ONE-DIMENSIONAL REPRESENTATIONS OF $U(p,q)$ AND THE HOWE CORRESPONDENCE

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ABSTRACT. We explicitly determine the theta lifts of all one-dimensional representations of $U(p,q)$ in terms of Langlands parameters, and determine exactly which lifts are unitary. Moreover, we show that such a lift is unitary if and only if it is a weakly fair derived functor modules of the form $A_{\mathfrak{q}}(\lambda)$. Finally we show that the correspondence of unitary representations behaves well with respect to associated cycles.

1. INTRODUCTION

Let (G,G') be a reductive dual pair in $Sp(2n,\mathbb{R})$. Let $\widetilde{G}$ and $\widetilde{G}'$ be the preimages of G and G' in $Mp = Mp(2n,\mathbb{R})$, the connected double cover of Sp. Write $\mathrm{Irr}(\widetilde{G})$ for the set of equivalence classes of irreducible Harish-Chandra modules for G, and likewise for $\mathrm{Irr}(\widetilde{G}')$. Two representations $\pi \in \mathrm{Irr}(\widetilde{G})$ and $\pi' \in \mathrm{Irr}(\widetilde{G}')$ are said to correspond if $\pi \otimes \pi'$ is a quotient of a fixed oscillator representation for Mp; in this case π and π' are said to occur in the correspondence. Howe proved that the map $\pi \mapsto \pi'$ is well-defined and bijective when restricted to those representations which occur [H5]. Hence we obtain a map

$$\theta : \mathrm{Irr}(\widetilde{G}) \longrightarrow \mathrm{Irr}(\widetilde{G}') \cup \{0\},$$

where $\theta(\pi) = 0$ if π does not occur in the correspondence. This map depends on a choice of one of the two oscillator representations for Mp.

Because of its role in the construction of automorphic forms, it is of considerable interest to compute the map θ as explicitly as possible. This has been accomplished in some cases, most notably when G and G' are complex [AB1]; when, roughly speaking, G and G' have the same size [M], [AB2], [P1], [P2]; whenever $n \geq p+q$ for the $(Sp(p,q), O^*(2n))$ dual pair [LPTZ]; and for type II dual pairs [M], [AB1], [LPTZ]. In complete generality, very little is known explicitly. Among the available abstract results, perhaps the most powerful asserts that when (G,G') is a type I dual pair in the stable range (i.e., roughly speaking, when G' is twice as large as G), then every representation of G occurs and, moreover, if π is unitary, then so is $\theta(\pi')$ ([Li1], [M]).

The present paper deals with the dual pair $\big(U(p,q), U(r,s)\big)$ and gives a complete description of $\theta(\chi)$, whenever χ is a one-dimensional representation of $U(p,q)$. It is important to clarify what we mean by a 'complete description' in this context. We determine exactly when $\theta(\chi) \neq 0$ in terms of the well-known (and explicit) correspondence of K-types in the space of joint harmonics (see Section 3 for the relevant definitions). In this case we give a description of the Langlands parameters of $\theta(\chi)$ (see Section 8); these representations will generally be very singular. While Langlands parameters are, in some sense, complete information, they fall short of determining the unitarity of $\theta(\chi)$. Worse yet, there exist nice families of singular unitary representations whose Langlands parameters do not look like families. To remedy this, we completely determine the unitarity of $\theta(\chi)$. In the cases when $\theta(\chi)$ is in fact unitary, we explicitly identify it as a derived functor module (more precisely, as a weakly fair $A_{\mathfrak{q}}(\lambda)$ module — see Section 2.3). This computation is very clean, and clearly indicates the family of unitary representations obtained as theta lifts of one-dimensional representations. The following summarizes these results; see Section 4 for more complete statements.

Theorem 1.1. *Let $\chi \in \mathrm{Irr}(\widetilde{U}(p,q))$ be a one-dimensional representation. Then $\theta(\chi) \neq 0$ if and only if χ occurs in the space of joint harmonics. There is a simple procedure to specify the Langlands parameters of $\theta(\chi)$. Moreover, $\theta(\chi)$ is non-zero and unitary if and only if $\theta(\chi)$ is a weakly fair derived functor module of the form $A_{\mathfrak{q}}(\lambda)$; here $\mathfrak{q}$ and λ are explicitly computable.*

The precise statement in Theorem 4.1 below is notable both for the simplicity of its answer, and because it suggests how to lift derived functor modules for arbitrary dual pairs. (We return to this below.) We now briefly sketch our proof. The first step is to determine occurrence, i.e., for fixed p and q and a genuine one-dimensional representation χ of $\widetilde{U}(p,q)$, find all r, s such that $\theta(\chi) \neq 0$. (To express the dependence of the lift on r and s, we denote this representation $\theta_{r,s}(\chi)$.) Using the mixed model [H1] of ω one can show that whenever a representation $\pi \in \mathrm{Irr}(\widetilde{U}(p,q))$ has a non-zero theta lift to $\widetilde{U}(r,s)$, then $\theta_{r+k,s+k}(\pi)$ will be non-zero for all positive integers k. This fact is called persistence (Kudla). It is convenient to divide the collection of groups $\{U(r,s)\}$ into Witt towers: for any integer k, we call the set $\{U(r,s) | r-s=k\}$ the k^{th} Witt tower. Because every representation occurs in the stable range, persistence implies that for any $\pi \in \mathrm{Irr}(\widetilde{U}(p,q))$ there is a well-defined first occurrence in each Witt tower; this is the non-zero theta lift of π to the group of least rank. So in order to determine all occurrences of χ it is sufficient to determine the first occurrence in each Witt tower. Because of the following result (which is essentially our Theorem 5.11), the same is true in many cases for determining the Langlands parameters of each lift.

