrix} & x_k & y_k & \\ x_k & & & y_k \\ -y_k & & & -x_k \\ & -y_k & -x_k & \end{pmatrix}.$$

Given positive integers m, r and s such that $2m+r+s=n$, we define $\mathfrak{h}_{\mathbb{R}}^{m,r,s}$ to be the subset of $\mathfrak{g}_{\mathbb{R}}$ consisting of those matrices of the form

$$\begin{pmatrix} A & B \\ -B & -A^{atr} \end{pmatrix}$$

with

$$A = \begin{pmatrix} M_{11}^{(1)} &&&&&&&& \\ & \ddots &&&&&&& \\ && M_{11}^{(m)} &&&&&& \\ &&& 0 &&&&& \\ &&&& \ddots &&&& \\ &&&&& 0 &&& \\ &&&&&& a_1 && \\ &&&&&&& \ddots & \\ &&&&&&&& a_s \end{pmatrix}$$

and

$$B = \begin{pmatrix} &&&&&&&& M_{12}^{(1)} \\ &&&&&&& \iddots & \\ &&&&&& M_{12}^{(m)} && \\ &&&&& t_1 &&& \\ &&&& \iddots &&&& \\ &&& t_r &&&&& \\ && 0 &&&&&& \\ & \iddots &&&&&&& \\ 0 &&&&&&&& \end{pmatrix}.$$

The various entries x_j, y_j, t_j and a_j are arbitrary real numbers.

It will be convenient to single out the above element in $\mathfrak{h}_{\mathbb{R}}^{m,r,s}$. We will denote it by $H(z_1, \ldots, z_m; t_1, \ldots, t_r; a_1, \ldots, a_s)$ or $H(\underline{z}; \underline{t}; \underline{a})$; here $z_j = x_j + iy_j$.

Lemma 2.1. *The $\mathfrak{h}_{\mathbb{R}}^{m,r,s}$ (such that $2m+r+s=n$) exhaust the $G_{\mathbb{R}}$-conjugacy classes of Cartan subalgebras in $\mathfrak{g}_{\mathbb{R}}$.*

Clearly, $\mathfrak{h}_{\mathbb{R}}^{m,r,s}$ is θ-stable, and so we write its eigenspace decomposition as $\mathfrak{h}_{\mathbb{R}}^{m,r,s} = \mathfrak{t}_{\mathbb{R}}^{m,r,s} \oplus \mathfrak{a}_{\mathbb{R}}^{m,r,s}$ with $\dim_{\mathbb{R}}(\mathfrak{a}_{\mathbb{R}}^{m,r,s}) = m+s$. In particular, $\mathfrak{h}_{\mathbb{R}}^{0,n,0} = \mathfrak{t}_{\mathbb{R}}^{0,n,0}$ and $\mathfrak{h}_{\mathbb{R}}^{0,0,n} = \mathfrak{a}_{\mathbb{R}}^{0,0,n}$. As usual, we write $\mathfrak{h}^{m,r,s}$ for the complexification of $\mathfrak{h}_{\mathbb{R}}^{m,r,s}$, and $\mathfrak{t}_{\mathbb{R}}$ for the compact Cartan subalgebra $\mathfrak{h}^{0,n,0}$. We also view $\mathfrak{t}$ as our 'abstract' Cartan subalgebra $\mathfrak{h}_a$.

2.3. **Roots.** Consider the (complexified) anti-diagonal Cartan subalgebra $\mathfrak{h}^{0,n,0}$. For $1 \leq j \leq n$, define the linear functional $e_j \in (\mathfrak{h}^{0,n,0})^*$ via

$$e_j(H(t_1, \ldots, t_n)) = it_j.$$

The functionals $\{\pm e_j \pm e_k, \pm 2e_j\}$ are roots of $\mathfrak{h}^{0,n,0}$ in $\mathfrak{g}$. We choose the standard positive roots

$$\Delta_a^+ = \Delta_{0,n,0}^+ = \{e_j \pm e_k \mid j < k\} \cup \{2e_j\}.$$

Suppose α is a noncompact imaginary root for $\mathfrak{h}$ in $\mathfrak{g}$. We write c^α for the Cayley transform through α. There is some ambiguity in the normalization of c^α; rather than give the normalization explicitly, we simply note that it is specified by (2.1) below.

In any case, we can explicitly relate $\mathfrak{h}_{\mathbb{R}}^{m,r,s}$ to $\mathfrak{h}^{0,n,0}$ as follows:

$$\mathfrak{h}^{m,r,s} = c^{m,r,s}(\mathfrak{h}^{0,n,0}); \qquad c^{m,r,s} = \prod_{1\le j\le m} c^{e_{2j-1}+e_{2j}} \circ \prod_{n-s+1\le j\le n} c^{2e_j}.$$

(There is something to check in this definition; namely, that in each appearance of some c^α, α is actually a noncompact imaginary root for the domain of c^α.) Transposing this isomorphism allows us to define a choice of positive roots $\Delta^+_{m,r,s} = [(c^{m,r,s})^{tr}]^{-1}(\Delta^+_{0,n,0})$ for the roots of $\mathfrak{h}^{m,r,s}$ in $\mathfrak{g}$. When it is clear from the context, we will be sloppy and write $e_j \in (\mathfrak{h}^{m,r,s})^*$ for the transport (via the transpose of $c^{m,r,s}$) of the functional $e_j \in \mathfrak{h}^{0,n,0}$.

With this understood, one checks that for $H = H(\underline{z};\underline{t};\underline{a})$,

$$\begin{aligned} e_{2j-1}(H) &= x_j + iy_j, && 1\le j\le m;\\ e_{2j}(H) &= x_j - iy_j, && 1\le j\le m;\\ e_j(H) &= it_j, && 2m+1\le j\le 2m+r;\\ e_j(H) &= a_j, && n-s\le j\le n. \end{aligned} \tag{2.1}$$

Hence we compute

$$\begin{aligned} \theta(e_{2j-1}) &= -e_{2j}, && 1\le j\le m;\\ \theta(e_j) &= e_j, && 2m+1\le j\le 2m+r;\\ \theta(e_j) &= -e_j, && 2m+r+1\le j\le n; \end{aligned} \tag{2.2}$$

from which we can easily compute the real, complex, and imaginary roots in $\Delta^+_{m,r,s}$ (see Lemma 6.14 below).

Finally, for $1\le j\le n$, define elements

$$H_j = H(\overbrace{0,\dots,0}^{j-1}, -i, 0, \dots, 0) \in i\mathfrak{t}_{\mathbb{R}}.$$

(So, in the canonical normalization, H_j is the coroot of e_{2j}.) Unlike the case of roots, we will not transport the element H_j to other Cartan subalgebras, and instead always consider H_j as an element of $i\mathfrak{t}_{\mathbb{R}}$.

2.4. **The metaplectic group and its center.** Let $\widetilde{G_{\mathbb{R}}} := Mp(2n,\mathbb{R})$ be the metaplectic group of rank n. It is a nontrivial central extension of order two of of $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$, defined by an explicit cocycle (see for example [Tor]). For the remainder of Section 2, we will let $\widetilde{G_{\mathbb{R}}} = Mp(2n,\mathbb{R})$ and $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$.

We denote the projection $\widetilde{G_{\mathbb{R}}} \longrightarrow G_{\mathbb{R}}$ by $\mathbf{pr}$, and write $\mathbf{z}$ for the nontrivial element of $\widetilde{G_{\mathbb{R}}}$ in $\mathbf{pr}^{-1}(\{\mathrm{Id}_{2n}\})$. We will use the following notational convention. Preimages (in $\widetilde{G_{\mathbb{R}}}$) of subgroups of $G_{\mathbb{R}}$ will be denoted by adding a tilde. For instance, we write $K_{\mathbb{R}}$ for the maximal compact subgroup of $G_{\mathbb{R}}$ and set $\widetilde{K}_{\mathbb{R}} := \mathbf{pr}^{-1}(K_{\mathbb{R}})$. Note that $\widetilde{K}_{\mathbb{R}}$ is a maximal compact subgroup of $\widetilde{G_{\mathbb{R}}}$. Here is a list of some easy but important structure theoretic facts.

Lemma 2.2. *Recall the elements $H_j \in i\mathfrak{t}_{\mathbb{R}}$ (defined at the end of Section 2.3).*

(a) *The element* $\mathbf{z} \in \widetilde{G_{\mathbb{R}}}$ *is central; it has order 2. Moreover, independent of* j, *we have*

$$\mathbf{z} = \exp_{Mp}(2i\pi H_j),$$

(b) *The center* $Z(\widetilde{G_{\mathbb{R}}})$ *has order four. We may write* $Z(\widetilde{G_{\mathbb{R}}}) = \{\mathbf{e}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ *with*

$$\mathbf{x} = \exp_{Mp}(i\pi(H_1 + \ldots + H_n)), \quad \textit{and } \mathbf{y} = \mathbf{zx}.$$

Moreover,
(i) *if* n *is odd, then* $\mathbf{x}^2 = \mathbf{z}$ *and* $Z(\widetilde{G_{\mathbb{R}}}) \simeq \mathbb{Z}/4$*;*
(ii) *if* n *is even, then* $\mathbf{x}^2 = \mathbf{e}$*, and* $Z(\widetilde{G_{\mathbb{R}}}) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

2.5. **Cartan subgroups in $G_{\mathbb{R}}$ and $\widetilde{G_{\mathbb{R}}}$, Weyl groups.** Recall the notation of the previous section; in particular, $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$ and $\widetilde{G_{\mathbb{R}}} = Mp(2n, \mathbb{R})$. Let $H_{\mathbb{R}}^{m,r,s}$ denote the centralizer in $G_{\mathbb{R}}$ of $\mathfrak{h}_{\mathbb{R}}^{m,r,s}$. We have

$$H_{\mathbb{R}}^{m,r,s} \simeq (\mathbb{C}^{\times})^m \times (S^1)^r \times (\mathbb{R}^{\times})^s,$$

which has 2^s connected components. The Weyl group $W_{\mathbb{R}}^{m,r,s} = N_{G_{\mathbb{R}}}(T)/Z_{G_{\mathbb{R}}}(T)$ is isomorphic to

$$(S_m \ltimes W(D_2)^m) \times S_r \times W(C_s);$$

here $W(C_k) \simeq S_k \ltimes (\mathbb{Z}/2)^k$ is the complex Weyl group of type C_k, and similarly for $W(D_2)$. In other words, $W_{\mathbb{R}}^{m,r,s}$ acts by negation, complex conjugation and permutation of the first m coordinates of $H(\underline{z}; \underline{t}; \underline{a}) \in \mathfrak{h}_{\mathbb{R}}^{m,r,s}$; S_r acts by permuting the next r coordinates; and $W(C_s)$ acts by permutation and sign changes on the last s coordinates. We write $W^{m,r,s}$ for the complex Weyl group of $\mathfrak{h}^{m,r,s}$ in $\mathfrak{g}$. In the next section, we realize these groups concretely inside S_{2n}.

Let $\widetilde{H}_{\mathbb{R}}^{m,r,s}$ denote the centralizer in $\widetilde{G_{\mathbb{R}}}$ of $\mathfrak{h}_{\mathbb{R}}^{m,r,s}$. Because $\widetilde{G_{\mathbb{R}}}$ is a central extension of $G_{\mathbb{R}}$, we have that $\mathbf{pr}^{-1}(H_{\mathbb{R}}^{m,r,s}) \subset \widetilde{H}_{\mathbb{R}}^{m,r,s}$. In fact, it turns out that $\mathbf{pr}^{-1}(H_{\mathbb{R}}^{m,r,s}) = \widetilde{H}_{\mathbb{R}}^{m,r,s}$, and $\widetilde{H}_{\mathbb{R}}^{m,r,s}$ is an abelian double cover of $H_{\mathbb{R}}^{m,r,s}$. Moreover, we can canonically identify $W_{\mathbb{R}}^{m,r,s}$ with $N_{\widetilde{G_{\mathbb{R}}}}(\mathfrak{h}_{\mathbb{R}}^{m,r,s})/Z_{\widetilde{G_{\mathbb{R}}}}(\mathfrak{h}_{\mathbb{R}}^{m,r,s})$. There is a little subtlety in describing the connected components of $\widetilde{H}_{\mathbb{R}}^{m,r,s}$ which we now explain.

Let $m_j = \exp_{Mp}(i\pi H_j) \in \widetilde{T}_{\mathbb{R}}$. From Lemma (2.2) above, we see that m_j has order 4. Assume now that $s \neq 0$ or $s = m = 0$, and define $F^{m,r,s}$ to be the group generated by the m_j for $n - s < j \leq n$. (Note that $\mathbf{z} \in F^{m,r,s}$.) On the other hand, if $s = 0$ and $m \neq 0$, then we define $F^{m,r,s}$ to be $\{\mathbf{e}, \mathbf{z}\}$. In either case, we have the direct product decomposition

$$\widetilde{H}_{\mathbb{R}}^{m,r,s} = F^{m,r,s} \times \exp_{Mp}(\mathfrak{h}_{\mathbb{R}}^{m,r,s}).$$

Thus $\widetilde{H}_{\mathbb{R}}^{m,r,s}$ has either 4^s components (if $s \neq 0$ or $s = m = 0$) or exactly 2 components (if $s = 0$ and $m \neq 0$).

2.6. **Weyl groups.** We need to discuss the Weyl groups $W_{\mathbb{R}}^{m,r,s}$ (introduced in the previous section) in a little more explicit detail. Let $G' = GL(2n, \mathbb{C})$, write $\mathfrak{h}'$ for the diagonal Cartan subalgebra in $\mathfrak{gl}(2n, \mathbb{C})$, and H' for the centralizer in G' of $\mathfrak{h}'$. Let $W'_{\mathbb{C}}$ denote the Weyl group $N_{G'}(H')/Z_{G'}(H')$. With these choices, there is a standard way to identify $W'_{\mathbb{C}}$ with S_{2n}.

Now consider $\mathfrak{h} = \mathfrak{h}^{0,0,n}$ (notation as in Section 2.2). Write H for the centralizer in G of $\mathfrak{h}$, and $W_{\mathbb{C}} = W^{0,0,n}$ for the Weyl group $N_G(H)/Z_G(H)$. The inclusion of normalizers $N_{G_{\mathbb{R}}}(H_{\mathbb{R}}^{0,0,n}) \hookrightarrow N_G(H)$ identifies $W_{\mathbb{R}}^{0,0,n}$ with $W_{\mathbb{C}}$, and the isomorphisms $c^{m,r,s}$ (notation as in Section 2.2) allow us to consider each $W_{\mathbb{R}}^{m,r,s}$ as a subgroup of $W_{\mathbb{C}}$. Finally, it is not difficult to verify that $N_{G'}(H') \subset N_G(H)$, and in fact this inclusion descends to an inclusion $W_{\mathbb{C}} \subset W'_{\mathbb{C}}$. Hence we can consider each $W_{\mathbb{R}}^{m,r,s}$ as a subgroup of a fixed S_{2n}.

In order to describe explicitly $W_{\mathbb{R}}^{m,r,s}$ as a subgroup of S_{2n}, we need to introduce a little notation. Given $w \in S_{2n}$, let F denote the number of fixed-points of w, i.e. $F = \{j \mid w(j) = j\}$. For each $j \in F$, choose a sign $\epsilon_j \in \mathbb{Z}/2$, and write ϵ for the tuple $(\epsilon_j)_{j \in F}$. We say that the pair (w, ϵ) is an involution with signed fixed points. We can consider the normalizer of (w, ϵ) in $W_{\mathbb{C}} \subset S_{2n}$,

$$N_{W_{\mathbb{C}}}(w, \epsilon) = \{x \in W_{\mathbb{C}} \mid xwx^{-1} = w, \quad \epsilon_{x(j)} = \epsilon_j, \text{ for all } j \in F\}.$$

The point of this section is that $W_{\mathbb{R}}^{m,r,s}$ can be written as such a normalizer. This is the content of Lemma 2.3, but first we need some notation. We define a particular involution with signed fixed points called the *canonical orbit diagram* $o^{m,r,s}$. (The terminology 'orbit diagram' will be explained in Section 6.3.) Explicitly $o^{m,r,s}$ is specified by

$$\begin{aligned}
&\sigma(j) = 2n + 1 - j - 1, \quad \text{for } 1 \leq j \leq 2m \text{ and } j \text{ odd},\\
&\sigma(j) = 2n + 1 - j + 1, \quad \text{for } 1 \leq j \leq 2m \text{ and } j \text{ even},\\
&\sigma(j) = j, \quad 2m + 1 \leq j \leq 2m + r \text{ or } 2n - 2m - r + 1 \leq j \leq 2n - 2m,\\
&\sigma(j) = 2n + 1 - j, \quad \text{else, i.e. for } n - s \leq j \leq n + 1 + s,\\
&\quad \epsilon_j = +, \quad \text{for } 2m + 1 \leq j \leq 2m + r,\\
&\quad \epsilon_j = -, \quad \text{for } 2n - 2m - r + 1 \leq j \leq 2n - 2m.
\end{aligned}$$

For instance, $o^{2,1,1}$ looks like

Lemma 2.3. *Identify $W_{\mathbb{C}}$ and $W^{m,r,s}$ with subgroups of S_{2n} as explained above. Then*

$$W_{\mathbb{C}} = N_{S_{2n}}(o^{0,0,n});$$

and, moreover,

$$W_{\mathbb{R}}^{m,r,s} = N_{W_{\mathbb{C}}}(o^{m,r,s}).$$

2.7. **An outer automorphism.** Consider the matrix

$$A = \begin{pmatrix} 0 & iJ \\ iJ & 0 \end{pmatrix} \in Sp(2n, \mathbb{C});$$

here J is defined as in Section 2.1. Conjugation by A defines an outer automorphism of $Sp(2n, \mathbb{R})$ of order two which we denote by ι. It lifts to an order two automorphism of $Mp(2n, \mathbb{R})$ denoted $\widetilde{\iota}$. On the level of roots (Section 2.3), ι induces the mapping $\epsilon_j \mapsto -\epsilon_j$. The reason that $\widetilde{\iota}$ will be important for us is the following trivial observation.

Proposition 2.4. *Suppose $\gamma \mapsto \check{\gamma}$ is a bijection satisfying the requirements of Theorem 0.1. Then the bijection $\gamma \mapsto \widetilde{\iota}(\check{\gamma})$ also satisfies the requirements of the theorem.*

3. Preliminaries

3.1. **Genuine modules.** As mentioned above, our main results on $Mp(2n,\mathbb{R})$ rely on an inductive reduction to its Levi factors, and so we need to enlarge the class of groups under consideration. With this in mind, define $\mathcal{L}(Mp)$ to be the class of groups formed by $Mp(2n,\mathbb{R})$ and the normalizers $\tilde{L}_{\mathbb{R}} = \mathrm{Norm}(Mp(2n,\mathbb{R}),\mathfrak{q})$ where $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{sp}(2n,\mathbb{C})$.

Notation for Section 3. When we write $\widetilde{G_{\mathbb{R}}}$, we will always mean a group in $\mathcal{L}(Mp)$. We will always denote $\mathbf{pr}(\widetilde{G_{\mathbb{R}}})$ by $G_{\mathbb{R}}$. (This generalizes the notation of Section 2 where $\widetilde{G_{\mathbb{R}}} = Mp(2n,\mathbb{R})$ and $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$.) Moreover, $\widetilde{K}_{\mathbb{R}}$ and $K_{\mathbb{R}}$ will denote maximal compact subgroups in $\widetilde{G_{\mathbb{R}}}$ and $G_{\mathbb{R}}$, and $\widetilde{T}_{\mathbb{R}}$ and $T_{\mathbb{R}}$ will denote their respective maximal tori. (Since we always have $K_{\mathbb{R}} = \mathbf{pr}(\widetilde{K}_{\mathbb{R}})$, the notation is consistent.)

An important observation to make is that $\mathcal{L}(Mp)$ is closed in an appropriate sense. Note that we may (and do) assume that the Cartan involution of $\widetilde{G_{\mathbb{R}}}$ is obtained by restricting the one for $Mp(2n,\mathbb{R})$.