Theorem 1.2. *Suppose $p+q \leq r+s$, and π is any irreducible admissible representation of $\widetilde{U}(p,q)$ such that $\theta_{r,s}(\pi) \neq 0$. Then there is a simple procedure to determine the Langlands parameters of $\theta_{r+k,s+k}(\pi)$ from those of $\theta_{r,s}(\pi)$, and vice versa.*

For reasons made apparent below, it is important to single out a particular Witt tower. It follows from the results of [P1] and [P2] that for any $\pi \in \mathrm{Irr}(\widetilde{U}(p,q))$, the first occurrence is at rank less than or equal to $p+q$ in at most one Witt tower, which (if it exists) we will call the "good" tower. In any event, Theorem 1.2 reduces Theorem 1.1 to determining the first occurrence of a given character χ in terms of its Langlands parameters, checking which lifts are $A_{\mathfrak{q}}(\lambda)$ modules (hence unitary), and finally determining which lifts are nonunitary.

We begin by tabulating the Langlands parameters of theta lifts of one-dimensional representations of the compact group $\widetilde{U}(p,0)$. Because the Howe correspondence for compact dual pairs is well-known (e.g. [EHW], [A]), this is reasonably straightforward. Since every one-dimensional representation of $\widetilde{U}(p,q)$ is the Langlands subquotient of a representation induced from a one-dimensional representation on a cuspidal parabolic subgroup with Levi factor $\widetilde{U}(p-q,0) \times (\mathbb{C}^{\times})^q$, we can use the compact case, together with the induction principle, to determine all early occurrences of non-trivial χ in the good tower, as well as all first occurrences outside the good tower (for any χ) if they are below the stable range. This includes the Langlands parameters of the lifts. For the trivial representation this reduction to the compact case is not possible since this character lifts to *every* group in the good tower. Here we rely on the results of Lee and Zhu [LZ], who have determined all lifts of the trivial representation.

We are thus reduced to the case when the first occurrence (outside the good tower) is in the stable range. In fact we can compute all lifts of χ in the stable range as follows. We first produce a candidate for the stable range lift of χ; this is a very small (and generally singular) weakly fair $A_{\mathfrak{q}}(\lambda)$ module. We compute its associated varieties and conclude that it is of low rank. By the theory of low rank representations due to Howe and Li [Li1], we conclude that our candidate must occur. Then we use the infinitesimal character correspondence and the correspondence of associated varieties of primitive ideals (which is given by the orbit correspondence in this situation [Pr3]) to prove that our candidate lift is indeed the actual lift of χ. To finish the proof, we convert Langlands parameters to $A_{\mathfrak{q}}(\lambda)$ data (using [T]) and vice versa where appropriate, and use Parthasarathy's Dirac operator inequality [Pa] to prove that all lifts which are not of the form $A_{\mathfrak{q}}(\lambda)$ are non-unitary.

There are a couple of interesting complements worth mentioning. A conjecture of Vogan's asserts that any unitary representation of $U(p,q)$ with integral infinitesimal character is a weakly fair $A_{\mathfrak{q}}(\lambda)$ module. Since θ preserves the condition of having an integral infinitesimal character (see [Pr2]), and since unitarity is preserved under stable range lifting, Vogan's conjecture predicts that the stable range theta lift of a weakly fair $A_{\mathfrak{q}}(\lambda)$ module must again be a weakly fair $A_{\mathfrak{q}}(\lambda)$ module. (There is substantial evidence to make such prediction, with some obvious modifications[1], in the generality of any dual pair.) At any rate, Theorem 1.1 verifies this prediction for the simplest kind of weakly fair $A_{\mathfrak{q}}(\lambda)$ module for $U(p,q)$, i.e. the one dimensional ones.

A further interesting point is that in the stable range the computation of Theorem 1.1 behaves well with respect to associated varieties and the orbit correspondence. This is predicted by a conjecture of Howe's and is discussed in Section 6. But even more is true: the multiplicities in the associated cycle are also preserved (Corollary 6.5). (They are all one in our case.) This kind of phenomenon was first observed in [NOT]. The methods we use to compute multiplicities, which are a combination of asymptotic and geometric techniques, turn out to be quite useful. For instance, they allow one to recover the relevant results in [NOT] (Remark 6.6).

A final interesting feature of the correspondence is that the first occurrence in the good tower is always finite dimensional.

2. Preliminaries

2.1. **Square roots of characters.** Consider an arbitrary group G and a character χ of G. Define a two-fold covering of G (called the $\sqrt{\chi}$-covering or $\chi^{1/2}$-covering) as follows

$$\begin{aligned} G^{\sqrt{\chi}} := \{(g,z) \in G \times \mathbb{C} \mid z^2 = \chi(g)\} &\longrightarrow G \\ (g,z) &\longrightarrow g. \end{aligned}$$

Recall that a representation of $G^{\sqrt{\chi}}$ is called genuine if it does not factor to G. For instance, the character $\sqrt{\chi} = \chi^{1/2}$ defined by $\chi^{1/2}(g,z) = \chi(z)z$ is a genuine character of $G^{\sqrt{\chi}}$.

2.2. **Generalities.** Even though we will deal exclusively with $U(p,q)$, it is conceptually a little simpler to begin with a few more general results and definitions, which we will later apply to the $U(p,q)$ setting. So consider a linear real reductive group G, with maximal compact subgroup K. Assume that all Cartan subgroups of G are abelian. Write $\mathfrak{g}_o$ for the Lie algebra of G, $\mathfrak{g}$ for its complexification, and $G_{\mathbb{C}}$ for the corresponding complex group. Adopt similar notation for subgroups of G. Let τ denote the Cartan involution of G, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding complexified decomposition. Write $(\cdot,\cdot)$ for the trace form on $\mathfrak{g}$ (which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$) normalized so that short simple roots have length 2. Using the trace form, we will often identify $\mathfrak{g}$ and $\mathfrak{g}^*$. We use the same notation $(\cdot,\cdot)$ for the dual of the trace form.

Let $\mathcal{N}^*(\mathfrak{g})$ denote the cone of nilpotent elements in $\mathfrak{g}^*$, and set $\mathcal{N}^*(\mathfrak{p}) = \mathcal{N}^*(\mathfrak{g}) \cap \mathfrak{p}^*$. Recall that $K_{\mathbb{C}}$ acts with finitely many orbits on $\mathcal{N}^*(\mathfrak{p})$. Given a finite length $(\mathfrak{g},K)$ module X, let $\mathrm{AV}(X)$ denote the closed points of the support of the associated $S(\mathfrak{g})$ module attached to any K-invariant good filtration of X. This is a well-defined subvariety of $\mathfrak{p}^*$ which in fact is a (finite) union of closures of equidimensional $K_{\mathbb{C}}$ orbits on $\mathcal{N}^*(\mathfrak{p})$. If I is a primitive ideal in $\mathfrak{U}(\mathfrak{g})$, we write $\mathrm{AV}(I)$ for the associated variety of the Harish-Chandra bimodule $\mathfrak{U}(\mathfrak{g})/I$, and in this case $\mathrm{AV}(I)$ is the closure of a single nilpotent orbit on $\mathfrak{g}^*$. Finally one can refine the associated variety by keeping track of the rank of the corresponding associated graded object along the irreducible components of its support. The resulting invariant, which can be viewed as a linear combination of the irreducible components of the associated variety, is called the associated cycle; see [V5] for more details.

[1]Essentially the only modification is that one must also include representations cohomologically induced from minimal representations, and not just one dimensionals.

2.3. **Cohomological induction.** Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the complexification of a maximally compact τ-stable Cartan subalgebra $\mathfrak{h}_o$. Choose a τ-stable system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a τ-stable parabolic containing $\mathfrak{h}$ with $\Delta(\mathfrak{u}) \subset \Delta^+(\mathfrak{g}, \mathfrak{h})$. Let L denote the analytic subgroup of G corresponding to $\mathfrak{l} \cap \mathfrak{g}_o$. Let $2\rho(\mathfrak{u})$ denote the character of L defined by $l \mapsto \det(Ad(l)|_{\mathfrak{u}})$, and consider (see Section 2.1) the two-fold covering $L^{\sqrt{2\rho(\mathfrak{u})}}$. As a matter of notation, we set $L^{\rho(\mathfrak{u})} = L^{\sqrt{2\rho(\mathfrak{u})}}$. We let $\mathcal{R}^j$ denote the jth cohomological induction functor, normalized as in [V4, Chapter 5][2]. This functor takes genuine finite length $(\mathfrak{l}, (L \cap K)^{\rho(\mathfrak{u})})$ modules to finite length $(\mathfrak{g}, K)$ modules and *preserves* infinitesimal character: if a genuine $(\mathfrak{l}, (L \cap K)^{\rho(\mathfrak{u})})$ module X has infinitesimal character given by $\gamma \in \mathfrak{h}^*$, then so does $\mathcal{R}^j_{\mathfrak{q}}(X)$.

Suppose now that $\mathbb{C}_\lambda$ is a one dimensional genuine $(\mathfrak{l}, (L \cap K)^{\rho(\mathfrak{u})})$ module whose differential restricted to $\mathfrak{h}$ is given by $\lambda \in \mathfrak{h}^*$. Recall our fixed choice of positive roots Δ^+. We say that λ is in the (weakly) good range for $\mathfrak{q}$ if $\lambda + \rho(\mathfrak{l})$ is (weakly) dominant for Δ^+. We say that λ is in the (weakly) fair range for $\mathfrak{q}$ if λ is (weakly) dominant for Δ^+. Finally for $S = \dim(\mathfrak{u} \cap \mathfrak{k})$, we set

$$A_{\mathfrak{q}}(\lambda) = \mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}_\lambda).$$

Here are the main properties of these modules.