Lemma 3.1. *Fix $\widetilde{G_{\mathbb{R}}} \in \mathcal{L}(Mp)$ and suppose $\mathfrak{q}$ is a θ-stable parabolic in Lie$(\widetilde{G_{\mathbb{R}}})_{\mathbb{C}}$. Then Norm$(\widetilde{G_{\mathbb{R}}},\mathfrak{q}) \in \mathcal{L}(Mp)$.*

The notation $\widetilde{T}_{\mathbb{R}}$ was used in the previous section to denote a fixed compact Cartan subgroup in $Mp(2n,\mathbb{R})$. The notation (outlined above) indicates that in this section $\widetilde{T}_{\mathbb{R}}$ denotes a compact Cartan subgroup in $\widetilde{G_{\mathbb{R}}}$. The next lemma insures that in fact these notations are consistent.

Lemma 3.2. *Let $\widetilde{G_{\mathbb{R}}}$ be in the class $\mathcal{L}(Mp)$ and adhere to the notations outlined above for Section 3. Then $\widetilde{K}_{\mathbb{R}}$ is the intersection of $\widetilde{G_{\mathbb{R}}}$ with $\widetilde{K}_{\mathbb{R}}^{Mp}$, the maximal compact subgroup in $Mp(2n,\mathbb{R})$. Consequently, $\widetilde{T}_{\mathbb{R}}$ is a compact Cartan subgroup of $Mp(2n,\mathbb{R})$. Hence by conjugating with $\widetilde{K}_{\mathbb{R}}^{Mp}$, we may assume that $\widetilde{T}_{\mathbb{R}} = \widetilde{H}^{0,n,0}$ (with notation as in Section 2.5).*

Remark 3.3. In the setting of the lemma, and when it is appropriate, we will still consider $\mathfrak{h}_a = \mathfrak{t}$ as our abstract Cartan subalgebra.

Finally, note that it follows directly from the definition (or, alternatively, the above lemma) that each group in $\mathcal{L}(Mp)$ (and each of its Cartan subgroups) contains $Z(Mp(2n,\mathbb{R})) = \{\mathbf{e},\mathbf{x},\mathbf{y},\mathbf{z}\}$. In particular, $Z(Mp(2n,\mathbb{R})) \subset Z(\widetilde{G_{\mathbb{R}}})$. We utilize these central elements as follows.

Definition 3.4. Fix $\widetilde{G_{\mathbb{R}}} \in \mathcal{L}(Mp)$, and let X be a module in $\mathcal{HC}(\mathfrak{g},\widetilde{K})$. We say X admits a central character if each element of $Z(\widetilde{G_{\mathbb{R}}})$ acts by a scalar. In this case, we obtain a homomorphism $\chi_X : Z(\widetilde{G_{\mathbb{R}}}) \longrightarrow \mathbb{C}^{\times}$. We call X genuine if $\chi_X(\mathbf{z}) = -1$, and denote the full subcategory of $\mathcal{HC}(\mathfrak{g},\widetilde{K})$ of genuine modules by $\mathcal{HC}(\mathfrak{g},\widetilde{K})^{\mathrm{gen}}$.

Now let $\widetilde{H}_{\mathbb{R}}$ be a θ-stable Cartan subgroup of $\widetilde{G}_{\mathbb{R}}$. Let $\gamma = (\Gamma,\overline{\gamma})$ be a λ_a-pseudocharacter for $\widetilde{H}_{\mathbb{R}}$. We say that γ is genuine if $\Gamma(\mathbf{z}) = -1$, and write $(H_{\mathbb{R}})^{\mathrm{gen}}_{\lambda_a}$ for the subset of genuine λ_a-pseudocharacters for $\widetilde{H}_{\mathbb{R}}$.

Fix a pseudocharacter $\gamma = (\Gamma,\overline{\gamma})$. Since $X(\gamma)$ is induced from a discrete series whose lowest K-type has highest weight Γ, and since the action of a central element commutes (in a suitable sense) with induction, we make the following conclusion.

Lemma 3.5. *The standard and irreducible modules $X(\gamma)$ and $\overline{X}(\gamma)$ are genuine if and only if γ is.*

Hence we can restate Theorem 1.3 for genuine modules.

Corollary 3.6. *Fix $\widetilde{G_{\mathbb{R}}} \in \mathcal{L}(Mp)$. The $\widetilde{K}_{\mathbb{R}}$-conjugacy classes of genuine pseudocharacters with regular infinitesimal character λ_a parameterize the irreducible objects in $\mathcal{HC}(\mathfrak{g}, \widetilde{K})^{gen}_{\lambda_a}$.*

3.2. **Half-integral infinitesimal character.** The main results of this paper concern only a certain kind of infinitesimal character, and we make that restriction precise now. Suppose Λ is a genuine character of the compact Cartan subgroup $\widetilde{T}_{\mathbb{R}} = \widetilde{H}^{0,n,0}$ (notation as above). Then its differential $\lambda = d\Lambda$ takes purely imaginary values on $\mathfrak{t}_{\mathbb{R}}$, and we can write $\lambda = \sum_j b_j e_j$ (Notation as in Section 2.3). The condition that Λ is a genuine character of $\widetilde{T}_{\mathbb{R}}$ means that each $b_i \in \frac{1}{2} + \mathbb{Z}$. More generally, we say that $\lambda \in \mathfrak{t}^*$ is half-integral if its restriction to $\mathfrak{t}$ is of the above form.

Recall that the infinitesimal character of an irreducible admissible representation of $\widetilde{G_{\mathbb{R}}}$ is specified by a Weyl group orbit $W_a \cdot \lambda_a \in \mathfrak{h}_a^*/W_a$. It is convenient to single out a dominant representative in this orbit. For this purpose, we fix the positive root system Δ_a^+ for the roots of $\mathfrak{h}_a$ in $\mathfrak{g}$ by restricting the positive root system $\Delta_a^+(Mp(2n,\mathbb{R}))$ of Section 2.3. We then define dominance with respect to this choice of positive roots.

The next lemma is part of Harish-Chandra's characterization of the discrete series.

Lemma 3.7. *The infinitesimal character of any genuine discrete representation of $\widetilde{G_{\mathbb{R}}} \in \mathcal{L}(Mp)$ is given by (the Weyl group orbit) of a dominant, regular, half-integral $\lambda_a \in (\mathfrak{h}^a)^*$.*

Remark 3.8. Consider the special case when $\widetilde{G_{\mathbb{R}}} = Mp(2n,\mathbb{R})$, and consider a dominant, half-integral, regular element $\lambda_a \in (\mathfrak{h}^a)^*$. In the basis $\{e_1, \dots, e_n\}$, we can write

$$\lambda_a = \left(\frac{2a_1-1}{2}, \dots, \frac{2a_n-1}{2}\right)$$

with $a_1 > a_2 > \dots > a_n > 0$, $a_i \in \mathbb{N}$. In particular, note that the simple roots which are integral for such a λ_a are precisely the short roots

$$S_{\text{int}} = \{e_i - e_{i+1} \mid i = 1, \dots, n\} \cup \{e_{n-1} + e_n\}.$$

More generally, we have the following lemma, whose simple verification we leave to the reader.

Lemma 3.9. *Let $\widetilde{G_{\mathbb{R}}}$ be a group in $\mathcal{L}(Mp)$, $\widetilde{H}_{\mathbb{R}}$ a θ-stable Cartan subgroup of $\widetilde{G_{\mathbb{R}}}$, and let γ be a genuine λ_a pseudocharacter for $\widetilde{H}_{\mathbb{R}}$ (of half-integral infinitesimal character). We have the following conclusions:*

(a) *The integral root system $R(\overline{\gamma})$ consists of the short roots in $\Delta(\mathfrak{g}, \mathfrak{h})$; in particular, it is θ-stable.*

(b) *The positive root system Δ_a^+ contains at most two simple long roots. If such a root does not exist, then (a) implies that $W_a = W(\lambda_a)$. If such roots do exist, then W_a is generated by $W(\lambda_a)$ and the simple reflections through the long simple roots.*

(c) *Finally, if α is a long root in $\Delta(\mathfrak{g}, \mathfrak{h})$, then*

$$m = 2\frac{\langle \alpha, \overline{\gamma} \rangle}{\langle \alpha, \overline{\gamma} \rangle} \in \mathbb{Z} + \frac{1}{2}$$

and $m\alpha$ is a weight of a finite-dimensional representation of $\mathfrak{g}$.

3.3. **Coherent continuation, cross action and Cayley transforms.** Fix $\widetilde{G_{\mathbb{R}}} \in \mathcal{L}(Mp)$ and consider a dominant, regular, half-integral element $\lambda_a \in (\mathfrak{h}_a)^*$. The point of this section is to develop the theory of coherent continuation, cross action, and Cayley transforms. Each of these concepts can be defined in terms of simple roots. When the simple root is integral (equivalently, according to Lemma 3.9(b), when the simple root is short) the appropriate definitions reduce to those given in [Vgr] for linear groups. However, for a simple long root, certain phenomena (different from the linear theory) present themselves. From the geometric viewpoint, this amounts to the observation that there is an extra case to consider in the geometric partition of [LV, Lemma 3.5]. At any rate, most of the interesting phenomena can be seen already in the case of $Mp(2,\mathbb{R})$ and for this reason, the reader may find the example of Section 4 helpful.

We begin by recalling the coherent continuation action of the integral Weyl group $W(\lambda_a)$ on $\mathbb{K}\mathcal{HC}(\mathfrak{g}, \widetilde{K})_{\lambda_a}$. For applications which we consider below, we find it convenient to extend this action to the entire complex Weyl group W_a. By Lemma 3.9(c), this amounts (at most) to defining the action of a long simple root. The one detail that we will need to check (Proposition 3.10 below) is that this extended action is good (in the sense that the action of $W(\lambda_a)$ is) for computing translation functors.

Fix a coherent family Θ (for $\widetilde{T}_{\mathbb{R}}$) based at an irreducible representation $X \in \mathrm{ob}\mathcal{HC}(\mathfrak{g}, \widetilde{K})_{\lambda_a}$. For $w \in W_a$, we define

$$w \cdot [X] = \Theta(w^{-1}\lambda_a) \in \mathbb{K}\mathcal{HC}(\mathfrak{g}, \widetilde{K})_{\lambda_a}. \tag{3.1}$$

When w actually is an element of $W(\lambda_a)$, this action and its properties are carefully explained in [Vgr, Chapter 7]. (One word of warning is in order here: some of the results in Chapter 7 do not hold for the extended action of W_a.)

Suppose we are in the setting where $W(\lambda_a) \subsetneq W_a$. According to Lemma 3.9, let α be the unique simple long root in W_a. The point of extending the coherent continuation action to W_a is that the action of s_α keeps track of the (nonintegral) wall-cross over the wall defined by α.

Proposition 3.10. *Retain the setting of the previous paragraph. Then*

$$s_\alpha \cdot [X] = [\psi_\alpha(X)];$$

here ψ_α is the nonintegral wall crossing functor described in Convention 1.8.

Proof. This follows directly from the definition of coherent families, together with the last assertion of Lemma 3.9. □

The next topic we address is Cayley transforms through noncompact imaginary roots. Again we stress that the theory is different for short roots and long roots, although the difference is more easily seen in the setting of *inverse* Cayley transforms described below.

Let $(\gamma, \widetilde{H}_{\mathbb{R}})$ be a λ_a-pseudocharacter of $\widetilde{G_{\mathbb{R}}}$, and let β be a noncompact root in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Let $\widetilde{H}_{\mathbb{R}}^{\beta}$ be the Cartan subgroup associated to the standard Cayley transform c^β. The type (I or II) of β is defined in [Vgr, Definition 8.3.4]. The Cayley transform $c^\beta(\gamma)$ of γ by β is defined in [Vgr, Definition 8.3.6], and with the notation there, $c^\beta(\gamma)=\{\gamma^\beta\}$ if γ is type I, $c^\beta(\gamma)=\{\gamma_+^\beta,\gamma_-^\beta\}$ if γ is type II, where γ^β, $\gamma_\pm^\beta$ are pseudocharacters of $\widetilde{H}_{\mathbb{R}}^\beta$.

The difference alluded to above manifests itself in the following guise: for a long noncompact imaginary root β (which, in fact, is always type I), γ is the unique pseudocharacter such that $c^\beta(\gamma)=\gamma^\beta$. This is to be contrasted with the type I setting in the linear case, where there always exists $\delta\neq\gamma$ such that $c^\beta(\delta)=\gamma^\beta$. We will explain this in more detail below.

Next we need to recall some of the material required to define inverse Cayley transforms of pseudocharacters by real roots satisfying the parity condition. We refer to [Vgr, Section 8.3] for omitted details.

Definition 3.11. Suppose that $\widetilde{H}_{\mathbb{R}}$ is a θ-stable Cartan subgroup of $\widetilde{G_{\mathbb{R}}}$, and $\alpha\in\Delta(\mathfrak{g},\mathfrak{h})$ is a real root. Choose root vectors X_α, $X_{-\alpha}$ in $\mathfrak{g}_{\mathbb{R}}$ such that $[X_\alpha,X_{-\alpha}]=H_\alpha$, where $H_\alpha\in\mathfrak{h}_{\mathbb{R}}$ is the coroot of α. Let $c_\alpha=\mathrm{Ad}(\xi_\alpha)$, where $\xi_\alpha=\exp(\frac{i\pi}{4}(X_\alpha+X_{-\alpha}))$ is an element of the adjoint group of $\mathfrak{g}$. Define $\mathfrak{h}_\alpha:=c_\alpha(\mathfrak{h})$. This is a θ-stable Cartan subalgebra of $\mathfrak{g}$ defined over $\mathbb{R}$, and $\beta={}^{tr}c_\alpha^{-1}(\alpha)$ is a noncompact imaginary root in $\Delta(\mathfrak{g},\mathfrak{h}_\alpha)$. Define $m_\alpha=\exp_{\widetilde{G_{\mathbb{R}}}}(i\pi H_\beta)\in\widetilde{G_{\mathbb{R}}}$; here H_β is the β-coroot in $i(\mathfrak{h}_\alpha)_{\mathbb{R}}$. One can check that m_α is an element of $\widetilde{H}_{\mathbb{R}}\cap(\widetilde{H}_\alpha)_{\mathbb{R}}$, which depends on the choices only up to a replacement of m_α by m_α^{-1}.

Definition 3.12. In the setting above, let $\epsilon_\alpha=\pm1$ be the sign defined in [Vgr, Definition 8.3.11]. Let $\gamma=(\Gamma,\overline{\gamma})$ be a λ_a-pseudocharacter for $\widetilde{H}_{\mathbb{R}}$. We say that α satisfies the parity condition with respect to γ if and only if

$$\Gamma(m_\alpha)=\epsilon_\alpha\exp\left(\pm i\pi2\frac{\langle\alpha,\overline{\gamma}\rangle}{\langle\alpha,\alpha\rangle}\right). \tag{3.2}$$

Lemma 3.13. *Let $\widetilde{G_{\mathbb{R}}}$ be a group in $\mathcal{L}(Mp)$, and $\widetilde{H}_{\mathbb{R}}$ a θ-stable Cartan subgroup of $\widetilde{G_{\mathbb{R}}}$. Let γ be a genuine λ_a-pseudocharacter for $\widetilde{H}_{\mathbb{R}}$. Suppose that α is a real root in $\Delta(\mathfrak{g},\mathfrak{h})$. If α is short, then $m_\alpha^2=\mathbf{e}$, thus the left-hand side of* (3.2) *is equal to $+1$ or -1, and since α is integral, then the right-hand-side of* (3.2) *takes only one value, equal to $+1$ or -1. If α is a long root, then $m_\alpha^2=\mathbf{z}$, thus since $\Gamma(\mathbf{z})=-1$ the left-hand side of* (3.2) *is equal to $+i$ or $-i$, and the right-hand-side of* (3.2) *takes values $+i$ and $-i$. Therefore, the parity condition is always satisfied for long real roots.*

Proof. Up to conjugacy, we can take $\widetilde{H}_{\mathbb{R}}$ to be one of the fixed Cartan subgroups $\widetilde{H}^{m,r,s}$ in $Mp(2n,\mathbb{R})$ (Section 2.5), and make explicit computations there. ☐

Remark 3.14. Notice that the last assertion is 'dual' to the fact that a long imaginary root is always noncompact.

Suppose we are in the setting of the previous lemma, and that α is a simple *short* real root in $\Delta^+(\overline{\gamma})$ satisfying the parity condition. Type I and II, and the pseudocharacter inverse Cayley transform $c_\alpha(\gamma)$ are defined in [Vgr, Definition 8.3.16]. With the notations there $c_\alpha(\gamma)=\{\gamma_\alpha\}$ in the type II case and $c_\alpha(\gamma)=\{\gamma_\alpha^+,\gamma_\alpha^-\}$ in the type I case. Let us just notice here that these definitions extend easily in our nonlinear setting because we require α to be simple in $\Delta^+(\overline{\gamma})$.

Now consider the case of a nonintegral simple root. According to Lemma 3.9, this case arises only when $W(\lambda_a)\subsetneq W_a$. In this case, let α be the simple long root in $\Delta^+(\overline{\gamma})$. By the Lemma 3.12, α satisfies the parity condition with respect to γ. Let $(H^\alpha)_\mathbb{R}$ be the Cartan subgroup defined by the map c_α of Definition 3.12, and let us denote by $\tilde{\alpha}$ the noncompact imaginary root ${}^{tr}(c_\alpha)^{-1}(\alpha)$ of $\mathfrak{h}_\alpha$ in $\mathfrak{g}$. Since α satisfies the parity condition, we can write

$$\Gamma(m_\alpha) = \epsilon\epsilon_\alpha \exp\left(i\pi 2\frac{\langle\alpha,\overline{\gamma}\rangle}{\langle\alpha,\alpha\rangle}\right).$$

Notice that m_α is well-defined only up to inverse, and so $\epsilon = \pm 1$ will be replaced by $-\epsilon$ for choices leading to m_α^{-1} instead of m_α. We now define $\gamma_\alpha = (\Gamma_\alpha, \overline{\gamma}_\alpha)$ by

$$\begin{gathered}\overline{\gamma}_{\alpha|\mathfrak{h}\cap\mathfrak{h}^\alpha} = \overline{\gamma}_{|\mathfrak{h}\cap\mathfrak{h}^\alpha},\\ \langle\tilde{\alpha}, \overline{\gamma}_\alpha\rangle = \epsilon\langle\alpha,\overline{\gamma}\rangle,\\ \Gamma_{\alpha|T_\mathbb{R}A^\alpha_\mathbb{R}\cap H^\alpha_\mathbb{R}} = \Gamma_{|T_\mathbb{R}A^\alpha_\mathbb{R}\cap H^\alpha_\mathbb{R}},\\ \Gamma_\alpha(\exp(irH_{\tilde{\alpha}})) = e^{(m+d)ir},\end{gathered}$$

where $m = 2\frac{\langle\alpha,\overline{\gamma}\rangle}{\langle\alpha,\alpha\rangle}$ and d is defined as in [Vgr, Definition 8.3.15]. We check as in [Vgr, Lemma 8.3.16] that γ_α is well-defined. Notice that replacement of m_α by m_α^{-1} would lead to the same pseudocharacter γ. (This is to be contrasted with the setting of a type I real root in the linear case.) Thus we put $c_\alpha(\gamma) = \{\gamma_\alpha\}$.

Let us write the Hecht-Schmid identities in this context.