Theorem 2.1. (1) *$A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho(\mathfrak{l})$.*
(2) *If λ is in the good range for $\mathfrak{q}$, then $A_{\mathfrak{q}}(\lambda)$ is nonzero, irreducible, and unitary. The Langlands parameters of $A_{\mathfrak{q}}(\lambda)$ are explicitly computable.*
(3) *If λ is in the weakly good range, then $A_{\mathfrak{q}}(\lambda)$ is irreducible and unitary (but possibly zero).*
(4) *If λ is in the weakly fair range, then $A_{\mathfrak{q}}(\lambda)$ is unitary (but possibly reducible or zero).*
(5) *If $G = U(p,q)$ (or a square-root cover of $U(p,q)$) and λ is in the weakly fair range, then $A_{\mathfrak{q}}(\lambda)$ is irreducible and unitary (but possibly zero). The Langlands parameters (and in particular the nonvanishing) of $A_{\mathfrak{q}}(\lambda)$ are explicitly computable.*

Proof. The first statement is essentially built into the normalization of $\mathcal{R}_{\mathfrak{q}}$. The second, third, and fourth statements are due to Vogan, Zuckerman, and Vogan-Zuckerman (see [KnV], for instance, or the exposition of [T, Section 3]). The final statement summarizes the main results of [T] (see [T, Theorem 3.1] and [T, Theorem 7.9]). □

2.4. **The orbit correspondence for reductive dual pairs.** Fix a reductive dual pair $G' \times G'' \subset G := Sp(2n, \mathbb{R})$, consider $\mathfrak{g}' \oplus \mathfrak{g}'' \subset \mathfrak{g}$ and the corresponding complexification $\mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{g}''_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$. Assume (as we may) that $\mathfrak{p}', \mathfrak{p}'' \subset \mathfrak{p}$. Consider the minimal orbit $\mathcal{O}^{\min}_{\mathbb{C}}$ in $\mathcal{N}^*(\mathfrak{g}_{\mathbb{C}})$, and fix one of the two irreducible components (say $\mathcal{O}^{\min}$) of $\mathcal{O}^{\min}_{\mathbb{C}} \cap \mathfrak{p}^*$. Write $\mu'_{\mathbb{C}}$ and $\mu''_{\mathbb{C}}$ for the restriction map from $\mathcal{O}^{\min}_{\mathbb{C}}$ to $(\mathfrak{g}'_{\mathbb{C}})^*$ and $(\mathfrak{g}''_{\mathbb{C}})^*$. (More precisely, the inclusion of $\mathfrak{g}'_{\mathbb{C}}$ into $\mathfrak{g}_{\mathbb{C}}$ gives $\mathfrak{g}^*_{\mathbb{C}} \to (\mathfrak{g}'_{\mathbb{C}})^*$ which we then restrict to $\mathcal{O}^{\min}_{\mathbb{C}}$ to obtain $\mu'_{\mathbb{C}}$, and likewise for $\mu''_{\mathbb{C}}$.) Similarly write μ' and μ'' for the restriction of $\mathcal{O}^{\min}$ to $(\mathfrak{p}')^*$ and $(\mathfrak{p}'')^*$. It is easy to check that

$$\mu''_{\mathbb{C}} \circ (\mu'_{\mathbb{C}})^{-1}(\mathcal{N}^*(\mathfrak{g}'_{\mathbb{C}})) \subset \mathcal{N}^*(\mathfrak{g}''_{\mathbb{C}})$$
$$\mu'' \circ (\mu')^{-1}(\mathcal{N}^*(\mathfrak{p}')) \subset \mathcal{N}^*(\mathfrak{p}'').$$

In the stable range, the following result is well-known, and was observed independently by a number of people.

Proposition 2.2. *Assume (G', G'') is a type I dual pair in the stable range with G' the smaller member. Fix an orbit $\mathcal{O}'$ of $K'_{\mathbb{C}}$ on $\mathcal{N}^*(\mathfrak{p}')$. Then $\mu'' \circ (\mu')^{-1}(\overline{\mathcal{O}'})$ is the closure of a single orbit $\mathcal{O}''$ of $K''_{\mathbb{C}}$ on $\mathcal{N}^*(\mathfrak{p}'')$. Moreover, if we set $\mathcal{O}'_{\mathbb{C}} = G'_{\mathbb{C}} \cdot \mathcal{O}'$ and $\mathcal{O}''_{\mathbb{C}} = G''_{\mathbb{C}} \cdot \mathcal{O}''$, then $\mu''_{\mathbb{C}} \circ (\mu'_{\mathbb{C}})^{-1}(\overline{\mathcal{O}'_{\mathbb{C}}}) = \overline{\mathcal{O}''_{\mathbb{C}}}$.*

Remark 2.3. Recall that the definition of μ' and μ'' depends on a choice of minimal orbit $\mathcal{O}^{\min}$. The two different choices of $\mathcal{O}^{\min}$ lead to two different maps $\mathcal{O}' \mapsto \mathcal{O}''$.

[2]This normalization differs from the more standard one of, say, [KnV]. It has the advantage, however, of making the statement of Theorem 4.1 much cleaner.

Example 2.4. We compute the orbit $\mathcal{O}''$ (and hence also $\mathcal{O}''_{\mathbb{C}}$) appearing in the proposition for the dual pair $(G', G'') = \big(U(p,q), U(r,s)\big)$; here the stable range assumption means $\min\{r,s\} \geq p+q$. Recall (see e.g. [CM, Chapter 9]) that the orbits of $K'_{\mathbb{C}}$ on $\mathcal{N}^*(\mathfrak{p}')$ are parameterized by signed Young diagrams of signature (p,q). Such a diagram consists of a Young diagram of size $p+q$ with p boxes labeled '+' and q boxes labeled '−' arranged in an alternating fashion along rows, modulo the equivalence relation of interchanging rows of equal length. As a matter of notation, such a diagram will be denoted by the sequence $(1^+)^{n_1^+}(1^-)^{n_1^-}, \dots, (k^+)^{n_k^+}(k^-)^{n_k^-}, \dots$, where n_k^ϵ is the number of rows of length k that begin with a box labeled ϵ. For instance, the zero orbit for $U(p,q)$ is parameterized by $(1^+)^p(1^-)^q$ and the two minimal orbits are given by $(1^+)^{p-1}(1^-)^{q-1}(2^+)^1$ and $(1^+)^{p-1}(1^-)^{q-1}(2^-)^1$.

Now fix notation as in Proposition 2.2, and suppose $\mathcal{O}'$ is parameterized by

$$(1^+)^{n_1^+}(1^-)^{n_1^-}, \dots, (k^+)^{n_k^+}(k^-)^{n_k^-}, \dots.$$

Then (for one of the choices in Remark 2.3) the diagram parameterizing $\mathcal{O}''$ is obtained by adding a single box of the required sign to each row-end of the diagram parameterizing $\mathcal{O}'$, and arranging the remaining signs in length-one rows. More precisely, $\mathcal{O}''$ is parameterized by

$$(1^+)^N(1^-)^M(2^+)^{n_1^+}(2^-)^{n_1^-}, \dots, (k{+}1^+)^{n_k^+}(k{+}1^-)^{n_k^-}, \dots,$$

where N and M are uniquely specified so that the number of plus (resp. minus) signs in the resulting diagram is r (resp. s). Note that the stable range hypothesis insures that N and M are positive. (For the other choice of minimal orbit, the diagram for $\mathcal{O}''$ is obtained by adding signs to the beginning (rather than end) of each row in the diagram parameterizing $\mathcal{O}'$.)

Theorem 2.5 (Przebinda [Pr3]). *Retain the hypothesis and notation of Proposition 2.2, but exclude the case of $(Sp(2n,\mathbb{R}), O(2n,2n))$. If X' is a genuine irreducible unitary representation of $\widetilde{G}'$ such that* $\mathrm{AV}(\mathrm{Ann}(X)) = \overline{\mathcal{O}'_{\mathbb{C}}}$, *then* $\mathrm{AV}(\mathrm{Ann}(\theta(X))) = \overline{\mathcal{O}''_{\mathbb{C}}}$.

2.5. **Low rank representations.** Suppose G is a finite cover of a linear reductive group (so $\mathfrak{g}$ can be viewed as a subalgebra of matrices). In this case, we can define the rank of an element $A \in \mathfrak{g}$, denoted $\mathrm{rk}(A)$, to be the rank of A as a matrix. By identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ using the trace form, we can also define the rank of an element of $\mathfrak{g}^*$. Given a finite length $(\mathfrak{g}, K)$ module X, we define the rank of $\mathrm{AV}(X)$ as the maximal possible rank of an element in $\mathrm{AV}(X)$

$$\mathrm{rk}(\mathrm{AV}(X)) := \max\{\mathrm{rk}(A) \mid A \in \mathrm{AV}(X)\}.$$

Finally define the rank of X to be the minimum of the $\mathrm{rk}(\mathrm{AV}(X))$ and the real rank of G (which we henceforth denote $\mathrm{rk}(G)$),

$$\mathrm{rk}(X) = \min\{\mathrm{rk}(\mathrm{AV}(X)), \mathrm{rk}(G)\}.$$

A representation X is said to be of *low rank* if $\mathrm{rk}(X) < \mathrm{rk}(G)$ (i.e. if $\mathrm{rk}(\mathrm{AV}(X)) < \mathrm{rk}(G)$). If X is irreducible and unitary, theorems of Howe and Li imply that this definition is known to coincide with the Howe's original notion of rank [H3].

Theorem 2.6. [Li2] *Let G' be the isometry group of a nondegenerate sesquilinear Hermitian form on a vector space over a division algebra, and let X' be an irreducible low rank representation of G'. Then there exists a type I reductive dual pair (G, G') in the stable range (with G the smaller member), a character δ of G', and an irreducible unitary representation X of G such that*

$$\theta(X) = X' \otimes \delta.$$

2.6. **Generalities about $U(p,q)$ and covers.** Let p and q be non-negative integers and set $p+q = n$. Explicitly we define $G = U(p,q)$ to be the set of $n \times n$ complex matrices preserving the Hermitian form of signature (p,q) defined by the diagonal matrix with p diagonal entries $+1$ followed by q entries -1, and choose $K \cong U(p) \times U(q)$ to be the maximal compact subgroup of $U(p,q)$ embedded diagonally. Fix the diagonal torus T of K. Then the roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ (in obvious notation) are $\Delta = \Delta(\mathfrak{g} : \mathfrak{t}) = \{\pm(e_i - e_j) | 1 \leq i < j \leq n\}$. Fix a system of positive compact roots $\Delta_c^+ = \{e_i - e_j | 1 \leq i < j \leq p \quad \text{or} \quad p+1 \leq i < j \leq n\}$.