Proposition 3.15 ([V1], Theorem 4.4). *Let $\widetilde{G_\mathbb{R}}$ be a group in $\mathcal{L}(Mp)$, and $\widetilde{H}_\mathbb{R}$ a θ-stable Cartan subgroup of $G_\mathbb{R}$. Let γ be a genuine λ_a-pseudocharacter for $\widetilde{H}_\mathbb{R}$. Suppose that there exist a simple long imaginary root, say α, in $\Delta^+(\overline{\gamma})$. By Lemma 3.9(b), α is necessarily unique; moreover, α is noncompact imaginary and type I. Define c^α, $\widetilde{H}^\alpha_\mathbb{R}$, γ^α as above. Write $n = 2\frac{\langle\alpha,\overline{\gamma}\rangle}{\langle\alpha,\alpha\rangle}$. Then*

$$[X(\gamma)] + s_\alpha \cdot [X(s_\alpha \times \gamma)] = X(\gamma^\alpha);$$

here $s_\alpha \times \gamma$ is, by definition, the λ_a-pseudocharacter $(\Gamma - (n+1)\alpha, \overline{\gamma} - n\alpha)$, and we are using the extended coherent continuation action of equation (3.1) *and Proposition 3.10.*

The pseudocharacter $s_\alpha \times \gamma$ in Proposition 3.15 above is an example of pseudocharacter obtained by cross action. In fact, the proposition defines the cross action for a reflection in the simple long root when such a root exists. We can appeal to [Vgr, Definition 8.3.1] for the cross action of reflections in the simple short (i.e. integral) roots. Hence given an element $w \in W(\mathfrak{g},\mathfrak{h})$ and a λ_a-pseudocharacter γ for $\widetilde{H}_\mathbb{R}$, we can consider $w \times \gamma$ which is a new λ_a-pseudocharacter. As usual, we transport the action of $W(\mathfrak{g},\mathfrak{h})$ to one of W_a.

Alternatively, to define the pseudocharacter $w \times \gamma$ (even for $w \in W_a$ but not $W(\lambda_a)$), we could simply have used [Vgr, Definition 8.3.1]. The final assertion of Lemma 3.9 implies that the definition makes sense in the extended setting. The reader may check that the definition in Proposition 3.15 agrees with that of [Vgr, Definition 8.3.1].

Here is a collection of easy facts about the cross action that we will need below.

Lemma 3.16. *Let $s \in W(\mathfrak{g},\mathfrak{h})$ be a reflection with respect to a simple root α in $\Delta^+(\overline{\gamma})$, and write $s \times \gamma = (s \times \Gamma, s \times \overline{\gamma})$. By definition we have $s \times \overline{\gamma} = s \cdot \overline{\gamma}$. Furthermore, we have:*

(a) *If α is long, then $s \times \Gamma(\mathbf{x}) = -\Gamma(\mathbf{x})$; in particular, $\overline{X}(\gamma)$ and $\overline{X}(s \times \gamma)$ have different central character.*
(b) *If α is short, then $s \times \Gamma(\mathbf{x}) = \Gamma(\mathbf{x})$. Moreover, $\overline{X}(\gamma)$ and $\overline{X}(s \times \gamma)$ have the same central character.*
(c) *Suppose α is not imaginary. Then $s \times \gamma = (\Gamma - n\alpha, \overline{\gamma} - n\alpha)$, where $n = 2\frac{\langle \alpha, \overline{\gamma}\rangle}{\langle \alpha,\alpha\rangle}$.*
(d) *If α is compact imaginary, then $s \times \gamma = (\Gamma - (n-1)\alpha, \overline{\gamma} - n\alpha)$.*
(e) *If α is noncompact imaginary, then $s \times \gamma = (\Gamma - (n+1)\alpha, \overline{\gamma} - n\alpha)$.*
(f) *Finally, suppose α is long imaginary, and denote by $\tilde{\alpha}$ its Cayley transform, and $\tilde{s}$ the corresponding reflection. Then γ^α, $s \times \gamma$, $(s \times \gamma)^\alpha$, $\tilde{s} \times \gamma^\alpha$ are defined, and $\tilde{s} \times \gamma^\alpha = (s \times \gamma)^\alpha$. (See* [Vgr, Lemma 8.3.7] *for the corresponding statement in case α is short.)*

3.4. **Nonintegral wall-crosses.** For the record, we state a result of Vogan's describing translation functors across a nonintegral wall.

Theorem 3.17 ([V1], Corollary 4.8 and Lemma 4.9)**.** *Let $\widetilde{G_{\mathbb{R}}}$ be a group in $\mathcal{L}(Mp)$ and let γ be a genuine λ_a-pseudocharacter of $\widetilde{G_{\mathbb{R}}}$. Suppose α is a long simple root in $\Delta^+(\overline{\gamma})$. Then, with the translation functor ψ_α chosen as in Convention 1.8 and $n = 2\frac{\langle \alpha, \overline{\gamma}\rangle}{\langle \alpha,\alpha\rangle}$,*

$$\psi_\alpha(\overline{X}(\gamma)) = \begin{cases} \overline{X}(s_\alpha \times \gamma) & \text{if } \alpha \text{ is complex,} \\ \overline{X}((s_\alpha \times \gamma)^\alpha) & \text{if } \alpha \text{ is imaginary,} \\ \overline{X}((s_\alpha \times \gamma)_\alpha) & \text{if } \alpha \text{ is real.} \end{cases}$$

3.5. **Reducibility of Standard Modules.** Using [Vgr, Lemmas 8.6.1–3], we deduce from Theorem 1.11 the following necessary condition for the reducibility of a standard module.

Proposition 3.18. *Fix half-integral infinitesimal character λ_a. Let $\widetilde{G_{\mathbb{R}}}$ be in $\mathcal{L}(Mp)$, $\widetilde{H}_{\mathbb{R}}$ a Cartan subgroup, and γ a genuine λ_a-pseudocharacter for $\widetilde{H}_{\mathbb{R}}$. The standard module $X(\gamma)$ is reducible only if there exists a simple root $\alpha \in \Delta^+(\overline{\gamma})$ such that*
(a) *$\theta(\alpha) \notin \Delta^+(\overline{\gamma})$;*
(b) *if α is real, then α satisfies the parity condition with respect to γ.*

The point of this result is that it is stated only in terms of simple roots, and this will be crucial in the proof of 1.14. We do not, however, give a necessary and sufficient condition purely in terms of simple roots.

4. $Mp(2,\mathbb{R})$

The inductive proofs of our main results eventually rely on detailed information about three dimensional groups. In the linear case, $SL(2,\mathbb{R})$ and $SL^{\pm}(2,\mathbb{R})$ suffice. In our setting we must also include $Mp(2,\mathbb{R})$. The point of this section is to recall the theory of $Mp(2,\mathbb{R})$ at regular half-integral infinitesimal character.

So let $\widetilde{G_{\mathbb{R}}} = Mp(2,\mathbb{R})$ and let $\mathfrak{g}$ denote its complexified Lie algebra $\mathfrak{sp}(2,\mathbb{C})$. Write $\widetilde{T}_{\mathbb{R}} = \widetilde{H}^{0,1,0}$ (Notation 2.5) for the compact Cartan in $\widetilde{G_{\mathbb{R}}}$ and $\widetilde{A}_{\mathbb{R}} = \widetilde{H}^{0,0,1}$ for the split one. They form a system of representatives of the two conjugacy classes of Cartan subgroups in $\widetilde{G_{\mathbb{R}}}$. The roots of $\mathfrak{t}$ in $\mathfrak{g}$ are $\{\pm\alpha\}$, where $\alpha = 2e_1$. Recall that we have fixed a Cayley transform $c^{0,0,1} : \mathfrak{t} \to \mathfrak{a}$. We will denote by $\hat{\alpha}$ the transport of α to $\mathfrak{a}$ by $c_{0,0,1}$. Thus $\Delta(\mathfrak{a},\mathfrak{g}) = \{\pm\hat{\alpha}\}$, and $\hat{\alpha}$ is real. We fix infinitesimal character $\lambda_a = a\alpha \in i\mathfrak{t}^*_{\mathbb{R}}$ with $a \in \mathbb{N} + \frac{1}{2}$. Note that here the integral Weyl group is trivial, while the complex Weyl group is S_2.

The next results explicitly describe the parameterization of irreducible Harish-Chandra modules with infinitesimal character λ_a.

Proposition 4.1. *There are four $\widetilde{K}_{\mathbb{R}}$-conjugacy classes of genuine λ_a-pseudocharacter in $Mp(2,\mathbb{R})$. The following pseudocharacters form a system of representatives of these classes.*

(i) $\gamma^+_{\mathrm{ds}} = (\Gamma^+_{\mathrm{ds}}, \overline{\gamma}^+_{\mathrm{ds}}) \in (\widetilde{T}_{\mathbb{R}})'_{\lambda_a}$*; here* $\overline{\gamma}^+_{\mathrm{ds}} = a\alpha$ *and* Γ^+_{ds} *is determined by the condition that* $d\Gamma^+_{\mathrm{ds}} = (a+1)e_1$.

(ii) $\gamma^-_{\mathrm{ds}} = (\Gamma^-_{\mathrm{ds}}, \overline{\gamma}^-_{\mathrm{ds}}) \in (\widetilde{T}_{\mathbb{R}})'_{\lambda_a}$*; here* $\overline{\gamma}^-_{\mathrm{ds}} = -a\alpha$ *and* Γ^-_{ds} *is determined by the condition* $d\Gamma^-_{\mathrm{ds}} = -(a+1)e_1$.

(iii) $\gamma^+_{\mathrm{ps}} = (\Gamma^+_{\mathrm{ps}}, \overline{\gamma}^+_{\mathrm{ps}}) \in (\widetilde{A}_{\mathbb{R}})'_{\lambda_a}$*; here* $\overline{\gamma}^+_{\mathrm{ps}} = a\hat{\alpha}$ *and* Γ^+_{ps} *is determined by requiring* $d\Gamma^+_{\mathrm{ps}} = \overline{\gamma}^+_{\mathrm{ps}}$ *and* $\Gamma^+_{\mathrm{ps}}(\mathbf{x}) = i^{2a}$.

(iv) $\gamma^-_{\mathrm{ps}} = (\Gamma^-_{\mathrm{ps}}, \overline{\gamma}^-_{\mathrm{ps}}) \in (\widetilde{A}_{\mathbb{R}})'_{\lambda_a}$*; here* $\overline{\gamma}^-_{\mathrm{ps}} = a\hat{\alpha}$ *and* Γ^-_{ps} *is determined by requiring* $d\Gamma^-_{\mathrm{ps}} = \overline{\gamma}^-_{\mathrm{ps}}$ *and* $\Gamma^-_{\mathrm{ps}}(\mathbf{x}) = i^{2a+2}$.

The first two pseudocharacters have length zero, while the latter two have length one. Moreover, we have the following character formulas:

$$[\overline{X}(\gamma^+_{\mathrm{ds}})] = [X(\gamma^+_{\mathrm{ds}})],$$
$$[\overline{X}(\gamma^+_{\mathrm{ps}})] = [X(\gamma^+_{\mathrm{ps}})] - [X(\gamma^+_{\mathrm{ds}})];$$

the analogous formulas hold for $X(\gamma^-_{\mathrm{ds}}), X(\gamma^-_{\mathrm{ps}})$*, etc.*

Note that there are two blocks of representations of $Mp(2,\mathbb{R})$ and that this partition is determined by the sign of the action of the element $\mathbf{x}$. The reader may now confirm Theorem 0.1 using the definitions

$$(\gamma^{\pm}_{\mathrm{ds}})^{\vee} = \gamma^{\mp}_{\mathrm{ps}};\ \ (\gamma^{\pm}_{\mathrm{ps}})^{\vee} = \gamma^{\mp}_{\mathrm{ds}};$$

$$\epsilon_{\gamma\gamma'} = \begin{cases} +1 & \text{if } \gamma = \gamma', \\ -1 & \text{if } \gamma \neq \gamma'. \end{cases}$$

One may note that, on the level of irreducible representations, the bijection described above coincides with the nonintegral wall cross ψ_α (Theorem 3.17), but this coincidence is a very special feature of the $Mp(2,\mathbb{R})$ setting. In fact, according to Lemma 6.23, we can compose the above bijection with $\widetilde{\iota}$ to obtain another choice of the duality; namely,

$$(\gamma^{\pm}_{\mathrm{ds}})^{\vee} = \gamma^{\pm}_{\mathrm{ps}};\ \ (\gamma^{\pm}_{\mathrm{ps}})^{\vee} = \gamma^{\pm}_{\mathrm{ds}}.$$

It is worth mentioning a few words about the geometric equivalent of Proposition 4.1. In this setting the complex flag variety is isomorphic to $\mathbb{P}^1$, $\widetilde{K}$ acts through K, and hence the description of $\widetilde{K}$ orbits on X is the familiar one: there is a unique open orbit, say Q_{ps}, which contains in its closure the two distinct points, say $Q^{\pm}_{\mathrm{ds}}$, which constitute the other two $\widetilde{K}$ orbits. Recall our fixed dominant half-integral infinitesimal character λ_a, and write $\mathcal{D}_{\lambda_a}$ for the sheaf of G homogeneous λ_a-twisted differential operators on X. (By abuse of notation, we also let $\mathcal{D}_{\lambda_a}$ denote the $\widetilde{K}$-homogeneous sheaf of operators on a $\widetilde{K}$ orbit Q induced by the inclusion of the orbit into X.) There is a unique irreducible $\widetilde{K}$ homogeneous $\mathcal{D}_{\lambda_a}$-connection on each orbit $Q^{\pm}_{\mathrm{ds}}$, and these correspond to the two discrete series representations in

Proposition 4.1(i–ii). On the other hand, there are four irreducible $\widetilde{K}$ homogeneous $\mathcal{D}_{\lambda_a}$-connections on Q_{ps}, but only two of these, say$\mathcal{L}^{\pm}_{\mathrm{ps}}$, give rise to genuine representations; these correspond to parts (iii) and (iv) of Proposition 4.1. The key geometric feature here is that the connection $\mathcal{L}^{+}_{\mathrm{ps}}$ extends to Q^{+}_{ds} but not to Q^{-}_{ds} (and similarly for $\mathcal{L}^{-}_{\mathrm{ps}}$). This is to be contrasted with the case of $SL(2,\mathbb{R})$ where an irreducible K-homogeneous connection on the open orbit either extends to both closed orbits, or to neither (but never to just one); see [LV, Lemma 3.5], for instance. It is this difference which accounts for the variations on the linear theory which we introduced in Section 3.3.

Let s be the reflection with respect to α. As examples of some of the general definitions of Section 3.3, we have the following relations between the four (equivalence classes of) pseudocharacters.

$$\gamma^{+}_{\mathrm{ds}} = (\gamma^{+}_{\mathrm{ps}})_{\alpha};\;\; \gamma^{+}_{\mathrm{ds}} = s \times \gamma^{-}_{\mathrm{ds}};$$

$$\gamma^{+}_{\mathrm{ps}} = (\gamma^{+}_{\mathrm{ds}})^{\alpha};\;\; \gamma^{+}_{\mathrm{ps}} = s \times \gamma^{-}_{\mathrm{ps}}.$$

Similar equations are obtained by reversing the signs $+$ and $-$.

Next we include some cohomology computations. There are two $\widetilde{K}$ conjugacy classes of proper θ-stable parabolics in $\mathfrak{g}$. As representatives, we can take $\mathfrak{q} = \mathfrak{t} \oplus \mathfrak{u}$ and $\bar{\mathfrak{q}} = \mathfrak{t} \oplus \bar{\mathfrak{u}}$, with $\mathfrak{u} = \mathfrak{g}^{\alpha}$. Since all of the modules $\overline{X}(\gamma)$ in this section are all essentially irreducible Verma modules for $\mathfrak{sl}(2,\mathbb{C})$, the next result is very easy.

Proposition 4.2. *Retain the notations outlined above. We have*

$$H^i(\mathfrak{u}, \overline{X}(\gamma^{+}_{\mathrm{ps}})) = \begin{cases} \Gamma^{+}_{\mathrm{ds}} \otimes e^{-\alpha} & i = 0, \\ 0 & i \neq 0; \end{cases} \qquad H^i(\bar{\mathfrak{u}}, \overline{X}(\gamma^{+}_{\mathrm{ps}})) = \begin{cases} 0 & i \neq 1, \\ \Gamma^{+}_{\mathrm{ds}} & i = 1. \end{cases}$$

We leave it to the reader to formulate the analogous statements for $\overline{X}(\gamma^{-}_{\mathrm{ps}})$. At any rate, we have now given enough details so that the reader can easily check the validity of Theorem 1.14 for $Mp(2,\mathbb{R})$.

5. Proof of theorem 1.14

Recall that $\widetilde{G_{\mathbb{R}}}$ is a group in $\mathcal{L}(Mp)$, and that we have fixed a half-integral infinitesimal character λ_a. The proof is similar to the one of Proposition 7.2 in [V2]. It proceeds by induction on the dimension of $\mathfrak{g}$, for $\widetilde{G_{\mathbb{R}}}$ in $\mathcal{L}(Mp)$, and then by induction on pseudocharacter length. The induction step in the proof differs from the one in [V2] because the root α that reduces the length of a nonminimal pseudocharacter need not be integral. In that case, the induction makes use of nonintegral wall-crossing translation functors. The key point that we have to check is these functors preserve (in a suitable sense) the parity conditions in the statement of the theorem. In contrast, when α is in fact integral, replacing l^I by l doesn't affect the arguments in [V2] for purely formal reasons (which we sketch below).

We begin by establishing the theorem in the case that $\dim \mathfrak{g}$ is minimal. This is straightforward.

Lemma 5.1. *Suppose $\mathfrak{g}$ is of the form*

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha},$$

where α is a root in $\Delta(\mathfrak{sp}(2n,\mathbb{C}),\mathfrak{t})$. If $\widetilde{G_{\mathbb{R}}}$ is linear, then the length function l coincides with the integral length l^I, and Theorem 1.14 is a special case of [V2, Proposition 7.2]. *If $\widetilde{G_{\mathbb{R}}}$ is not linear, then α is necessarily long, $\widetilde{G_{\mathbb{R}}} \simeq Mp(2,\mathbb{R})$, and Theorem 1.14 follows from the considerations of Section 4.*

Let us now go back to the induction steps of the proof. The notations are as in the statement of the theorem. Consider first (a). If $X(\gamma^1)$ is a standard irreducible, (a) is a consequence of Theorem 6.13 in [V2]. So suppose $X(\gamma^1)$ is not a standard irreducible. Then by Proposition 3.18, we can find a simple root $\alpha \in \Delta^+(\overline{\gamma}^1)$ such that $\theta(\alpha) \notin \Delta^+(\overline{\gamma}^1)$ and if α is real, then α satisfies the parity condition. Let α^2 be the root in $\Delta^+(\overline{\gamma}^2)$ corresponding to α under the isomorphism $i_{\overline{\gamma}^1,\overline{\gamma}^2}$.

If α is not integral, by [V2, Proposition 4.4], $\overline{X}^L(\gamma^2_{\mathfrak{q}})$ is a composition factor of $H^i(\mathfrak{u}, \overline{X}(\gamma^1))$ if and only if $\psi^{\mathfrak{l}}_\alpha(\overline{X}^L(\gamma^2_{\mathfrak{q}}))$ is a composition factor of $H^i(\mathfrak{u}, \psi_\alpha(\overline{X}(\gamma^1)))$. Let

$$n = 2\frac{\langle \alpha, \overline{\gamma}^1\rangle}{\langle \alpha,\alpha\rangle}.$$

Then $n\alpha$ is an extremal weight for some finite dimensional representation of $\mathfrak{g}$. By Theorem 3.17, we have

$$\psi_\alpha(\overline{X}(\gamma^1)) = \begin{cases} \overline{X}((s_\alpha \times \gamma)_\alpha) & \text{if } \alpha \text{ is real satisfying the parity condition,} \\ \overline{X}(s_\alpha \times \gamma) & \text{if } \alpha \text{ is complex with } \theta(\alpha) \notin \Delta^+(\overline{\gamma}). \end{cases} \tag{5.1}$$

Let us denote by γ' the pseudocharacter appearing in the right-hand side of these equalities, so that $l(\gamma') = l(\gamma^1) - 1$.