Consider the dual pair $U(p,q)\times U(r,s)\subset Sp(2(p+q)(r+s),\mathbb{R})$. In the notation of Section 2.1, its preimage in the metaplectic group is isomorphic to $U(p,q)^{\sqrt{\det^{r-s}}}\times U(r,s)^{\sqrt{\det^{p-q}}}$. When the context is clear, to simplify notation we will write $\widetilde{U}(p,q)$ for $U(p,q)^{\sqrt{\det^{r-s}}}$ and likewise for $\widetilde{U}(r,s)$. Again if the context is clear and H is a subgroup of $U(p,q)$, $H^{\sqrt{\det^{r-s}}}$ will be denoted $\widetilde{H}$ (and similarly for subgroups of $U(r,s)$).

Recall that the cover $\widetilde{U}(p,q)$ is split if and only if $r-s$ is even (and similarly for $\widetilde{U}(r,s)$ if $p-q$ is even). In this case, we will identify the genuine admissible dual of $\widetilde{U}(p,q)$ with the admissible dual of $U(p,q)$ (by tensoring with the nontrivial character of $\mathbb{Z}/2$). In general, the one-dimensional representations of $\widetilde{U}(p,q)$ are those of the form $\det^{\frac{k}{2}}$ with $k\in\mathbb{Z}$, and the genuine ones will be those with $k\equiv r-s(mod\,2)$. (Sometimes we will write $\det^{\frac{k}{2}}_{p,q}$ for the character $\det^{\frac{k}{2}}$ of $\widetilde{U}(p,q)$ in order to avoid confusion about the group in question.) Applied to the case of the torus, for instance, one concludes that if σ is a genuine irreducible representation of $\widetilde{K}\subset\widetilde{U}(p,q)$, then the highest weight of σ will have entries in $\mathbb{Z}+\frac{r-s}{2}$.

We will use the following terminology: If G_1 is a reductive Lie group with maximal compact subgroup K_1 and σ is an irreducible representation of K_1, we will refer to σ as a K_1-type, as a K-type for G_1, or, if there is no confusion likely about the group in question, simply as a K-type. Moreover, we will identify K-types with their highest weights. If X is a representation of G_1, we will refer to a lowest K-type (in the sense of Vogan) of σ as an LKT.

2.7. **Langlands parameters for $\widetilde{U}(p,q)$.** We use the parameterization of [V3] (see §3 of [P1] for more details). Let $\pi\in\mathrm{Irr}(\widetilde{U}(p,q))$. We can write π as the unique irreducible quotient of an induced representation $Ind_P^{\widetilde{U}(p,q)}(\rho\otimes\chi)$. Here $P=MN$ is a cuspidal parabolic subgroup of $\widetilde{U}(p,q)$ with Levi factor $M\cong\widetilde{U}(p-m,q-m)\times(\mathbb{C}^\times)^m$ for some nonnegative integer m, ρ is a genuine limit of discrete series representation of $\widetilde{U}(p-m,q-m)$, and χ is a character of $(\mathbb{C}^\times)^m$. For π to be genuine, ρ must be a genuine representation of $\widetilde{U}(p-m,q-m)$. We require that ρ and χ satisfy conditions F-1 (non-singularity w.r.t. simple compact roots) and F-2 (the non-parity condition) of [V3]. The representation ρ is determined by a Harish-Chandra parameter λ and a system of positive roots Ψ, and we sometimes write $\rho=\rho(\lambda,\Psi)$. If $m=p=q$ then ρ is one of the two characters of $\widetilde{U}(0,0)=\mathbb{Z}/2\mathbb{Z}$. For π to be genuine, ρ must be the sign representation. We write $sgn=\rho(\lambda_{sgn},\Psi_{sgn})$. The character χ may be given by an m-tuple of integers $\mu=(\mu_1,\mu_2,\dots,\mu_m)$ and an m-tuple of complex numbers $\nu=(\nu_1,\dots,\nu_m)$ in the following way:

$$\chi\left(\prod_{j=1}^m r_je^{i\theta_j}\right)=\prod_{j=1}^m r_j^{\nu_j}e^{i\mu_j\theta_j} \tag{2.7}$$

The representation π is uniquely determined by the parameters m, λ, Ψ, μ, and ν.

Definition 2.8. If π is an irreducible admissible representation of $\widetilde{U}(p,q)$, we call the parameters (m,λ,Ψ,μ,ν) (as above) the Langlands parameters of π. Given a set of parameters (m,λ,Ψ,μ,ν) satisfying conditions F-1 and F-2 of [V3], we let $\pi(m,\lambda,\Psi,\mu,\nu)$ denote the irreducible representation of $\widetilde{U}(p,q)$ with Langlands parameters (m,λ,Ψ,μ,ν).

Example 2.9. The Langlands parameters of $\det^{\frac{k}{2}}_{p,q}$ are (m,λ,Ψ,μ,ν) with

$$m=min\{p,q\}, \tag{2.10}$$

$$\lambda=\left(\tfrac{k+|p-q|-1}{2},\tfrac{k+|p-q|-3}{2},\dots,\tfrac{k-|p-q|+1}{2}\right), \tag{2.11}$$

$$\mu=(k,k,\dots,k), \tag{2.12}$$

$$\nu=(|p-q|+1,|p-q|+3,\dots,|p-q|+2q-1), \tag{2.13}$$

and Ψ is the system of roots uniquely determined by λ.

2.8. **Notation for Theorem 4.1.** We need to define precisely the notation that appears in Theorem 4.1 below. This is conceptually very simple (and doubtlessly overly detailed for the expert), but nonetheless requires a little careful bookkeeping because of the covers involved. In any case, this section should be read in conjunction with the statement of Theorem 4.1. So assume the dual pair $\big(U(p,q),U(r,s)\big)$ is fixed and adopt the notation of Section 2.6. Assume further that $r \geq p$ and $s \geq q$. Recall that the ($K_{\mathbb{C}}$ conjugacy classes of) τ-stable parabolic subalgebras of $\mathfrak{u}(r,s)$ are parameterized by sequences of pairs of integers $\{(p_1,q_1),\ldots,(p_l,q_l)\}$ such that $\sum_i p_i = r$ and $\sum_i q_i = s$. (See, for example, [T, Section 2.1].) For the statement of Theorem 4.1(1), we are interested in the parabolic $\mathfrak{q}$ corresponding to the sequence $\{(p,q),(r-p,s-q)\}$. The corresponding Levi subgroup of $\widetilde{U}(r,s)$ is then

$$L \simeq \big(U(p,q)\times U(r-p,s-q)\big)^{\det^{(p+q)/2}\otimes\det^{(p+q)/2}}.$$

A direct calculation shows that $2\rho(\mathfrak{u}) = \det^{r+s-(p+q)}\otimes\det^{p+q}$. For applications to cohomological induction, we are thus interested in representations (and in particular one dimensional characters $\mathbb{C}_\lambda$) of $L^{\rho(\mathfrak{u})}$, which are genuine for the covering coming from $\rho(\mathfrak{u})$. For such a character $\mathbb{C}_\lambda$, we can thus form $A_{\mathfrak{q}}(\lambda)$ (with notation as in Section 2.3).

We now make this a little more explicit. The simplest case occurs when $p+q \equiv r+s \equiv 0 \mod 2$ (and hence also $(p+q+r+s)\equiv 0 \mod 2$) . In this case $L \simeq \big(U(p,q)\times U(r-p,s-q)\big)\times\mathbb{Z}/2$ and $L^{\rho(\mathfrak{u})} \simeq L\times\mathbb{Z}/2$ again. So a typical character of $L^{\rho(\mathfrak{u})}$ looks like $\mathbb{C}_\lambda = (\det^a_{p,q}\otimes\det^b_{r-p,s-q}\otimes\epsilon_1\otimes\epsilon_2)$ with $a,b\in\mathbb{Z}$. This character is genuine for the $\rho(\mathfrak{u})$ cover if and only if ϵ_2 is the nontrivial character sgn of $\mathbb{Z}/2$, and the corresponding $A_{\mathfrak{q}}(\lambda)$ is genuine for $\widetilde{U}(r,s)$ if and only if $\epsilon_1 = \mathrm{sgn}$. In this case, we write

$$A_{\mathfrak{q}}(\det^a_{p,q}\otimes\det^b_{r-p,s-q})$$

for the more precise $A_{\mathfrak{q}}((\det^a_{p,q}\otimes\det^b_{r-p,s-q})\otimes\mathrm{sgn}\otimes\mathrm{sgn})$. For instance, when $b=0$ these genuine representations of $\widetilde{U}(r,s)$ appear in the statement of Theorem 4.1(1).

Next we consider the case when $p+q$ is even, $r+s$ is odd. In this case, one can check that

$$L^{\rho(\mathfrak{u})} \simeq \big(U(p,q)^{\sqrt{\det}}\times U(r-p,s-q)\big)\times\mathbb{Z}/2.$$

A typical character of $L^{\rho(\mathfrak{u})}$ is of the form $(\det^{a/2}\otimes\det^b\otimes\epsilon)$ with $a,b\in\mathbb{Z}$, which is genuine for the $\rho(\mathfrak{u})$ cover if and only if a is odd. The corresponding $A_{\mathfrak{q}}(\lambda)$ is genuine for $\widetilde{U}(r,s)$ if and only if $\epsilon = \mathrm{sgn}$, in which case we write

$$A_{\mathfrak{q}}(\det^{a/2}\otimes\det^b)$$

for the more precise $A_{\mathfrak{q}}(\det^{a/2}\otimes\det^b\otimes\mathrm{sgn})$. When $b=0$ and a is odd, for instance, these genuine representations of $\widetilde{U}(r,s)$ appear in the statement of Theorem 4.1(1).