Case 1: α^2 is complex. Then, by (5.1),

$$\psi^{\mathfrak{l}}_\alpha(\overline{X}^L(\gamma^2_{\mathfrak{q}})) = \overline{X}^L(\gamma^2_{\mathfrak{q}} - n\alpha^2).$$

Furthermore, $\gamma^2 - n\alpha^2$ is a pseudocharacter in $(\widehat{H}^2)'$ and $(\gamma^2 - n\alpha^2)_{\mathfrak{q}} = \gamma^2_{\mathfrak{q}} - n\alpha^2$. The inductive hypothesis applied to $\overline{X}(\gamma')$ gives

$$(l(\gamma') - l(\gamma^2 - n\alpha^2)) - (l_{\mathfrak{q}}(\gamma^2 - n\alpha^2) - i) = 0 \mod 2.$$

Since $l(\gamma^2 - n\alpha^2) = l(\gamma^2) \pm 1$, and $l_{\mathfrak{q}}(\gamma^2 - n\alpha^2) = l_{\mathfrak{q}}(\gamma^2)$, we obtain (a).

Case 2: α^2 is real, satisfying the parity condition in $\Delta(\mathfrak{l}, \mathfrak{h}^2)$ or non-compact imaginary in $\Delta(\mathfrak{l}, \mathfrak{h}^2)$. To fix the notations, assume, that α^2 is noncompact imaginary, the other case being entirely similar. Then

$$\psi^{\mathfrak{l}}_\alpha(\overline{X}^L(\gamma^2_{\mathfrak{q}})) = \overline{X}^L((\gamma^2_{\mathfrak{q}} - n\alpha^2)^{\alpha^2}).$$

Furthermore, $(\gamma^2 - n\alpha^2)^{\alpha^2}$ is a pseudocharacter in $((\widehat{H}^2)^{\alpha^2})'$ and $((\gamma^2 - n\alpha^2)^{\alpha^2})_{\mathfrak{q}} = (\gamma^2_{\mathfrak{q}} - n\alpha^2)^{\alpha^2}$. Since $l((\gamma^2 - n\alpha^2)^{\alpha^2}) = l(\gamma^2) + 1$, and $l_{\mathfrak{q}}(\gamma^2 - n\alpha^2) = l_{\mathfrak{q}}(\gamma^2)$, we get (a) from the inductive hypothesis. To prove the last equality, we use Lemma 9.5.10 of [Vgr], with l^I replaced by l (the proof given there applies).

Case 3: α^2 is non-compact imaginary not in $\Delta(\mathfrak{l}, \mathfrak{h}^2)$. Then

$$\psi^{\mathfrak{l}}_\alpha(\overline{X}^L(\gamma^2_{\mathfrak{q}})) = \overline{X}^L(\gamma^2_{\mathfrak{q}} - n\alpha^2).$$

In that case $\gamma^2 - n\alpha^2 = (\Gamma^2 - n\alpha^2, \overline{\gamma}^2 - n\alpha)$ is not a pseudocharacter of $(\widehat{H}^2)'$, but $\gamma'' = s \times \gamma = (\Gamma^2 - (n+1)\alpha^2, \overline{\gamma}^2 - n\alpha)$ is, and $\gamma''_{\mathfrak{q}} = \gamma^2_{\mathfrak{q}} - n\alpha^2$. Since $l(\gamma'') = l(\gamma^2)$ and $l_{\mathfrak{q}}(\gamma'') = l_{\mathfrak{q}}(\gamma^2) \pm 1$, we get (a) from the inductive hypothesis.

Notice α^2 cannot be real not satisfying the parity condition because of Lemma 3.13. Similarly, α^2 cannot be real not in $\Delta(\mathfrak{l}, \mathfrak{h}^2)$ because $\mathfrak{q}$ is θ-stable.

Suppose now that α is integral. As mentioned above, the proof in [V2] implies that (a) holds for γ^1 and γ^2. Similarly, since parts (b)–(e) of Theorem 1.14 are statements that involve only integral roots, their proofs also follow from arguments of [V2]. $\square$

Let us say a few words about the algorithm computing composition series. It is essentially the same as in [Vgr, Chapter 9] with the following small adjustments. First of all, in step (c), Theorem 3.18 implies that we we may assume that α is actually simple in $\Delta^+(\overline{\gamma})$. The proof of (c) given in [V2, Theorem 7.3] is then still valid in our context.

Now consider step (a); i.e. suppose we want to compute composition series of $X(\gamma)$, γ not minimal. Then there exists a simple root $\alpha \in \Delta^+(\overline{\gamma})$ with α either real satisfying the parity condition, or complex such that $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$. If α is integral, we follow [Vgr], except that we replace the integral length l^I by l everywhere. Now, if α is nonintegral and complex, there are no significant changes in the algorithm. So let us consider the case where α is real and nonintegral. We write the Hecht-Schmid character identities of Proposition 3.15 as

$$[X(\gamma_\alpha)] + s_\alpha \cdot [X(s_{\tilde{\alpha}} \times \gamma_\alpha)] = [X(\gamma)], \tag{5.2}$$

with $\tilde{\alpha} = c_\alpha \cdot \alpha$. Since $l(\gamma_\alpha) = l(s_{\tilde{\alpha}} \times \gamma_\alpha) = l(\gamma) - 1$, we know the composition series of these two standard modules. We deduce from (5.2) the composition series of $X(\gamma)$, using [Vgr, Lemma 7.3.20].

Finally, consider step (b). When α is integral, the arguments in [Vgr] remain unchanged. When α is nonintegral, we use Theorem 1.7, and [V2, Proposition 4.4] to compute the cohomology of $\overline{X}(\gamma)$.

6. Representation theory of $Mp(2n, \mathbb{R})$

In this section we explicitly parameterize the irreducible admissible genuine representations of the metaplectic group whose infinitesimal character is regular and half-integral. The parameterization is in terms of certain combinatorially defined objects called diagrams, which are simply a convenient way to organize the data of a pseudocharacter. In terms of this parameterization, we work out the general theory of Section 3 explicitly (see Lemmas 6.17 and 6.29, for example) and state a sharper version of Theorem 0.1 in Theorem 6.34.

To get started, we need to define the combinatorial sets that will parameterize pseudocharacters for $Mp(2n, \mathbb{R})$.

6.1. Metaplectic diagrams: definitions and examples.

Let

$$\Sigma(n) = \{\sigma \in S_n \mid \sigma^2 = 1\}$$

denote the set of involutions in the symmetric group S_n. Let $\Sigma_\pm(n) = \Sigma(n) \times (\mathbb{Z}/2\mathbb{Z})^n$, which we will view primarily as a set and not a group. Define $D_0(2n)$ to be the subset consisting of those pairs $(\sigma, (\epsilon_i)) \in \Sigma_\pm(2n)$ subject to the restrictions

$$\begin{cases} \text{1. (Symmetry of signs)} & \epsilon_{2n+1-j} = -\epsilon_j; \\ \text{2. (Symmetry of involution)} & \sigma(2n+1-j) = 2n+1-\sigma(j); \text{ and} \\ \text{3. (Orientations of non-fixed points)} & \text{If } \sigma(j) \neq j, \text{ then } \epsilon_j \epsilon_{\sigma(j)} = -1. \end{cases}$$

A coordinate-free version of the first two conditions reads: (1′) $\epsilon_{w_o j} = -\epsilon_j$; and (2′) $w_o \sigma w_o = \sigma$, where w_o denotes the long element in S_{2n}.

We find it convenient to think of the elements in $D_0(2n)$ pictorially, i.e. as some kind of diagrams. For instance, if $\sigma = (1\ 11)(2\ 12)(34)(58)(9\ 10)$ and $\epsilon = (-,-,+,-,-,+,-,+,+,-,+,+)$. we think of (σ, ϵ) as

Here an arrow (or orientation) between j and $\sigma(j)$ points to the index k such that $\epsilon_k = +$; i.e. our arrows start at a $-$ and end at a $+$.

For reasons that will be clear in Section 6.2 below, we want to forget about certain orientations of elements in $D_0(2n)$. This amounts to making some identifications among the elements in $D_0(2n)$. At first we want to forget only about the arrows between pairs j and $\sigma(j)$ where $\sigma(j) \neq 2n+1-j$; for instance, we want to identify the following two elements in $D_0(4)$:

So we introduce an equivalence relation on $D_0(2n)$ generated by $(\sigma, \epsilon) \sim (\sigma, \epsilon')$ if

$$\epsilon_j = \epsilon'_j, \quad \text{for all } j \text{ such that } \sigma(j) \in \{j, 2n+1-j\}.$$

We define the set of *metaplectic diagrams* (or just *diagrams*) to be the equivalence classes of this equivalence relation, and denote this set by $D(2n)$. Pictorially, for instance, we will denote the equivalence class consisting of the above two elements of $D_0(4)$ as

It will be convenient to forget about all the orientations of an element in $D(2n)$, so we define a further equivalence on representatives $(\sigma, \epsilon) \in D(2n)$ generated by $(\sigma, \epsilon) \sim' (\sigma, \epsilon')$ if

$$\epsilon_j = \epsilon'_j, \quad \text{for all } j \text{ such that } j = \sigma(j).$$

Because of Lemma 6.5 below, we call the equivalence classes of this relation the *orbit diagrams* for $Mp(2n, \mathbb{R})$ (or $Sp(2n, \mathbb{R})$). We denote this set by $OD(2n)$, and write *orb* for the projection $D(2n) \longrightarrow OD(2n)$, and call $orb(\gamma)$ the underlying orbit of γ. Note that the data of an element is $OD(2n)$ is simply an involution (subject to the symmetry condition (2) above) together with a choice of signs for its fixed points (subject to the symmetry condition (1) above). This is consistent with the canonical orbit diagram introduced in Section 2.6. Pictorially, for instance, the image of the above element of $D(12)$ under *orb* will be drawn as

Associated to each $\gamma \in D(2n)$, represented by a pair (σ, ϵ) say, we can associate three numbers $m(\gamma), r(\gamma)$, and $s(\gamma)$ which depend only on σ (and hence only on

$orb(\gamma) \in OD(2n)$). We define

$$\begin{aligned} m(\gamma) &= \frac{1}{4}\#\{j \mid \sigma(j) \notin \{j, 2n+1-j\}\ \}, \\ r(\gamma) &= \frac{1}{2}\#\{j \mid \sigma(j) = 2n+1-j\}, \\ s(\gamma) &= \frac{1}{2}\#\{j \mid \sigma(j) = j\}. \end{aligned}$$

Because of conditions (1) and (2) in the definition of $D_0(2n)$, we have that $2m(\gamma) + r(\gamma) + s(\gamma) = n$. We say that $\gamma \in D(2n)$ is *of type* (m, r, s) if $m = m(\gamma), r = r(\gamma)$, and $s = s(\gamma)$, and denote the set of all type (m, r, s) diagrams (or orbit diagrams) as $D(2n)^{m,r,s}$ (or $OD(2n)^{m,r,s}$). For example, the diagrams listed above are in $D(4)^{1,0,0}$ and $D(12)^{2,1,1}$.

In the next section, we will think of the elements of $D(2n)^{m,r,s}$ parameterizing pseudocharacters for $\widetilde{H}_{\mathbb{R}}^{m,r,s}$, and we will need to define an action of $W_{\mathbb{C}} \subset S_{2n}$ on them. This is the reason for the next definition.

Definition 6.1. Recall the identification of $W_{\mathbb{C}}$ as a subgroup of S_{2n} (Lemma 2.3). We define an action (denoted by $\times$) of $W_{\mathbb{C}}$ on $S_{2n} \times (\mathbb{Z}/2)^n$ via

$$w \times (\sigma, \epsilon) = (w\sigma, \epsilon'), \qquad \text{with } \epsilon'_{w(j)} = \epsilon_j.$$

Clearly this descends to $D(2n)$ and $OD(2n)$.

This will turn out to be the cross action on pseudocharacters; see Section 6.6 below.

6.2. **Diagrams and pseudocharacters.** Fix $\lambda_a \in (\mathfrak{h}^a)^*$ and suppose λ_a is half-integral, dominant, and regular (see Remark 3.8). We are in a position to describe a bijection between $D(2n)$ and conjugacy classes of λ_a-pseudocharacters for $Mp(2n, \mathbb{R})$. In particular, we will associate to each $\delta \in D(2n)^{m,r,s}$ a $\widetilde{K}_{\mathbb{R}}$ conjugacy class of pseudocharacters $\gamma = (\Gamma, \bar{\gamma})$ (with notation as in Section 1.2). (In fact, we will think of the Cartan subgroup for γ as being fixed, so that $\widetilde{K}_{\mathbb{R}}$ conjugacy reduces to $W^{m,r,s}$ conjugacy.) The basic idea is that the data of $orb(\delta)$ determines $\bar{\gamma}$ and $d\Gamma$, while the extra data of the orientations of the arrows of δ nails down Γ globally. (As is explained in Section 6.3, this is equivalent to the fact that $orb(\delta)$ specifies a $\widetilde{K}$ orbit on the flag variety while the data of the orientations of δ specify a local system.)

Recall our realization of the complex Weyl group $W_{\mathbb{C}}$ and the real Weyl groups $W_{\mathbb{R}}^{m,r,s}$ inside S_{2n} (Section 2.6), and the the canonical orbit diagram $o^{m,r,s}$ introduced in Section 2.6.

Lemma 6.2. *The action of* $W_{\mathbb{C}}$ *on* $OD(2n)^{m,r,s}$ *(Definition 6.1) is transitive. Hence we obtain a bijection*

$$\begin{aligned} W_{\mathbb{C}}/W_{\mathbb{R}}^{m,r,s} &\longrightarrow OD(2n)^{m,r,s} \\ wW_{\mathbb{R}}^{m,r,s} &\longrightarrow w \times o^{m,r,s}. \end{aligned}$$

Proof. The transitivity is clear from the definitions. The final assertion follows from Lemma 2.3. $\square$

Recall that we are trying to build a $W^{m,r,s}$ conjugacy class of λ_a-pseudocharacters, say $W^{m,r,s} \cdot \gamma$, from the data of $\delta \in D(2n)$. Given $\delta \in D(2n)^{m,r,s}$, the lemma says that we can find a $w \in W_{\mathbb{C}}$ (defined only up to $W_{\mathbb{R}}^{m,r,s}$) so that

$$orb(\delta) = w \times o^{m,r,s}.$$

The pair $(\widetilde{H}_{\mathbb{R}}^{m,r,s}, w\lambda)$ is thus well-defined up to $W^{m,r,s}$ conjugacy. We define the second component of the pseudocharacter to be $\bar{\gamma} = w\lambda$, and it remains only to define Γ on $\widetilde{H}_{\mathbb{R}}^{m,r,s} = F^{m,r,s} \times \exp_{Mp} \mathfrak{h}_{\mathbb{R}}^{m,r,s}$. The compatibility condition (1.1) specifies the differential $d\Gamma$ and hence we need only specify the value of Γ on $F^{m,r,s}$. Thus far we have only utilized the data of $orb(\delta)$, but now the extra data of the orientations of δ must come into play.

First consider the case where $m \neq 0$ but $s = 0$. (Actually there are no orientations involved in this case.) Here $F^{m,r,s}$ is generated by $\mathbf{z}$, and we define $\Gamma(\mathbf{z}) = -1$, as we must in order to obtain a genuine pseudocharacter.

Next assume either $(m, s) \neq (0, 0)$ or $m = 0$. In the latter case, if $s = 0$, then $\widetilde{H}_{\mathbb{R}}^{m,r,s} = \widetilde{T}_{\mathbb{R}}$ is connected and Γ is specified already by its differential; hence, in either case, we may assume s is nonzero. Recall the elements $m_{n+1-j} = \exp_{Mp}(i\pi H_{n+1-j})$, $1 \leq j \leq s$, which generate $F^{m,r,s}$ (see Section 2.5). Write (σ, ϵ) for a representative of $\delta \in D(2n)^{m,r,s}$, and enumerate the s element set

$$\{j \mid 1 \leq j \leq n \text{ such that } \sigma(j) = 2n+1-j\}$$

as $\{k_s < k_{s-1} < \cdots < k_1\}$. We define

$$\Gamma(m_{n-j+1}) := \epsilon_{k_j} \exp(d\Gamma(i\pi H_{n-j+1})); \tag{6.1}$$

here we use our Cayley transforms $c^{m,r,s}$ to make sense of evaluating $d\Gamma \in (\mathfrak{h}^{m,r,s})^*$ on $H_j \in i\mathfrak{t}_{\mathbb{R}}$. The above equation defines Γ. Observe that for λ_a regular and half-integral $\Gamma(\mathbf{z}) = -1$, so the pseudocharacter constructed here is genuine.

At last we can state the result for which we have been aiming. Once we unwind the definitions, its proof is essentially obvious.

Proposition 6.3. *Fix a regular half-integral infinitesimal character λ_a. The above description implements a bijection between $D(2n)^{m,r,s}$ and $W^{m,r,s}$ conjugacy classes of genuine λ_a-pseudocharacters for $\widetilde{H}_{\mathbb{R}}^{m,r,s}$.*

Notation 6.4. Given $\delta \in D(2n)$ and half-integral infinitesimal character λ_a, we write $X_{\lambda_a}(\delta)$ and $\overline{X}_{\lambda_a}(\delta)$ for the standard and irreducible genuine Harish-Chandra modules of infinitesimal character λ_a attached to the λ_a-pseudocharacter corresponding (via Proposition 6.3) to δ. If it is clear from the context, we may omit the subscript λ_a from the notation.

6.3. **Diagrams and Beilinson-Bernstein parameters.** In this section, we give an equivalent interpretation of the parameter set $D(2n)$. In a little more detail, let $\mathcal{D}_{\lambda_a}$ denote the sheaf of G homogeneous λ_a twisted differential operators on the complex flag variety X. (We give X the complex structure specified by $G/\overline{B}$, where $\overline{B}$ is constructed from the *negative* roots.) As in Section 4, we also let $\mathcal{D}_{\lambda_a}$ denote the corresponding sheaf on any $\widetilde{K}$ orbit on X. Given $\delta \in D(2n)$, we first describe a $\widetilde{K}$ orbit Q_δ on X (Lemma 6.5), and then an irreducible $\widetilde{K}$ homogeneous $\mathcal{D}_{\lambda_a}$-connection on Q_δ denoted $\mathcal{L}_\delta$. It turns out (Proposition 6.8) that the correspondence $\delta \mapsto \mathcal{L}_\delta$ is a bijection and that the Harish-Chandra module $\overline{X}(\delta)$ (Notation 6.4) is isomorphic to the one attached to $\mathcal{L}_\delta$ by the Beilinson-Bernstein theory.

The next lemma explains the terminology 'orbit diagrams' that we introduced in the preceding section.

Lemma 6.5. *The set of $\widetilde{K}$ orbits on the complex flag variety X is parameterized by the set $OD(2n)$.*

Proof. The $\widetilde{K}$ action on X factors though K, so the K orbits on X coincide with the $\widetilde{K}$ orbits. The result for K orbits can be found in [MaO]; see also [Ya, Corollary 3.2.6]. (In the latter paper, the set $OD(2n)$ is denoted $\mathcal{C}(Sp(2n,\mathbb{R}))$.) For completeness, we describe the orbit $Q = \widetilde{K} \cdot \mathfrak{b}$ parameterized by $\delta' \in OD(2n)^{m,r,s}$. Lemma 6.2 attaches a coset $wW^{m,r,s}$ to δ', and then we define

$$\underline{=}\mathfrak{h}^{m,r,s} \oplus \sum_{\alpha \in w\Delta_+^{m,r,s}} \mathfrak{g}^{-\alpha}.$$

□

Notation 6.6. Given $\delta \in D(2n)$, let Q_δ denote the orbit parameterized by $orb(\delta)$ in Lemma 6.5.

For future reference, we include a dimension computation. (For explicit examples, see Proposition 4.1 or Example 6.22 below.)

Lemma 6.7. *Fix half-integral infinitesimal character λ_a. Consider $\delta \in D(2n)$ be represented by (σ, ϵ), and let γ denote the corresponding λ_a-pseudocharacter (Proposition 6.3). Recall the pseudocharacter length $l(\gamma)$ of Definition 1.1. Then (in Notation 6.6)*

$$\dim(Q_\delta) = l(\gamma) + \frac{1}{2}n(n-1),$$

and

$$\begin{aligned} 2l(\gamma) = &\sum_{j\,:\,j<\sigma(j)} (\sigma(j) - j - \#\{k \mid k < j < \sigma(k) < \sigma(j)\}) \\ &+ \#\{j \mid j \leq n < \sigma(j) \leq 2n{+}1{-}j\}. \end{aligned}$$

Proof. The first assertion can be found in the proof of Lemma 4.5 in [V3]. The second follows from the definition and Lemma 6.14 below; see also [Ya, Proposition 3.4.2]. □

For a fixed $\delta \in D(2n)^{m,r,s}$, we now describe an irreducible $\widetilde{K}$ homogeneous $\mathcal{D}_{\lambda_a}$-connection on Q_δ. Let $\mathfrak{b}_\delta \in Q_\delta$ denote the Borel subalgebra defined in the proof of Lemma 6.5, and write B_δ for the corresponding subgroup of G. Recall that $H^{m,r,s} \subset B_\delta$. For simplicity of notation, temporarily write $H_\mathbb{R}$ for $H_\mathbb{R}^{m,r,s}$, and $\widetilde{H}_\mathbb{R}$ for its preimage under $\mathbf{pr}$. Next recall that we can build an irreducible $\widetilde{K}$ homogeneous connection on Q_δ from a character, say Λ, of $\widetilde{T}_\mathbb{R} = \widetilde{H}_\mathbb{R} \cap \widetilde{K}$. Explicitly, the complexification and subsequent restriction of Λ to $Z_{\widetilde{K}}(\mathfrak{b}_\delta) = \mathbf{pr}^{-1}(K \cap B_\delta)$ define a $\widetilde{K}$-equivariant line bundle $\widetilde{K} \times_{Z_{\widetilde{K}}(\mathfrak{b}_\delta)} \mathbb{C}_\Lambda$, and the sheaf of differential operators of this bundle is a connection supported on $Q_\delta = \widetilde{K}/Z_{\widetilde{K}}(\mathfrak{b}_\delta)$. Finally, recall the the character Γ_δ of $\widetilde{H}_\mathbb{R}$ constructed just before Proposition 6.3. The basic idea is to build the connection we are seeking from the restriction of Γ to $\widetilde{T}_\mathbb{R}$, but there are some delicate ρ-shifts involved which we now recall.

Set $\Delta_\delta^+ = w\Delta^{m,r,s}$ as in the proof of Lemma 6.5, and recall from [V3, Lemma 2.6] that there exists a character Φ_δ of $\widetilde{T}_{\mathbb{R}}$ (which is trivial on $F^{m,r,s}$) with differential

$$d\Phi_\delta = [\rho(\Delta_\delta^+) + \rho_{\mathrm{imag}}(\Delta_\delta^+) - 2\rho_{\mathrm{cpt,\,imag}}(\Delta_\delta^+)]|_{\mathfrak{t}_{\mathbb{R}}}.$$

Then we define a character $\Lambda_\delta = \Gamma_\delta \otimes \Phi_\delta^{-1}$ on $\widetilde{T}_{\mathbb{R}}$; note that by construction, $d\Lambda_\delta + \rho(\Delta_\delta^+) = w\lambda_a$. Write $\mathcal{L}_\delta$ for the corresponding irreducible $\widetilde{K}$ homogeneous $\mathcal{D}_{\lambda_a}$-connection on Q_δ. (Note that in the definition of $\mathcal{L}_\delta$, we made a choice of the representative of the coset $wW^{m,r,s}$; in fact, $\mathcal{L}_\delta$ is independent of this choice.)

Once we unwind the definitions, it is fairly easy to see that the map $\delta \mapsto \mathcal{L}_\delta$ is bijective. The Beilinson-Bernstein theory describes how to pass from $\mathcal{L}_\delta$ to an irreducible Harish-Chandra module for $Mp(2n,\mathbb{R})$ with infinitesimal character λ_a. We write $L_{\lambda_a}(\delta)$ for this representation. (Keep in mind that we choose to localize at the dominant weight λ_a, but with respect to the opposite complex structure on X, so $L_{\lambda_a}(\delta)$ appears as sections of the appropriate $\mathcal{D}_{\lambda_a}$ module.) The main point is that we have arranged the bijection $\delta \mapsto \mathcal{L}_\delta$ to be compatible with Proposition 6.3, at least for a fixed infinitesimal character.

Proposition 6.8. *Fix half-integral infinitesimal character*

$$\lambda_a = (\frac{2n-1}{2}, \frac{2n-3}{2}, \dots, \frac{1}{2}) \in (\mathfrak{h}^a)^*.$$

Then (in the notation described just before the proposition and Notation 6.4),

$$\overline{X}_{\lambda_a}(\delta) = L_{\lambda_a}(\delta).$$

To get a statement at arbitrary half-integral infinitesimal character, one must make a further twist of Φ_δ along the lines of [V3, Proposition 2.7a]. Since we do not need it here, we leave such a formulation to the reader.

6.4. **Central character.** Using the parameterization of Proposition 6.3, we can now explicitly compute representation theoretic information on the level of diagrams. We will need this information in various guises; for orientation, here is one of the results for which we are aiming.

Proposition 6.9. *Fix regular half-integral infinitesimal character λ_a. There are two blocks of genuine representations of $\widetilde{G_{\mathbb{R}}}$ of infinitesimal character λ_a and they are characterized by the central action of $\mathbf{x}$ (defined in Section 2.4): on one block $\mathbf{x}$ acts by a nonzero scalar C, and on the other block it acts by $-C$. Moreover, each block is an equivalence class of the equivalence relation generated by the cross action (in the integral Weyl group) and Cayley transforms.*

The proposition is proved in Section 6.6 below.

It is clearly desirable to read off the action of $\mathbf{x}$ on $X_{\lambda_a}(\delta)$ (Notation 6.4) directly from the diagram δ. This is the content of the next proposition in certain special cases. First we isolate a trivial lemma.

Lemma 6.10. *Recall the elements H_j of Section 2.3. As usual let ρ denote the half-sum of the positive roots Δ_a^+ and view ρ as an element of $(i\mathfrak{t}_{\mathbb{R}})^*$. Then*

$$\exp(i\pi \sum_j \rho(H_j)) = (-1)^N,$$

where N denotes the smallest integer greater than or equal to $n/2$.

Proposition 6.11. *Fix regular dominant half-integral infinitesimal character* $\lambda_a = \sum_{j=1}^{n} \lambda_j e_j$ *with* $\lambda_i > \cdots > \lambda_n$, *let* c *denote the constant* $\exp(i\pi \sum_j \lambda_j)$, *and let* N *denote the smallest integer greater than or equal to* $n/2$.

(a) *Let* $\delta \in D(2n)^{0,n,0}$ *be represented by the pair* (e, ϵ), *and let*

$$m = \#\{j \mid 1 \le j \le n \text{ and } \epsilon_j = -\}.$$

Then the central element $\mathbf{x}$ *(Section 2.4) acts on the discrete series representation* $X_{\lambda_a}(\delta)$ *(Notation 6.4) by the scalar* $(-1)^{m+N}c$.

(b) *Let* $\delta \in D(2n)^{0,0,n}$ *be represented by the pair* (w_o, ϵ) *(where* w_o *is the long element in* S_{2n}*), and define* p *as in (a). Then* $\mathbf{x}$ *acts on the principal series representation* $X_{\lambda_a}(\delta)$ *(Notation 6.4) by* $(-1)^m c$.

(c) *Suppose* $n = 2m$ *and let* $\delta \in D(2n)^{m,0,0}$ *be represented by* (σ, δ). *(This* m *has nothing to do with the one defined in part (a).) Write*

$$M = \#\{j \mid 1 \le j \le n, \text{ and } \sigma(j) > 2n+1-j\}.$$

Then $\mathbf{x}$ *acts on* $X_{\lambda_a}(\delta)$ *by the scalar* $(-1)^M c$.

Proof. In each case, write $\gamma = (\Gamma, \overline{\gamma})$ for the pseudocharacter corresponding to δ by Proposition 6.3. Consider first (a). Recall that the character Γ of $\widetilde{T}_{\mathbb{R}}$ is the highest weight of a lowest $\widetilde{K}$ type of $X(\gamma)$. Hence we need only compute $\Gamma(\mathbf{x})$. By definition, $\mathbf{x} = \exp_{Mp}(i\pi \sum_j H_j)$, and since $\widetilde{T}_{\mathbb{R}}$ is connected, we conclude

$$\begin{aligned}\Gamma(\mathbf{x}) &= \exp\left(i\pi \sum_j d\Gamma(H_j)\right)\\ &= \exp\left(i\pi \sum_j (\overline{\gamma} + \rho - 2\rho_c)(H_j)\right),\end{aligned}$$

here the second equality follows from the compatibility condition 1.1 on $d\Gamma$. Since, $\sum_j \rho_c(H_{2e_j}) = 0$, Lemma 6.10 implies that the sums appearing in the above equation reduce to

$$(-1)^N \exp\left(i\pi \sum_{j=1}^{n} \epsilon_j \lambda_j\right).$$

A little easy manipulation leads to the formula in the lemma.

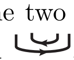
Next consider (b). Since $\mathbf{x}$ is central, it follows from the definition of the induced action that $\mathbf{x}$ acts on $X_{\lambda_a}(\delta)$ by $\Gamma(\mathbf{x})$. Equation 6.1 gives

$$\Gamma(m_j) = -\epsilon_j \exp(i\pi\lambda_j).$$

Since $\mathbf{x} = \prod_j m_j$, part (b) now follows.

Part (c) follows in exactly the same way, and so we omit the details. $\square$

Remark 6.12. Combining the above proposition with Theorem 1.3, it is not difficult to compute the scalar by which $\mathbf{x}$ acts on $X_{\lambda_a}(\delta)$ for arbitrary δ. The precise statement is quite cumbersome, so we do not include it here.

Example 6.13. For instance, consider $Mp(4, \mathbb{R})$ and consider infinitesimal character $(\frac{3}{2}, \frac{1}{2})$. The element $\mathbf{x}$ acts by -1 on the on the two discrete and principal series representations parameterized by $++--$ and , and by $+1$ on those parameterized by $+-+-$ and  in $D(4)$.

6.5. **Cayley transforms.** In this section, we compute Cayley transforms on the level of diagram. To get started, we need the following classification of roots on the level of diagrams. The reader is invited to consult the example following the lemma.

Lemma 6.14. *Fix $\delta \in D(2n)^{m,r,s}$, let (σ, ϵ) be a representative for δ, and let w be any element in the coset of $W^{m,r,s}$ specified by Lemma 6.2. Consider a simple root $\alpha' \in \Delta_a^+$. For definiteness, define an index j according to whether $\alpha' = e_j - e_{j+1}$ or $\alpha' = e_{2j}$ (so in the latter case j necessarily equals n). Then for the root $\alpha := w \cdot (c_{m,r,s}^{tr})^{-1}(\alpha')$ (which is simple for $w \cdot \Delta_+^{m,r,s}$) we have the following conclusions:*

(a) *α is compact imaginary if and only if α is short, $\sigma(j) = j$, $\sigma(j+1) = j+1$, and $\epsilon_j = \epsilon_{j+1}$;*
(b) *α is type I noncompact imaginary if and only if one of the following cases holds:*
  (i) *α is short, $\sigma(j) = j$, $\sigma(j+1) = j+1$, and $\epsilon_j \neq \epsilon_{j+1}$,*
  (ii) *α is long and $\sigma(n) = n$ (and so $\epsilon_n \neq \epsilon_{n+1}$ and $\sigma(n+1) = n+1$);*
(c) *α is type II noncompact imaginary if and only if α is short and $\sigma(j) = (2n+1-j)-1$ (and hence $\sigma(j+1) = (2n+1-(j+1))+1$);*
(d) *α is type I real if and only if one of the following cases holds:*
  (i) *α is short, and $\sigma(j) = j+1$, or*
  (ii) *α is long, and $\sigma(n) = n+1$;*
(e) *α is type II real if and only if α is short, $\sigma(j) = (2n+1-j)$, $\sigma(j+1) = (2n+1-(j+1))$, and $\epsilon_j = -\epsilon_{j+1}$;*
(f) *α is complex in the remaining cases, which we can enumerate as follows:*
  (i) *α is short and $\{\sigma(j), \sigma(j+1)\} \neq \{j, j+1\}$ and $\{\sigma(j), \sigma(j+1)\} \neq \{2n+1-j, 2n+1-(j+1)\}$. In this case $\theta(\alpha) > 0$ if and only if*
    (A) *$\sigma(j) \leq n$ and $\sigma(j+1) > \sigma(j)$, or*
    (B) *$\sigma(j) > n$ and $\sigma(j+1) < 2n+1-\sigma(j)$;*
  (ii) *α is long and $\{\sigma(n), \sigma(n+1)\} \neq \{n, n+1\}$. In this case $\theta(\alpha) > 0$ if and only if $\sigma(n) < n$ (and so $\sigma(n+1) > n+1$).*

Proof. This follows directly from the definitions and Equations (2.1) and (2.2) in Section 2.3. ☐

Example 6.15. We give examples of the cases arising in the lemma.

(a) Let $\alpha' = e_1 - e_2$, and consider the diagram $\delta = + + - - \in D(4)$. Construct α as in Lemma 6.14. Then α is compact imaginary for $w \cdot \Delta_+^{0,2,0}$. (In the following statements, we will use the abbreviated terminology "$e_1 - e_2$ is compact imaginary for δ".)
(b) We have the following examples of type I noncompact imaginary roots:
  (i) $e_1 - e_2$ (short) type I noncompact imaginary for $+ - + - \in D(4)$.
  (ii) $2e_1$ is (long) type I noncompact imaginary for $+- \in D(2)$.
(c) $e_1 - e_2$ is type II noncompact imaginary for .
(d) We have the following examples of type I real roots:
  (i) $e_1 - e_2$ is (short) type I real for .
  (ii) $2e_2$ is (long) type I real for .
(e) $e_1 - e_2$ is type II real for .
(f) (i) The following roots are short and complex:
    (A) $e_1 - e_2$ is complex for ; after applying θ, the root is positive.
    (B) $e_1 - e_2$ is complex for ; after applying θ, the root is negative.
  (ii) $2e_2$ is long, complex, and after applying θ negative for .

Next we describe the operations of Cayley transforms and inverse Cayley transforms on the level of diagrams.

Definition 6.16. Suppose $\delta \in D(2n)^{m,r,s}$ and let (σ, ϵ) be a representative of δ. Suppose α is noncompact imaginary and simple for δ. If α is type I, we are going to define a new element $\delta^\alpha \in D(2n)^{m,r-1,s+1}$; and if α is type II, we will define two elements $\delta^\alpha_\pm \in D(2n)^{m,r-1,s+1}$. These will be called the *Cayley transform of δ through α*. In each case $(\sigma^\alpha, \epsilon^\alpha)$ (or $(\sigma^\alpha_\pm, \epsilon^\alpha_\pm)$) will denote a representative of the corresponding diagram. We proceed case by case.

(a) $\alpha = e_j - e_{j+1}$ (short) type I. We define

$$\begin{aligned} \sigma^\alpha(k) &= k \text{ for } k \notin \{j, j+1, 2n+1-j, 2n+1-(j+1)\}, \\ \sigma^\alpha(j) &= j+1, \quad \sigma^\alpha(j+1) = j, \\ \sigma^\alpha(2n+1-j) &= 2n+1-(j+1), \quad \sigma^\alpha(2n+1-(j+1)) = 2n+1-j, \text{ and} \\ \epsilon^\alpha &= \epsilon. \end{aligned}$$

For example, in $Mp(4,\mathbb{R})$ consider the diagram represented by (σ, ϵ) with σ trivial and $\epsilon = (+,-,+,-)$. Pictorially, we can write $\delta = + - + -$, and we have

$$(+-+-)^{e_1-e_2} \quad = $$ .

(b) $\alpha = 2e_n$ (long) type I. We define

$$\begin{aligned} \sigma^\alpha(k) &= \sigma(k) \text{ for } k \notin \{n, n+1\}, \\ \sigma^\alpha(n) &= n+1, \sigma^\alpha(n+1) = n, \\ \epsilon^\alpha_j &= \epsilon_j \text{ for } j \notin \{n, n+1\}, \text{ and} \end{aligned}$$

$$\begin{aligned} (\epsilon^\alpha_n, \epsilon^\alpha_{n+1}) = (+,-) &\text{ if } (\epsilon_n, \epsilon_{n+1}) = (-,+) \text{ and } \#\{k \mid \sigma(k) = k\} = 0 \pmod 4\text{; or} \\ &\text{if } (\epsilon_n, \epsilon_{n+1}) = (+,-) \text{ and } \#\{k \mid \sigma(k) = k\} = 2 \pmod 4; \end{aligned}$$

and

$$\begin{aligned} (\epsilon^\alpha_n, \epsilon^\alpha_{n+1}) = (-,+) &\text{ if } (\epsilon_n, \epsilon_{n+1}) = (+,-) \text{ and } \#\{k \mid \sigma(k) = k\} = 0 \pmod 4\text{; or} \\ &\text{if } (\epsilon_n, \epsilon_{n+1}) = (-,+) \text{ and } \#\{k \mid \sigma(k) = k\} = 2 \pmod 4. \end{aligned}$$

(The above parity considerations are manifestations of those discussed in Section 3.3 when Cayley transforms for long roots were defined.) For example, in $Mp(2,\mathbb{R})$, we have

$$(+-)^{2e_1} \quad = $$ ,

while in $Mp(4,\mathbb{R})$ we have

$$(++--)^{2e_2} \quad = $$ .

(Recall that our arrows begin at minus signs and end at plus signs.)

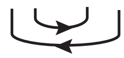
(c) $\alpha = e_j - e_{j+1}$ type II. We define

$$\begin{aligned} &\sigma^\alpha_\pm(k) = k \text{ and } (\epsilon^\alpha_\pm)_k = \epsilon_k \text{ for } k \notin \{j, j+1, 2n+1-j, 2n+1-j-1\}; \\ &\sigma^\alpha_\pm(j) = 2n+1-j, \quad \sigma^\alpha_\pm(j+1) = 2n+1-(j+1), \\ &\sigma^\alpha_\pm(2n+1-j) = j, \quad \sigma^\alpha_\pm(2n+1-(j+1)) = j+1. \end{aligned}$$

Finally, we define

$$((\epsilon_+^\alpha)_j, (\epsilon_+^\alpha)_{2n+1-j}) = (+,-); \text{ and } ((\epsilon_-^\alpha)_j, (\epsilon_-^\alpha)_{2n+1-j}) = (-,+);$$
$$((\epsilon_+^\alpha)_{j+1}, (\epsilon_+^\alpha)_{2n+1-(j+1)}) = (-,+); \text{ and } ((\epsilon_-^\alpha)_{j+1}, (\epsilon_-^\alpha)_{2n+1-(j+1)}) = (+,-).$$

For example, in $Mp(4,\mathbb{R})$, we have

()$^{e_1-e_2}_{+}$ = 

and

()$^{e_1-e_2}_{-}$ =  .

The point of the definition is that, under the correspondence of Proposition 6.3, we have really defined the representation theoretic Cayley transforms of Section 3.3 (at least for a fixed infinitesimal character).

Lemma 6.17. *Fix infinitesimal character $\lambda_a = (\frac{2n-1}{2}, \ldots, \frac{1}{2})$. Consider $\delta \in D(2n)$, suppose α is a noncompact imaginary root for δ, and write δ^α for the Cayley transform of the diagram δ defined above. Write γ for the λ_a-pseudocharacter parameterized by δ (Proposition 6.3), and γ' for the one parameterized by δ^α. Write γ^α for the pseudocharacter Cayley transform of Section 3.3. Then $\gamma' = \gamma^\alpha$.*

Proof. This is an exercise in tracking down the definitions. We omit the details. □

Remark 6.18. From the discussion in Section 3.3, we know that $X(\gamma)$ and $X(\gamma^\alpha)$ have the same central character. Using Proposition 6.11 and Definition 6.16, one can verify this directly. (The verification involves some rather delicate parity manipulations.)

Remark 6.19. By inverting Definition 6.16, one can define inverse Cayley transforms on the level of diagrams. As mentioned in Section 3.3, it is important to note that the inverse Cayley transform through a long type I real root (satisfying the parity condition) is single valued, unlike the linear type I case.

6.6. **The cross action.** In this section, we begin by observing that the action of W_a on $D(2n)$ coincides with the cross action of Section 3.3. More precisely, we have

Lemma 6.20. *Fix infinitesimal character $\lambda_a = (\frac{2n-1}{2}, \ldots, \frac{1}{2})$. Consider $\delta \in D(2n)$, $w \in W_a$, and write γ and γ' for the λ_a-pseudocharacters corresponding to δ and $w \times \delta$ via Proposition 6.3. Write $w \times \gamma$ for the pseudocharacter cross action of Section 3.3. Then $\gamma' = w \times \gamma$.*

Remark 6.21. Using Proposition 6.11, one can explicitly check that the cross action in the integral Weyl group preserves the central action of the element $\mathbf{x}$, and that the cross action in the long simple root e_{2n} always negates the scalar by which $\mathbf{x}$ acts. See Lemma 3.16.

Example 6.22. In Figure 1, we enumerate the set $D(4)$. By Proposition 6.3 and the Langlands classification (Theorem 1.3), $D(4)$ parameterizes the set of irreducible genuine representations of $Mp(4,\mathbb{R})$ with fixed regular half-integral infinitesimal character. Here we fix infinitesimal character $(\frac{3}{2}, \frac{1}{2})$, and will thus think of the vertices of the graph in Figure 1 as genuine irreducible representations with

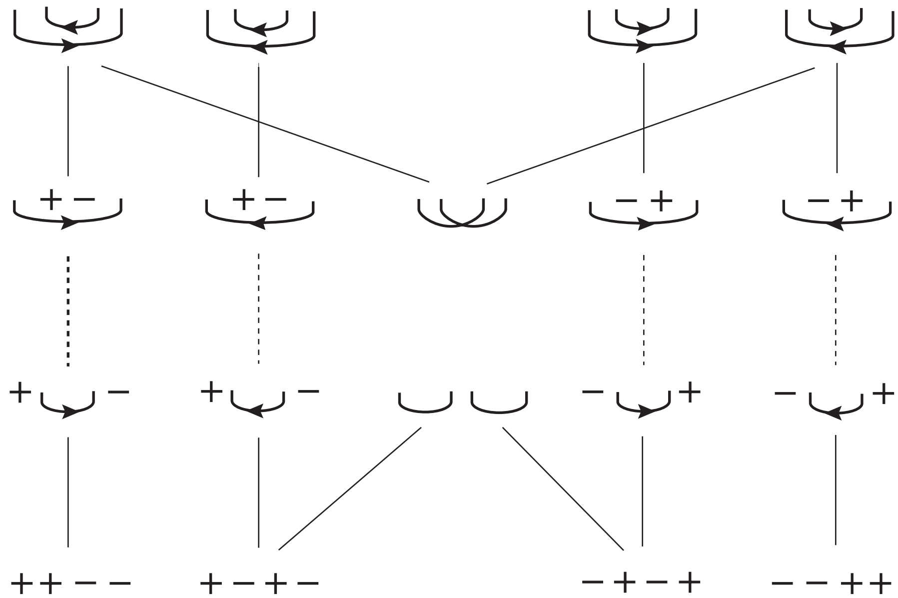


FIGURE 1. The set of Langlands parameters for $Mp(4,\mathbb{R})$ at regular half-integral infinitesimal character.

this infinitesimal character. We have arranged the length zero representations on the bottom line—these are the discrete series, with the holomorphic and anti-holomorphic ones in the bottom left and right corners. Representations in the first row have length one, those in the second have length two, and so on. Dotted lines indicate cross action in simple roots in the integral Weyl group which are simple for W_a (i.e. through $s_{e_1-e_2}$ here), while solid lines indicate Cayley transforms.

Note that there are two connected components of the graph. From Proposition 6.11 (see Example 6.13), we see that the discrete series in each component have opposite central character with respect to $\mathbf{x}$. Since Cayley transforms and the cross action of the integral Weyl group preserve central character (see Remarks 6.18 and 6.21), we conclude that the representations in each connected component have opposite central character with respect to $\mathbf{x}$. Hence a single block of representations in $Mp(4,\mathbb{R})$ cannot intersect both connected components nontrivially. On the other hand, two representations that are related by the cross action (in the integral Weyl group) and Cayley transforms lie in the same block; see the references in the proof of Proposition 6.9 below. Hence, if the intersection of a block with a connected component is nontrivial, it must be the entire connected component. We conclude that each connected component of the graph is a block of representations. This is consistent with Proposition 6.9.

Proof of Proposition 6.9. The translation principle reduces the proposition to the case of $\lambda_a = (\frac{2n-1}{2}, \dots, \frac{1}{2})$. In fact, the arguments of the preceding example are enough to handle this case. It is not difficult to see that one can always perform a sequence of Cayley transforms, inverse Cayley transforms, and integral Weyl group cross actions to arrive at a discrete series. The arguments of [Vgr, Corollary 8.2.9] apply to our setting to show that this sequence of operations preserves blocks. According to Proposition 6.11 and the arguments of the $Mp(4,\mathbb{R})$ example above, we need only show that we can move (via a sequence of Cayley transforms and cross actions) between any two given two discrete series representations with the same central character. This amounts to showing that via such a sequence, we can replace the first n signs of a diagram in $D(2n)^{0,n,0}$ by any other sequence of n signs, so long as the parity of the number of pluses is preserved. In turn we need only show how to exchange two adjacent inequivalent signs, and how to replace two adjacent identical signs by their negatives. We saw explicitly how to do this in $Mp(4,\mathbb{R})$ above, and the general case amounts to embedding this argument in $Mp(2n,\mathbb{R})$. The (easy) details are left to the reader. $\square$

For future reference, we now return to the outer automorphism $\widetilde{\iota}$ of Section 2.7. In the following lemma, we also let $\widetilde{\iota}$ denote the corresponding action on pseudocharacters.

Lemma 6.23. *On the level of pseudocharacters we have*

$$\widetilde{\iota}(\gamma) = w_o \times \gamma;$$

here w_o is the long element in S_{2n}. Moreover,

(a) *if n is even, $\mathbf{x}$ acts by the same scalar on $X(\gamma)$ and $X(\widetilde{\iota}(\gamma))$;*
(b) *if n is odd and $\mathbf{x}$ acts by the scalar c on $X(\gamma)$, then $\mathbf{x}$ acts by the scalar $-c$ on $X(\widetilde{\iota}(\gamma))$.*

Proof. The first statement is very easy. The latter two can be proved by examining how the conditions in Proposition 6.11 change under the action of $\widetilde{\iota}$. (The signs in (a) and (b) of that proposition change depending on the parity of n; the sign in (c) doesn't change.) $\square$

6.7. **Nonintegral wall crossing.** Fix infinitesimal character $\lambda_a = (\frac{2n-1}{2}, \ldots, \frac{1}{2})$. Let $\alpha = 2e_n$, and recall the (nonintegral) wall-crossing translation functor $\psi = \psi_\alpha$ defined in Convention 1.8. In terms of the Langlands classification, we recalled Vogan's computation of ψ_α in Theorem 3.17. Our goal here is to give this computation on the level of diagrams.

Definition 6.24. We define an involution $\psi : D(2n) \longrightarrow D(2n)$ specified by the following three cases (and the requirement that ψ^2 is the identity). In each case the pair (σ, ϵ) is a representative for $\delta \in D(2n)$.

(a) $\sigma(n) \notin \{n, n+1\}$. Then $\psi(\delta) = s_{2e_n} \times \delta$ (with notation as in Definition 6.1). For example,

$\psi\Big($  $\Big) =$  .

(b) $\sigma(n) = n$. Then $\psi(\delta)$ is represented by a pair (σ', ϵ') with $\sigma'(j) = \sigma(j)$ and $\epsilon'_j = \epsilon_j$ for $j \notin \{n, n+1\}$, $\sigma'(n) = n+1$, $\sigma'(n+1) = n$, and

$$(\epsilon^\alpha_n, \epsilon^\alpha_{n+1}) = (+,-) \text{ if } (\epsilon_n, \epsilon_{n+1}) = (+,-) \text{ and } \#\{k \mid \sigma(k) = k\} = 0 \pmod 4, \text{ or}$$
$$\text{if } (\epsilon_n, \epsilon_{n+1}) = (-,+) \text{ and } \#\{k \mid \sigma(k) = k\} = 2 \pmod 4,$$

and

$$(\epsilon^\alpha_n, \epsilon^\alpha_{n+1}) = (-,+) \text{ if } (\epsilon_n, \epsilon_{n+1}) = (-,+) \text{ and } \#\{k \mid \sigma(k) = k\} = 0 \pmod 4, \text{ or}$$
$$\text{if } (\epsilon_n, \epsilon_{n+1}) = (+,-) \text{ and } \#\{k \mid \sigma(k) = k\} = 2 \pmod 4.$$

For example, in $Mp(2,\mathbb{R})$, we have

$\psi(+-) =$  ,

while in $Mp(4,\mathbb{R})$ we have

$\psi(++--) =$ 

.

(c) $\sigma(n) = n+1$. This is covered by the preceding case and the stipulation that ψ is an involution.

Note that the second case of the definition formally resembles the definition of a Cayley transform on the level of diagrams (see Definition 6.16), but notice that the arrows are oriented in the *opposite* direction. At any rate, the next result justifies the definition.

Lemma 6.25. *Fix infinitesimal character $\lambda_a = (\frac{2n-1}{2}, \dots, \frac{1}{2})$, let $\alpha = 2e_n$, and recall the functor $\psi = \psi_\alpha$ (defined in Convention 1.8). Fix $\delta \in D(2n)$ and write γ and γ' for the λ_a-pseudocharacters corresponding (via Proposition 6.3) to δ and $\psi(\delta)$ (Definition 6.24). Then*

$$\psi(\overline{X}(\gamma)) = \overline{X}(\gamma').$$

Proof. This is a translation of Theorem 3.17 into the case at hand. We leave the details to the reader. ☐

Corollary 6.26. *Fix regular half-integral infinitesimal character λ_a and otherwise retain the setting of Lemma 6.25. Then ψ is an equivalence of categories between the full subcategories of $\mathcal{HC}(\mathfrak{g}, \widetilde{K})^{gen}_{\lambda_a}$ respectively generated by each of the two blocks (see Proposition 6.9) of genuine irreducible representations of $Mp(2n,\mathbb{R})$ with infinitesimal character λ_a.*

Proof. A translation principle reduces to a single infinitesimal character λ. All that remains to check is that $\mathbf{x}$ acts oppositely on $\overline{X}(\gamma)$ and $\psi(\overline{X}(\gamma))$. All the relevant details can be seen in the $Mp(4,\mathbb{R})$ case (Example 6.22). We leave the details of the general case to the reader. ☐

6.8. **Some τ-invariant computations.** Fix a half-integral infinitesimal character λ_a. From the data of a diagram $\delta \in D(2n)$, it is desirable to read off the τ-invariant (see Section 1.2) of the corresponding irreducible representation $\overline{X}_{\lambda_a}(\delta)$ (Notation 6.4). We describe how to do that in this section.

First we recall Vogan's fundamental computation ([V1, Corollary 4.13]).

Theorem 6.27. *Consider an irreducible Harish-Chandra module X (of infinitesimal character λ_a) of a reductive Lie group of the class considered in* [V1]. *In the notation of the Langlands classification (Section 1.2), write $X = \overline{X}(\gamma)$. Suppose α is a simple root in the integral root system $R^+(\gamma)$. Moreover, suppose α is simple in the entire root system $\Delta^+(\gamma)$. Then α is in the (abstract) τ-invariant of X if and only if*

(a) *α is compact imaginary for $\Delta^+(\gamma)$, or*
(b) *α is complex for $\Delta^+(\gamma)$ and $\theta(\alpha) \notin \Delta^+(\gamma)$, or*
(c) *α is real and satisfies the parity condition (Definition 3.12).*

Recall the set of simple integral roots S_{int} for regular half-integral infinitesimal character (Remark 3.8). Using the theorem, together with Lemma 6.14, we can easily determine whether a root $e_j - e_{j+1}$ is in the τ-invariant of $\overline{X}_\lambda(\delta)$ (Notation 6.4). (Now we are no longer speaking of the abstract τ-invariant, but rather the relative one defined with respect to the choices implicit in S_{int}.) Notice, however, that the theorem does not apply to the simple long root $e_{n-1} + e_n$ since it is not simple for the entire root system. Instead, we must apply the following lemma (which follows from the considerations of Section 4 in [V1]).

Lemma 6.28. *Let X be an irreducible Harish-Chandra module of regular half-integral infinitesimal character for $Mp(2n, \mathbb{R})$. Then $e_{n-1}+e_n$ is in the τ-invariant of X if and only if $e_{n-1} - e_n$ satisfies the conditions of Theorem 6.27 for the irreducible module $\psi(X)$ (where ψ is the nonintegral e_{2n}-wall crossing functor defined in Convention 1.8).*

Since we have explicitly computed ψ, Lemma 6.28 and Theorem 6.27 constitute a complete description of the τ-invariant of an irreducible Harish-Chandra module of half-integral infinitesimal character for $Mp(2n, \mathbb{R})$. For instance, the next result now follows easily.

Proposition 6.29. *Fix infinitesimal character $\lambda_a = (\frac{2n-1}{2}, \ldots, \frac{1}{2})$.*

(a) *Let $\delta \in D(2n)^{0,n,0}$ be represented by the pair (e, ϵ), and consider the corresponding discrete series $\overline{X}_{\lambda_a}(\delta)$. Then*
$$\tau(\overline{X}_{\lambda_a}(\delta)) = \{e_j - e_{j+1} \mid 1 \leq j \leq n,\ \epsilon_j = \epsilon_{j+1}\}.$$

(b) *Let $\delta \in D(2n)^{0,0,n}$ be represented by the pair (w_o, ϵ) (where w_o is the long element in S_{2n}) and consider the corresponding irreducible quotient of the principal series $\overline{X}_{\lambda_a}(\delta)$. Then*
$$\tau(\overline{X}_{\lambda_a}(\delta)) = \{e_j - e_{j+1} \mid 1 \leq j \leq n,\ \epsilon_j \neq \epsilon_{j+1}\} \cup \{e_{n-1} + e_n\}.$$

Corollary 6.30. *Fix half-integral infinitesimal character λ_a and let $\delta \in D(2n)^{0,n,0}$ be represented by the pair (e, ϵ). There exist exactly two Langlands quotients of principal series (of infinitesimal character λ_a) whose τ-invariant is complementary to the τ-invariant of $\overline{X}_{\lambda_a}(\delta)$. Explicitly, the two principal series representations are $\overline{X}_{\lambda_a}(\check{\delta})$ and its outer automorphism conjugate $\overline{X}_{\lambda_a}(\tilde{\iota}(\check{\delta}))$ (Notation 6.23) where $\check{\delta}$ is defined to be the diagram represented by the pair (w_o, ϵ). Moreover,*

(a) *if $n \equiv 1$ or 2 modulo 4, then if $\mathbf{x}$ acts by the scalar C on $\overline{X}_{\lambda_a}(\delta)$, it acts by $-C$ on $\overline{X}_{\lambda_a}(\check{\delta})$;*

(b) *if $n \equiv 0$ or 3 modulo 4, then $\mathbf{x}$ acts by the same scalar on $\overline{X}_{\lambda_a}(\delta)$ and $\overline{X}_{\lambda_a}(\check{\delta})$.*

Proof. This follows from Proposition 6.29, Proposition 6.11, and Lemma 6.23. ☐

6.9. **Definition of the duality.** In this section, we extend the definition of $\check{\delta}$ in Corollary 6.30 to define the full duality of Theorem 0.1 on the level of diagrams. This amounts to an involution
$$D(2n) \to D(2n)$$
$$\delta \mapsto \check{\delta}.$$

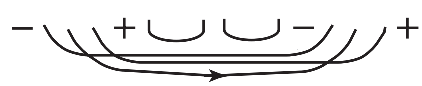
Definition 6.31. Write (σ, ϵ) for a representative of $\delta \in D(2n)$. We will define a representative $(\check{\sigma}, \check{\epsilon})$ for $\check{\delta}$ according to the following rules.

(a) If $\sigma(j) = j$ (and so $\sigma(2n–1–j) = 2n–1–j$ by condition (2) in Section 6.1), then $\check{\sigma}(j) = 2n–1–j$ and $\check{\sigma}(2n–1–j) = j$ while $\check{\epsilon}_j = -\epsilon_j$ and $\check{\epsilon}_{2n-1-j} = -\epsilon_{2n-1-j}$.
(b) If $\sigma(j) = 2n–1–j$ (and so $\sigma(2n–1–j) = j$), then $\check{\sigma}(j) = j$ and $\check{\sigma}(2n–1–j) = 2n–1–j$ while $\check{\epsilon}_j = -\epsilon_j$ and $\check{\epsilon}_{2n-1-j} = -\epsilon_{2n-1-j}$.
(c) If $\sigma(j) = k \notin \{j, 2n–1–j\}$ (and so $\sigma(2n–1–k) = 2n–1–j$), then

$$\begin{aligned} &\check{\sigma}(j) = 2n–1–k,\\ &\check{\sigma}(k) = 2n–1–j,\\ &\check{\sigma}(2n–1–j) = k,\\ &\check{\sigma}(2n–1–k) = j,\\ &\check{\epsilon}_l = \epsilon_l, \qquad \text{for } l = j, k, 2n–1–j, 2n–1–k. \end{aligned}$$

For instance, the dual of the diagram

is



(Again recall that our arrows point from minus signs to plus signs.)

Example 6.32. Consider again Example 6.22. The duality amounts to picking up one of the connected components of the graph given in Figure 1, turning it upside down, and overlaying on the other connected component. Note that, since the graph is symmetric about flipping the (vertical) center, there is an ambiguity in this process. For instance, we could match a given discrete series with either principal series. The symmetry is a manifestation of the outer automorphism $\widetilde{\iota}$ of Section 2.7, and will be explained more thoroughly below.

The next result, like Proposition 6.29, examines how the duality affects the central action of the element $\mathbf{x}$.

Proposition 6.33. *Fix half-integral infinitesimal character λ, and fix $\delta \in D(2n)$. Then we have the following conclusions:*
(a) *If $n \equiv 1$ or 2 modulo 4, then if $\mathbf{x}$ acts by the scalar C on $\overline{X}_{\lambda_a}(\delta)$, it acts by $-C$ on $\overline{X}_{\lambda_a}(\check{\delta})$.*
(b) *If $n \equiv 0$ or 3 modulo 4, then $\mathbf{x}$ acts by the same scalar on $\overline{X}_{\lambda_a}(\delta)$ and $\overline{X}_{\lambda_a}(\check{\delta})$.*

Proof. This follows exactly in the same way as Proposition 6.29. ☐

Here is a sharpened version of Theorem 0.1, which we prove in Section 8 below.

Theorem 6.34. *Fix half-integral infinitesimal character λ_a, write $B^{\pm}$ for the two blocks of $\mathcal{P}_{\lambda_a}$ (Proposition 6.9), and recall Notation 6.4. Suppose that $n \equiv 1$ or 2 modulo 4. Then the map $\gamma \mapsto \check{\gamma}$ (Definition 6.31) is a bijection between B^{ε} and $B^{-\varepsilon}$ such that if*

$$[\overline{X}_{\lambda_a}(\delta)] = \sum_{\delta' \in B^{\varepsilon}} c_{\delta\delta'} [X_{\lambda_a}(\delta')],$$

then

$$[X_{\lambda_a}(\check{\delta}')] = \sum_{\delta \in B^{\varepsilon}} \epsilon_{\delta\delta'} c_{\delta'\delta} [\overline{X}_{\lambda_a}(\check{\delta})];$$

here

$$\begin{aligned}\epsilon_{\delta\delta'} &= (-1)^{\dim(Q_\delta)-\dim(Q_{\delta'})} \qquad \textit{(Notation 6.6)}\\ &= -1^{l(\delta)-l(\delta')},\end{aligned}$$

which is computed explicitly in Lemma 6.7.

If $n \equiv 0$ or 3 modulo 4, then $\gamma \mapsto \check{\gamma}$ is a bijection between B^ε and B^ε (or $B^{-\varepsilon}$ and $B^{-\varepsilon}$) and the analogous statements hold.

Remark 6.35. According to Lemma 6.23, the map $\gamma \mapsto \widetilde{\iota}(\check{\gamma})$ gives another choice of the duality. Note that since a module and its dual must have complementary τ-invariants, Corollary 6.30 implies that these two bijections are the *only* choices of the duality. Next note that Lemma 6.23 implies that if n is even $\widetilde{\iota}$ preserves blocks, and that if n is odd $\widetilde{\iota}$ switches blocks. Hence we arrive at the following table.

	$(\ )^\vee$	$(\ )^\vee \circ \widetilde{\iota}$
$n \equiv 0\ (4)$	preserves blocks	preserves blocks
$n \equiv 1\ (4)$	switches blocks	preserves blocks
$n \equiv 2\ (4)$	switches blocks	switches blocks
$n \equiv 3\ (4)$	preserves blocks	switches blocks

Remark 6.36. We close by remarking on an equivalent formulation of Definition 6.36 closer to Vogan's original formulation in [V4]. By Proposition 6.29, the condition that a representation and its dual have complementary τ-invariant and complementary length, essentially specifies the duality on the discrete and principal series. (The modifier 'essentially' is needed only up to the two possible choices discussed in Remark 6.35.) We can move from any representation in a given block to a discrete series representation via a sequence of integral Weyl group cross actions and Cayley transforms. As is clear from the example of $Mp(4,\mathbb{R})$, there is an obvious dual sequence of cross actions and Cayley transforms. Applied to the dual of the discrete series, we arrive at the dual of the original representation in question. Since we do not need it here, we leave the details to the reader.

7. Kazhdan-Lusztig algorithm for $Mp(2n,\mathbb{R})$: closed form

7.1. **Bruhat $\mathcal{G}$-order.** Due to the presence of nonintegral noncompact imaginary roots, the definition of the Bruhat $\mathcal{G}$-order on the set of λ_a-pseudocharacter $\mathcal{P}_{\lambda_a}$ is a little bit more complicated than the one in [V3] for linear groups. Recall that we do not distinguish in the notations a pseudocharacter γ and its $K_{\mathbb{R}}$ conjugacy class in $\mathcal{P}_{\lambda_a}$.

Definition 7.1. Let γ and γ' be two elements of $\mathcal{P}_{\lambda_a}$, and $s \in S$. We write $\gamma' \xrightarrow{s} \gamma$ in the following cases:

(a) The simple root α in $\Delta^+(\overline{\gamma})$ corresponding to $s \in S$ is noncompact imaginary and short, and $\gamma' \in c^\alpha(\gamma)$.

(b) The simple root α in $\Delta^+(\overline{\gamma})$ corresponding to $s \in S$ is complex such that $\theta(\alpha) \in \Delta^+(\overline{\gamma})$ and $\gamma' = s \times \gamma$.

(c) The simple root α in $\Delta^+(\overline{\gamma})$ corresponding to $s \in S$ is noncompact imaginary and long, and $\gamma' = (s \times \gamma)^\alpha$.

Definition 7.2. The Bruhat $\mathcal{G}$-order is the smallest order relation on $\mathcal{P}_{\lambda_a}$ having the following properties:

(i) If $\gamma \in \hat{H}'_{\lambda_a}$ is a pseudocharacter, and α is a noncompact imaginary simple root in $\Delta^+(\overline{\gamma})$, then for all $\gamma' \in c^\alpha(\gamma)$, we have $\gamma < \gamma'$.

(ii) If $\gamma \in \hat{H}'_{\lambda_a}$ is a pseudocharacter, and α is a complex simple *short* root in $\Delta^+(\overline{\gamma})$, such that $\theta(\alpha) \in \Delta^+(\overline{\gamma})$, then $\gamma < s \times \gamma$.

(iii)(exchange condition) If $\gamma' \leq \delta'$, $\gamma \xrightarrow{s} \gamma'$ and $\delta \xrightarrow{s} \delta'$, then $\gamma \leq \delta$.

If s is the reflection with respect to the long root, $\gamma' \leq \delta'$, $\delta \xrightarrow{s} \delta'$, and $\gamma \xrightarrow{s} \gamma'$, then $\gamma \leq \delta$.

As a motivation for this complicated definition we state the following result.

Theorem 7.3. *Consider $\gamma, \delta \in \mathcal{P}_{\lambda_a}$.*

(a) *If $\gamma < \delta$ in the Bruhat $\mathcal{G}$-order, then $l(\gamma) < l(\delta)$. Moreover, if $(Q_\gamma, \mathcal{L}_\gamma)$ and $(Q_\delta, \mathcal{L}_\delta)$ are the Beilinson-Bernstein parameters corresponding to γ and δ (cf. Proposition 6.8), then $Q_\gamma \subset \overline{Q}_\delta$.*

(b) *Suppose γ, δ in $\mathcal{P}_{\lambda_a}$ and $\overline{X}(\gamma)$ occurs as a composition factor in $X(\delta)$. Then $\gamma \leq \delta$.*

Proof. The first part follows from Lemma 5.5 in [V3] the same way one proves the relation between the Bruhat order and the containment of Schubert varieties. The definition of the Bruhat $\mathcal{G}$-order is tailored exactly to fit the second assertion. The proof is by induction on $l(\delta)$. If $l(\delta) = 0$, δ is minimal, i.e. $X(\delta)$ is a discrete series representation, then the assertion is trivially seen to hold. Otherwise, we know from Proposition 3.18 that there exists a simple root α in $\Delta^+(\overline{\delta})$ such that either α is complex with $\theta(\alpha) \notin \Delta^+(\overline{\delta})$, or α is real, satisfying the parity condition. Let s be the corresponding reflection in the Weyl group. If α is a short root, the induction step is similar to the one given for linear groups. Notice that the differences in the definition of the Bruhat $\mathcal{G}$-order above with the one in [V3] are only in the use of the long simple root. So suppose that α is long, and say, complex. Then by Theorem 3.17 we have $X(\delta) = \psi_\alpha(X(s \times \delta))$. The hypothesis (ii) of the Theorem is satisfied if and only if $\psi_\alpha(\overline{X}(\gamma))$ occurs in $X(s \times \delta)$. Let $\gamma' \in \mathcal{P}_{\lambda_a}$ such that $\psi_\alpha(\overline{X}(\gamma)) = \overline{X}(\gamma')$. Since $l(s \times \delta) = l(\delta) - 1$, by induction hypothesis we have $\gamma' \leq s \times \delta$. The explicit form of γ' is known and corresponds to the different cases in the exchange condition above. We conclude that $\gamma \leq \delta$. In the case where α is real and long, the use of the Hecht-Schmid identity (Proposition 3.15) and the exchange condition lead us to the same conclusion. $\square$

7.2. **Kazhdan-Lusztig polynomials for the metaplectic group.** Let B be an abelian group containing an element u of infinite order. Let $\mathcal{M}$ (respectively $\mathcal{M}'$) be the free $\mathbb{Z}[u, u^{-1}]$-module (respectively B-module) with basis $\mathcal{P}_{\lambda_a}$. Recall the simple reflections

$$S = \{s_{e_i - e_{i+1}}, 1 \leq i \leq n-1;\ s_{2e_n}\};$$

and, for regular half-integral infinitesimal character, the simple integral reflections

$$S_{\mathrm{int}} = \{s_{e_i - e_{i+1}}, 1 \leq i \leq n-1;\ s_{e_{n-1}+e_n}\}.$$

By analogy with [V3, Definition 6.4] for $s \in S$, we now define operators T_s on the basis elements γ. (In the definition below, we denote by α the simple root in $\Delta^+(\overline{\gamma})$ corresponding to s.) Note that if α is a short root, the formulas below are the ones given in [V3].

Definition 7.4. (a1) If α is compact imaginary, then α is short $T_s\gamma = u\gamma$.

(a2) If α is real not satisfying the parity condition, $T_s\gamma = -\gamma$.

(b1) If α is complex and $\theta(\alpha) \in \Delta^+(\overline{\gamma})$, then $T_s\gamma = s \times \gamma$.

(b2 short) If α is short, complex and $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$, then

$$T_s\gamma = u(s \times \gamma) + (u-1)\gamma.$$

(b2 long) If α is long, complex and $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$, then

$$T_s\gamma = u(s \times \gamma).$$

(c1) If α is type II non-compact imaginary (and thus α is necessarily short), then

$$T_s\gamma = \gamma + \gamma_+^\alpha + \gamma_-^\alpha.$$

(c2) If α is short and real type II (and thus α is necessarily short) satisfying the parity condition, then

$$T_s\gamma = (u-1)\gamma - s \times \gamma + (u-1)\gamma_\alpha.$$

(d1 short) If α is type I non-compact imaginary short, then

$$T_s\gamma = s \times \gamma + \gamma^\alpha.$$

(d1 long) If α is type I non-compact imaginary long, then

$$T_s\gamma = (s \times \gamma)^\alpha + s \times \gamma.$$

(d2 short) If α is short and real type I, satisfying the parity condition, then

$$T_s\gamma = (u-2)\gamma + (u-1)(\gamma_\alpha^+ + \gamma_\alpha^-).$$

(d2 long) If α is long and real type I, satisfying the parity condition, then

$$T_s\gamma = -s \times \gamma + (u-1)(s \times \gamma)_\alpha.$$

By analogy with the linear case, one might expect that the $\mathbb{Z}[u, u^{-1}]$ algebra $\mathcal{H}$ generated by $\langle T_s \mid s \in S\rangle$ is isomorphic to $\mathcal{H}(W)$, the Hecke algebra of the complex Weyl group. This isn't quite true. What is true, however, is that $\mathcal{H}$ contains the Hecke algebra of the integral Weyl group. More precisely, we have the following result.

Proposition 7.5. *Extend the definitions of the various T_s (in Definition 7.4) to $\mathbb{Z}[u, u^{-1}]$-linear endomorphisms of $\mathcal{M}$, and write $\mathcal{H}$ for the algebra that they generate. We have the following conclusions:*

(a) *If $s \in S$ is short, then operator T_s satisfies*

$$(T_s - 1)(T_s + u) = 0.$$

(b) *If $s \in S$ is long, the operator T_s satisfies*

$$T_s^2 = u\, Id.$$

In particular, $\mathcal{H}$ is not isomorphic to the Hecke algebra of the complex Weyl group.

(c) *Define*

$$T_{e_{n-1}+e_n} = (u^{-\frac{1}{2}}T_{e_{2n}})^{-1}T_{e_{n-1}-e_n}u^{-\frac{1}{2}}T_{e_{2n}}.$$

Then the algebra $\langle T_s \mid s \in S_{\text{int}}\rangle$ is isomorphic to the Hecke algebra of the integral Weyl group.

Proof. To prove the proposition, one need only appeal to the definitions and check the relevant relations. For (a) and (b), this is quite easy. The final assertion involves a more complicated check, but it too is elementary. □

Remark 7.6. Note that the definition of $T_{e_{n-1}+e_n}$ in (c) formally mimics the definition of an integral wall-cross with respect to a simple integral root that is not simple in the whole roots system. (See the comments after the definition in Section 1.3.)

The next ingredient we need is a 'Verdier duality' endomorphism of $\mathcal{M}'$.

Proposition 7.7. *There exists at most one B-linear map $\mathbb{D} : \mathcal{M}' \to \mathcal{M}'$ with the following properties. Define $R_{\gamma,\delta}$ by*

$$\mathbb{D}(\delta) = u^{-l(\delta)} \sum_{\gamma\in\mathcal{D}} R_{\gamma,\delta}(u)\gamma : \tag{7.1}$$

(a) $\mathbb{D}(bm) = b^{-1}\mathbb{D}(m)$ *($m \in \mathcal{M}'$, $b \in B$).*

(b *short*) *If $s \in S$ is a reflection with respect to a short root and $m \in \mathcal{M}'$, then*

$$\mathbb{D}((T_s + 1)m) = u^{-1}(T_s + 1)\mathbb{D}(m).$$

(b *long*) *If $s \in S$ is the reflection with respect to the long root and $m \in \mathcal{M}'$, then*

$$\mathbb{D}((T_s)m) = u^{-1}T_s(\mathbb{D}(m)).$$

(c) $R_{\gamma,\gamma} = 1$.

(d) $R_{\gamma,\delta} \neq 0$ *only if* $\gamma \leq \delta$.

Suppose that $\mathbb{D}$ exists. There is an algorithm for computing the various $R_{\gamma,\delta}$; there are polynomials in u of degree at most $l(\delta) - l(\gamma)$. Furthermore:

(e) $\mathbb{D}^2$ *is the identity.*

(f) $\mathbb{D}$ *preserves* $\mathcal{M}$.

(g) *On $\mathcal{M}$, the specialization of $\mathbb{D}$ to $u = 1$ is the identity.*

Proof. Again, the proof is essentially in [V3], and we concentrate only on the differences. The proof is by induction on $l(\delta)$, and for fixed δ by downward induction on $l(\gamma)$. If δ is minimal, all the claims are trivial. If not, there exists a simple root α in $\Delta^+(\overline{\delta})$ such that either α is complex with $\theta(\alpha) \notin \Delta^+(\overline{\delta})$, or α is real, satisfying the parity condition. If α is a short root, the induction step is similar to the one given for linear groups. So suppose that α is long and complex. Then

$$l(s \times \delta) = l(\delta) - 1,$$
$$T_s(s \times \delta) = \delta.$$

The identity (b long) in Definition 7.4 for $m = s \times \delta$ can be written as

$$\mathbb{D}(\delta) = u^{-1}T_s(\mathbb{D}(s \times \delta)).$$

Equating the coefficient of γ in the expression of the left- and right-hand sides in terms of R polynomials gives formulas for $R_{\gamma,\delta}$. Suppose now that α is long and real. We have

$$l((s \times \delta)^\alpha) = l(\delta) - 1,$$
$$T_s((s \times \delta)^\alpha) = \delta.$$

We conclude by the same arguments. The rest of the proof is essentially no different than the linear case. □

We see that for combinatorial purpose, the role of this odd long root is not a bad thing: if it is imaginary or real, it is always type I, and the formulas tend to be simpler.

Corollary 7.8. *Suppose $\mathbb{D}$ exists, and suppose that for some $\delta \in \mathcal{P}_{\lambda_a}$ there is an element*

$$C_\delta = \sum_{\gamma \leq \delta} P_{\gamma,\delta}(u)\gamma$$

with the following properties:
(a) $\mathbb{D}(C_\delta) = u^{-l(\delta)} C_\delta$.
(b) $P_{\delta,\delta} = 1$.
(c) *If $\gamma \neq \delta$, then $P_{\gamma,\delta}$ is a polynomial in u of degree at most $\frac{1}{2}(l(\delta) - l(\gamma) - 1)$. Then $P_{\gamma,\delta}$ is computable. (In particular, C_δ is unique).*

The proof, which we omit, is similar to the one in [V3].

Lemma 7.9. *Suppose that $\delta \in \mathcal{P}_{\lambda_a}$, let $s \in S$ be the reflections with respect to the long root and $\alpha \in \Delta^+(\overline{\delta})$ the corresponding simple root. Suppose the elements C_δ, $C_{s\times\delta}$, etc., of the previous corollary exist.*
(a) *If α is complex, $\theta(\alpha) \in \Delta^+(\overline{\delta})$, then $T_s C_\delta = C_{s\times\delta}$.*
(b) *If α is complex, $\theta(\alpha) \notin \Delta^+(\overline{\delta})$, then $T_s C_\delta = u C_{s\times\delta}$.*
(c) *If α is imaginary, then $T_s C_\delta = C_{(s\times\delta)^\alpha}$.*
(c) *If α is real, then $T_s C_\delta = u C_{(s\times\delta)_\alpha}$.*

Proof. Suppose we are in case (a). The element $C_{s\times\delta}$ is characterized by the list of properties of the previous corollary. Thus, it is sufficient to show that $T_s C_\delta$ also satisfies them to prove the equality. We have

$$\mathbb{D}(T_s C_\delta) = u^{-1} T_s(\mathbb{D}(C_\delta)) = u^{-l(\delta)-1} T_s C_\delta = u^{-l(s\times\delta)} T_s C_\delta.$$

Furthermore,

$$\begin{aligned} T_s C_\delta &= T_s\delta + \sum_{\gamma<\delta} P_{\gamma,\delta}(u) T_s\gamma \\ &= s\times\delta + \sum_{\substack{\alpha \text{ real for } \gamma \\ \gamma<\delta}} P_{\gamma,\delta}(u)[-(s\times\gamma) + (u-1)(s\times\gamma)_\alpha] \\ &\quad + \sum_{\substack{\alpha \text{ imaginary for } \gamma \\ \gamma<\delta}} P_{\gamma,\delta}(u)[s\times\gamma + (s\times\gamma)^\alpha] \\ &\quad + \sum_{\substack{\alpha \text{ complex for } \gamma,\, \theta(\alpha)\in\Delta^+(\overline{\gamma}) \\ \gamma<\delta}} P_{\gamma,\delta}(u)[s\times\gamma] \\ &\quad + \sum_{\substack{\alpha \text{ complex for } \gamma,\, \theta(\alpha)\notin\Delta^+(\overline{\gamma}) \\ \gamma<\delta}} u P_{\gamma,\delta}(u)[s\times\gamma]. \end{aligned} \tag{7.2}$$

Using the exchange condition, all elements γ' appearing on the right-hand side are seen to be lower or equal to $s\times\delta$ in the Bruhat $\mathcal{G}$-order, and the polynomials satisfy the degree estimate in (c). Furthermore, $s\times\delta$ appears with coefficient 1. Thus the claim is proved. Case (b) follows from (a) and the equality $T_s^2 = u\,\mathrm{Id}$. Cases (c) and (d) are established similarly. ∎

Identifying the coefficient of γ' in the right-hand side of (7.2) with

$$C_{s\times\delta} = \sum_{\gamma'\leq s\times\delta} P_{\gamma',s\times\delta}(u)\,\gamma',$$

we conclude

(a) if α is real for γ',

$$P_{\gamma',s\times\delta} = -P_{s\times\gamma',\delta} + P_{(s\times\gamma')_\alpha,\delta};$$

(b) if α is imaginary for γ',

$$P_{\gamma',s\times\delta}(u) = P_{s\times\gamma',\delta}(u) + (u-1)P_{(s\times\gamma')^\alpha,\delta}(u);$$

(c) if α is complex for γ', with $\theta(\alpha)\in\Delta^+(\overline{\gamma}')$,

$$P_{\gamma',s\times\delta} = P_{s\times\gamma',\delta};$$

(d) if α is complex for γ', with $\theta(\alpha)\notin\Delta^+(\overline{\gamma}')$,

$$P_{\gamma',s\times\delta}(u) = u\,P_{s\times\gamma',\delta}(u).$$

Formulas corresponding to cases (b), (c), (d) of the corollary are obtained similarly. Summarizing, we have the following result.

Proposition 7.10. *In the setting above (α is the long simple root in Δ_a^+), we have the following identities.*

Case I: *If α is complex for δ, $\theta(\alpha)\notin\Delta^+(\overline{\delta})$, and*

(i) *α is complex for γ, $\theta(\alpha)\notin\Delta^+, then (\overline{\gamma})$*

$$P_{\gamma,\delta} = P_{s\times\gamma,s\times\delta};$$

(ii) *α is complex for γ, $\theta(\alpha)\in\Delta^+(\overline{\gamma})$, then*

$$u\,P_{\gamma,\delta}(u) = P_{s\times\gamma,s\times\delta}(u);$$

(iii) *α is real for γ, then*

$$P_{\gamma,\delta} = -P_{s\times\gamma,s\times\delta} + P_{(s\times\gamma)_\alpha,s\times\delta};$$

(iv) *α is imaginary for γ, then*

$$P_{\gamma,\delta}(u) = P_{s\times\gamma,s\times\delta}(u) + (u-1)P_{(s\times\gamma)^\alpha,s\times\delta}(u).$$

Case II: *If α is complex for δ, $\theta(\alpha)\in\Delta^+(\overline{\delta})$, and*

(i) *α is complex for γ, $\theta(\alpha)\notin\Delta^+(\overline{\gamma})$, then*

$$P_{\gamma,\delta}(u) = u\,P_{s\times\gamma,s\times\delta}(u);$$

(ii) *α is complex for γ, $\theta(\alpha)\in\Delta^+(\overline{\gamma})$, then*

$$P_{\gamma,\delta} = P_{s\times\gamma,s\times\delta};$$

(iii) *α is real for γ, then*

$$u\,P_{\gamma,\delta}(u) = -P_{s\times\gamma,s\times\delta}(u) + P_{(s\times\gamma)_\alpha,s\times\delta}(u);$$

(iv) *α is imaginary for γ, then*

$$u\,P_{\gamma,\delta}(u) = P_{s\times\gamma,s\times\delta}(u) + (u-1)P_{(s\times\gamma)^\alpha,s\times\delta}(u).$$

Case III: *If α is real for δ, and*

(i) *α is complex for γ, $\theta(\alpha)\notin\Delta^+(\overline{\gamma})$, then*

$$P_{\gamma,\delta}(u) = P_{s\times\gamma,(s\times\delta)_\alpha}(u); \tag{7.3}$$

(ii) α *is complex for* γ, $\theta(\alpha) \in \Delta^+(\overline{\gamma})$, *then*

$$P_{\gamma,\delta} = P_{s\times\gamma,(s\times\delta)_\alpha};$$

(iii) α *is real for* γ, *then*

$$P_{\gamma,\delta}(u) = -P_{s\times\gamma,(s\times\delta)_\alpha}(u) + P_{(s\times\gamma)_\alpha,(s\times\delta)_\alpha}(u);$$

(iv) α *is imaginary for* γ, *then*

$$P_{\gamma,\delta}(u) = P_{s\times\gamma,(s\times\delta)_\alpha}(u) + (u-1)P_{(s\times\gamma)^\alpha,(s\times\delta)_\alpha}(u).$$

Case IV: *If* α *is imaginary for* δ, *and*

(i) α *is complex for* γ, $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$, *then*

$$u\,P_{\gamma,\delta}(u) = P_{s\times\gamma,(s\times\delta)^\alpha}(u);$$

(ii) α *is complex for* γ, $\theta(\alpha) \in \Delta^+(\overline{\gamma})$, *then*

$$u\,P_{\gamma,\delta}(u) = P_{s\times\gamma,(s\times\delta)^\alpha}(u);$$

(iii) α *is real for* γ, *then*

$$u\,P_{\gamma,\delta}(u) = -P_{s\times\gamma,(s\times\delta)^\alpha}(u) + P_{(s\times\gamma)_\alpha,(s\times\delta)^\alpha}(u);$$

(iv) α *is imaginary for* γ, *then*

$$u\,P_{\gamma,\delta}(u) = P_{s\times\gamma,(s\times\delta)^\alpha}(u) + (u-1)P_{(s\times\gamma)^\alpha,(s\times\delta)^\alpha}(u).$$

Corollary 7.11. *The formulas of Proposition 7.10, cases I and III, together with the ones of Proposition 6.14 in* [V3], *the fact that* $P_{\gamma,\gamma} = 1$ *and the fact that* $P_{\gamma,\delta} \neq 0$ *only if* $\gamma \leq \delta$ *completely characterize the polynomials* $P_{\gamma,\delta}$. *More precisely, if* $P_{\gamma',\delta'}$ *is known when* $l(\delta') < l(\delta)$, *or* $l(\delta') = l(\delta)$ *and* $l(\gamma') > l(\gamma)$, *then these formulas determine* $P_{\gamma,\delta}$.

Proof. The argument is again the same as in [V3]. □

Notice that the formulas in cases II and IV are easily obtained from the ones in cases I and III respectively. The formulas in Proposition 6.14 of [V3] relate the Kazhdan-Lusztig polynomials $P_{\gamma,\delta}$ with polynomials $P_{\gamma',\delta'}$, where γ' and δ' are obtained from γ and δ using cross-action or Cayley transform with respect to simple integral reflections.

7.3. **Kazhdan-Lusztig algorithm.** In this section, we will use some results of [ABV], Section 17. The Langlands parameters there are not the ones we consider here, unfortunately. They are what the authors call equivalence classes of final limit characters, and we will refer to them as ABV-parameters. Of course, the two parametrizations are equivalent, and a procedure to obtain a pseudocharacter from a final limit character, or conversely, is described in Section 11 of [ABV]. If $\gamma \in \mathcal{P}_{\lambda_a}$, we will denote by $\tilde{\gamma}$ the corresponding ABV-parameter.

Let γ, δ in $\mathcal{P}_a$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be a representative of the K-orbit on the flag manifold associated to γ, with $\mathfrak{h}$ defined over $\mathbb{R}$ and θ-stable. Write d for the codimension of the orbit corresponding to δ in the flag manifold. Define

$$Q_{\gamma,\delta}(u) = \sum_{q\in\mathbb{Z}} u^{\frac{1}{2}(q-d)} \mathrm{mult}(\tilde{\gamma} \otimes \rho(\mathfrak{n}); H_q(\mathfrak{n}, \overline{X}(\delta))). \tag{7.4}$$

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$, chosen for γ as in [V3], (A.2). Set

$$r = l(\delta) - l(\gamma) - \dim \mathfrak{u} \cap \mathfrak{p}.$$

Then, by Corollary A.10 and Proposition 4.3 of [V3]

$$Q_{\gamma,\delta}(u) = \sum_{q\in\mathbb{Z}} u^{\frac{1}{2}(q+r)} \mathrm{mult}[\overline{X}^L(\gamma_{\mathfrak{q}}), H_q(\mathfrak{u}, \overline{X}(\delta))].$$

Theorem 7.12. *The polynomials $Q_{\gamma,\delta}$ defined above are the Kazhdan-Lusztig polynomials $P_{\gamma,\delta}$ of the previous section.*

Proof. The idea is to prove that the formulas in Corollary 7.11 are satisfied by the Q-polynomials. The proof is by induction on $l(\delta)$. So fix δ and suppose that $P_{\gamma',\delta'} = Q_{\gamma',\delta'}$ for all δ' such that $l(\delta') < l(\delta)$. Formulas coming from transformations with respect to an integral reflection $s \in S$ (the ones of Proposition 6.14 of [V3]) are obtained as in Lemma 7.8 of [V3]. So we concentrate on the formulas in Proposition 7.10. In that case, we won't need the induction hypothesis. Suppose we are in case I, (i). Using the translation functor ψ_α, we obtain

$$\begin{aligned}\mathrm{mult}[\overline{X}^L(\gamma_{\mathfrak{q}}), H_q(\mathfrak{u}, \overline{X}(\delta))] &= \mathrm{mult}[\psi_\alpha^{\mathfrak{l}} \cdot \overline{X}^L(\gamma_{\mathfrak{q}}), \psi_\alpha^{\mathfrak{l}} \cdot H_q(\mathfrak{u}, \overline{X}(\delta))] \\ &= \mathrm{mult}[\psi_\alpha^{\mathfrak{l}} \cdot \overline{X}^L(\gamma_{\mathfrak{q}}), H_q(\mathfrak{u}, \psi_\alpha \cdot \overline{X}(\delta))] \\ &= \mathrm{mult}[\overline{X}^L((s\times\gamma)_{\mathfrak{q}}), H_q(\mathfrak{u}, \overline{X}(s\times\delta))].\end{aligned}$$

Now, the θ-stable parabolic $\mathfrak{q}$ has the same properties with respect to $s\times\gamma$ as to γ, and $l(s\times\delta) - l(s\times\gamma) = l(\delta) - l(\gamma)$. Thus we obtain $Q_{s\times\gamma,s\times\delta} = Q_{\gamma,\delta}$, as we wish.

Suppose we are in case I, (ii). The difference with (i) is only that $l(s\times\delta) - l(s\times\gamma) = l(\delta) - l(\gamma) - 2$. This accounts for the factor u in the formula $Q_{s\times\gamma,s\times} = u\, Q_{\gamma.\delta}$.

Suppose we are in case I, (iii). Then,

$$\begin{aligned}\mathrm{mult}[\overline{X}^L(\gamma_{\mathfrak{q}}), H_q(\mathfrak{u}, \overline{X}(\delta))] &= \mathrm{mult}[\psi_\alpha^{\mathfrak{l}} \cdot \overline{X}^L(\gamma_{\mathfrak{q}}), \psi_\alpha^{\mathfrak{l}} \cdot H_q(\mathfrak{u}, \overline{X}(\delta))] \\ &= \mathrm{mult}[\psi_\alpha^{\mathfrak{l}} \cdot \overline{X}^L(\gamma_{\mathfrak{q}}), H_q(\mathfrak{u}, \psi_\alpha \cdot \overline{X}(\delta))] \\ &= \mathrm{mult}[\overline{X}^L((s\times\gamma_\alpha)_{\mathfrak{q}}), H_q(\mathfrak{u}, \overline{X}(s\times\delta))].\end{aligned}$$

The parabolic subalgebra $\mathfrak{q}$ does not satisfy the properties of [V3], (A.2) with respect to $s\times\gamma_\alpha$. Let $\mathfrak{q}_0 = l_\alpha + \mathfrak{u}_0 \subset \mathfrak{l}$ which satisfies these properties for $(s\times\gamma_\alpha)_{\mathfrak{q}}$ in $\tilde{L}$. Then

$$\mathfrak{q}_\alpha = l_\alpha + \mathfrak{u} + \mathfrak{u}_0 = l_\alpha + \mathfrak{u}_\alpha$$

satisfies the properties with respect to $s\times\gamma_\alpha$. In the proof of Lemma 7.8 in [V3], we find the following identity:

$$\begin{aligned}&\mathrm{mult}[\overline{X}^L((s\times\gamma_\alpha)_{\mathfrak{q}}), H_q(\mathfrak{u}, \overline{X}(s\times\delta))] \\ &\quad = \mathrm{mult}[\overline{X}^{L_\alpha}((s\times\gamma_\alpha)_{\mathfrak{q}_\alpha}), H_{q+\dim(\mathfrak{u}_0\cap\mathfrak{p})}(\mathfrak{u}_\alpha, \overline{X}(s\times\delta))] \\ &\qquad - \mathrm{mult}[\overline{X}^L((s\times\gamma)_{\mathfrak{q}}), H_{q+1}(\mathfrak{u}, \overline{X}(s\times\delta))].\end{aligned}$$

As above, straightforward computations gives the formula we want.

Suppose we are now in case I, (iv). Notice that the formula we want can be derived from the formulas in case I, (iii) and case II, (iii). We have already obtained the first one, and the second can be proved the same way. Thus we are done.

Formulas in case III are obtained similarly. □

Finally, we can state the main result of this section.

Theorem 7.13. *The value at* 1 *of the Kazhdan-Lusztig polynomials* $P_{\gamma,\delta}$ *gives the multiplicity of* $\overline{X}(\gamma)$ *in* $X(\delta)$. *More precisely, with the notations of Equation 1.3,*

$$M(\gamma,\delta)=(-1)^{l(\delta)-l(\gamma)}P_{\gamma,\delta}(1).$$

This is a consequence of the previous theorem, and [ABV, Theorem 16.21(d)].

8. Proof of Theorem 6.34

In this section, we at last prove Theorem 6.34. Given the theory of Section 7, the argument is essentially the same as the linear case.

Recall the $\mathbb{Z}[u,u^{-1}]$ algebra $\mathcal{H}$ (Proposition 7.5) and the $\mathcal{H}$ module $\mathcal{M}$. Partition the set of λ_a-pseudocharacters according to the action of the central element $\mathbf{x}$ and write

$$\mathcal{P}_{\lambda_a}=\mathcal{P}^+_{\lambda_a}\coprod\mathcal{P}^-_{\lambda_a}.$$

Correspondingly, as $\mathbb{Z}[u,u^{-1}]$ modules, we can write $\mathcal{M}=\mathcal{M}^+\oplus\mathcal{M}^-$. For orientation, we include the following result.

Proposition 8.1. *As an* $\mathcal{H}(W_{\mathrm{int}})$ *module (Proposition 7.5),* $\mathcal{M}$ *decomposes as*

$$\mathcal{M}=\mathcal{M}^+\oplus\mathcal{M}^-.$$

Proof. This follows from the formulas of Definition 7.4. The point is that when the reflection s corresponds to an integral root (i.e. a short one), the terms on the right side of the definition of $T_s\gamma$ all have the same central character as γ. □

Now we head toward the duality theorem. Define the dual $\mathbb{Z}[u,u^{-1}]$ module

$$\mathcal{M}^*=\mathrm{Hom}_{\mathbb{Z}[u,u^{-1}]}(\mathcal{M},\mathbb{Z}[u,u^{-1}]).$$

The transposed action of $\mathcal{H}$ on $\mathcal{M}$ defines an action of $\mathcal{H}^{\mathrm{op}}$ on $\mathcal{M}^*$; choosing an antiautomorphism of $\mathcal{H}$ (identifying $\mathcal{H}$ with $\mathcal{H}^{\mathrm{op}}$) makes $\mathcal{M}$ an $\mathcal{H}$ module. Concretely, for $\mu\in\mathcal{M}^*$ and for a simple reflection s we define

$$T_s\cdot\mu=[-u(T_s)^{-1}]^{tr}\cdot\mu; \tag{8.1}$$

here the invertibility of T_s follows directly from Proposition 7.5.

Write $\{\mu_\gamma\mid\gamma\in\mathcal{P}_{\lambda_a}\}$ for the basis of $\mathcal{M}^*$ dual to the basis $\mathcal{P}_{\lambda_a}$ of $\mathcal{M}$, and recall the bijection

$$\begin{gathered}\mathcal{P}^+_{\lambda_a}\longrightarrow\mathcal{P}^-_{\lambda_a}\\ \gamma\longrightarrow\check{\gamma}\end{gathered}$$

of Definition 6.31.

Proposition 8.2. *The* $\mathbb{Z}[u,u^{-1}]$ *linear isomorphism*

$$\begin{gathered}\mathcal{M}\longrightarrow\mathcal{M}^*\\ \gamma\longrightarrow\mu_{\check{\gamma}}\end{gathered}$$

is an isomorphism of $\mathcal{H}$ *modules.*

Proof. We have defined the $\mathcal{H}$ module structures on $\mathcal{M}$ and $\mathcal{M}^*$ explicitly (Definition 7.4 and Equation 8.1), and we have explicitly defined the map $\gamma\mapsto\check{\gamma}$ (Definition 6.31). So the verification of the theorem amounts to some rather serious bookkeeping. We omit the details. The reader is advised to consider the example of $Mp(4,\mathbb{R})$ (Example 6.32). □

The proof of Theorem 6.34 now proceeds exactly as the proof of Theorem 13.13 in [V4]. Consequently, we have the following stronger result.

Theorem 8.3. *Recall the Kazhdan-Lusztig polynomials $P_{\gamma\delta}$ of Section 7. The inverse of the matrix*

$$(P_{\gamma\delta})_{\gamma,\delta\in\mathcal{P}_{\lambda_a}}$$

is the matrix

$$(\epsilon_{\gamma\delta}P_{\check{\delta}\check{\gamma}})_{\gamma,\delta\in\mathcal{P}_{\lambda_a}};$$

here $\epsilon_{\gamma\delta}$ is defined as in Theorem 6.34.

References

[A] Adams, J., Lifting of Characters on Orthogonal and Metaplectic Groups, *Duke Math. J.*, **92**(1998), no. 1, 129–178. MR **99h:**22014

[AB] Adams, J. and Barbasch, D., Genuine Representations of the Metaplectic Group, *Compositio Math.*, **113**(1998), no. 1, 23–60. MR **99h:**22013

[ABV] Adams, J., D. Barbasch, and D. A. Vogan, Jr., *The Langlands Classification and Irreducible Characters for Real Reductive Groups*, Progress in Math, Birkhäuser (Boston), **104**(1992). MR **93j:**22001

[BBD] Beilinson, A., J. Bernstein and P. Deligne, Faisceaux Pervers, *Astérisque*, **100**(1982), 5–171. MR **86g:**32015

[GKM] Goresky, M., R. Kottwitz, and R. MacPherson, Discrete series characters and the Lefschetz formula for Hecke operators, *Duke Math. J.*, **89**(1997), no. 3, 477–554. MR **99e:**11064a; MR **99e:**11064b

[H] Herb, R., Discrete series characters and two-structures, *Trans. Amer. Math. Soc.*, **350**(1998), no. 8, 3341–3369. MR **98k:**22058

[La] Langlands, R.P., On the classification of irreducible representations of real algebraic groups, in *Representation theory and harmonic analysis on semisimple Lie groups*, Math. Survey Monographs, Amer. Math. Soc. (Providence), **33**(1989), 101–170. MR **91e:**22017

[LV] Lusztig, G. and Vogan, D., Singularities of closures of K-orbits on flag manifolds, *Invent. Math.* **71**(1983), 365–379. MR **84h:**14060

[MS] Mars, J.G.M. and Springer, T. A., Hecke Algebra Representations related to spherical varieties, *Representation Theory*, **2**(1998), 33–69. MR **99h:**20072

[MaO] Matsuki, T. and Oshima, T. *Embeddings of discrete series into principal series*, in *The orbit method in representation theory (Copenhagen, 1988)*, Progress in Math., Birkhäuser(Boston), Boston, MA, **82**(1990), 147–175. MR **92d:**22020

[Mi1] Miličić, D., Intertwining functors and irreducibility of standard Harish-Chandra sheaves, in *Harmonic analysis on reductive groups (Brunswick 1989)*, Progr. Math., Birkhäuser (Boston), **101**(1991), 209–222. MR **94c:**22013

[Mi2] Miličić, D., *Notes on Localization Theory*, preprint, http://www.math.utah.edu/~milicic/

[R] Renard, D., Endoscopy for $Mp(2n,\mathbb{R})$, *Amer. J. Math.* **121**(1999), no. 6, 1215–1243. CMP 2000:05

[Tor] Torasso, P., Sur le Caractère de la Représentation de Shale-Weil de $Mp(n,\mathbb{R})$ and $Sp(n,\mathbb{C})$, *Math. Ann.*, **252**(1980), 53–86. MR **82a:**22019

[V1] Vogan, D. A., Jr., Irreducible characters of semisimple Lie groups I, *Duke Math. J.*, **46**(1979), no. 1, 61–108. MR **80g:**22016

[V2] Vogan, D. A., Jr., Irreducible characters of semisimple Lie groups II: the Kazhdan-Lusztig conjectures, *Duke Math. J.*, **46**(1979), no. 4, 805–859. MR **81f:**22024

[V3] Vogan, D. A., Jr., Irreducible characters of semisimple Lie groups III: Proof of Kazhdan-Lusztig conjecture in the integral case, *Invent. Math.*, **71**(1983), no. 2, 381–417. MR **84h:**22036

[V4] Vogan, D. A., Jr., Irreducible characters of semisimple Lie groups IV: character multiplicity duality, *Duke Math. J.*, **49**(1982), no. 4, 943–1073. MR **84h:**22037

[Vgr] Vogan, D. A., Jr., *Representations of Real Reductive Lie Groups*, Progress in Math., Birkhäuser(Boston), **15**(1981). MR **83c:**22022

[Ya] Yamamoto, A., Orbits in the flag variety and images of the moment map for classical groups I, *Representation Theory*, **1**(1997), 329–494. MR **98j:**22024

University of Poitiers, Laboratoire de Mathématiques, BP 179, 86960 Futuroscope Cedex, France

E-mail address: `renard@mathlabo.univ-poitiers.fr`

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

E-mail address: `ptrapa@math.ias.edu`

Current address: Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138