A more interesting case is when $r+s\equiv p+q\equiv 1 \mod 2$. From the definitions,

$$L^{\rho(\mathfrak{u})} = \{(A,B,x,y)\mid (A,B)\in U(p,q)\times U(r-p,q-s), x,y\in\mathbb{C}^\times \text{ s.t. } \det(A)\det(B)=x^2, \det(B)=y^2\},$$

where projection on x defines a genuine character for the $\widetilde{U}(r,s)$ cover and projection on y defines a genuine character for the $\rho(\mathfrak{u})$ cover. The map $(A,B,x,y)\mapsto(A,B,(x/y),y)$ is an isomorphism onto

$$\{(A,B,z,y)\mid (A,B)\in U(p,q)\times U(r-p,q-s), z,y\in\mathbb{C}^\times \text{ s.t. } \det(A)=z^2, \det(B)=y^2\},$$

which is simply $U(p,q)^{\sqrt{\det}}\times U(r,s)^{\sqrt{\det}}$. With this identification, a typical character of L^ρ is of the form $\det^{a/2}\otimes\det^{b/2}$ with $a,b\in\mathbb{Z}$. One checks that this character is genuine for the $L^{\rho(\mathfrak{u})}$ cover if and only if exactly one of a and b is even and exactly one is odd, and the corresponding $A_{\mathfrak{q}}(\lambda)$ is genuine for $\widetilde{U}(r,s)$ if and only if b is even. When a is odd and $b=0$, for instance, these representations appear in the statement of Theorem 4.1(1).

Finally if $p+q$ is odd and $r+s$ is even, a similar argument shows

$$L^{\rho(\mathfrak{u})} \simeq \big(U(p,q)\times U(r-p,s-q)\big)^{\sqrt{\det}}\times\mathbb{Z}/2,$$

so that a typical character is of the form $(\det^{a/2} \otimes \det^{b/2} \otimes \epsilon)$ with $a, b \in Z$ both of the same parity. Such a character is genuine for the $\rho(\mathfrak{u})$ cover if $\epsilon = \mathrm{sgn}$ and in this case we write $A_{\mathfrak{q}}(\det^{a/2} \otimes \det^{b/2})$ for the more precise $A_{\mathfrak{q}}(\det^{a/2} \otimes \det^{b/2} \otimes \epsilon)$. This representation is genuine for $\widetilde{U}(r,s)$ if and only if a and b are even. For instance, when $b = 0$ such representations appear in the statement of Theorem 4.1(1).

Analogous notation applies to the modules that appear in Theorem 4.1(2–4) below. We leave it to the reader to supply the details of the above analysis for these cases.

3. The space of joint harmonics

The following discussion is in [H5]. Let (G, G') be a reductive dual pair in $Sp = Sp(2n, \mathbb{R})$, and let $\mathcal{F}$ be the Fock space of the oscillator representation of Mp. Recall that the $\widetilde{U}(n)$-finite vectors may be realized as the space of polynomials in n variables, in such a way that the action of $\widetilde{U}(n)$, and therefore, that of $\widetilde{K}$ and $\widetilde{K}'$ (here K and K' are maximal compact subgroups of G and G' respectively), preserves the degree. This allows us to associate to each $\widetilde{K}$- and $\widetilde{K}'$-type occurring in $\mathcal{F}$ a degree, which is the minimal degree of polynomials in the isotypic subspace.

There is a $\widetilde{K} \times \widetilde{K}'$-invariant subspace $\mathcal{H}$ of $\mathcal{F}$, the space of joint harmonics, with the following properties.

Theorem 3.1 (Howe). *There is a one-one correspondence of $\widetilde{K}$- and $\widetilde{K}'$-types on $\mathcal{H}$ with the following properties. Suppose π and π' are irreducible admissible representations of $\widetilde{G}$ and $\widetilde{G}'$ respectively, and $\pi \leftrightarrow \pi'$ in the correspondence for the dual pair (G, G'). Let σ be a $\widetilde{K}$-type occurring in π, and suppose that σ is of minimal degree among the $\widetilde{K}$-types of π. Then σ occurs in $\mathcal{H}$. Let σ' be the $\widetilde{K}'$-type which corresponds to σ in $\mathcal{H}$. Then σ' is a $\widetilde{K}'$-type of minimal degree in π'.*

Since one-dimensional representations contain only one K-type, Theorem 3.1 implies that in order for this representation to occur in the correspondence, the corresponding K-type must occur in the space of joint harmonics. We now determine for which choices of r and s the one-dimensional representation $\chi = \det^{\frac{k}{2}}$ occurs in the space $\mathcal{H}$. Let us first recall the (well known) correspondence of K-types in $\mathcal{H}$ (see [P1], e.g.).

Lemma 3.2. *The correspondence of K-types for $\widetilde{U}(p,q)$ and $\widetilde{U}(r,s)$ in the space of joint harmonics for the dual pair $\big(U(p,q), U(r,s)\big)$ is given as follows:*

(1) *If σ is a K-type for $\widetilde{U}(p,q)$, then σ occurs in $\mathcal{H}$ if and only if σ is of the form*

$$(3.3)\quad (\overbrace{\tfrac{r-s}{2}, \dots, \tfrac{r-s}{2}}^{p}; \overbrace{\tfrac{s-r}{2}, \dots, \tfrac{s-r}{2}}^{q}) + (a_1, a_2, \dots, a_x, 0, \dots, 0, b_1, \dots, b_y; c_1, c_2, \dots, c_z, 0, \dots, 0, d_1, \dots, d_v),$$

with $x + v \leq r$ and $y + z \leq s$. Then $\mu \leftrightarrow \mu'$, where μ' is the K-type for $\widetilde{U}(r,s)$ given by

$$(3.4)\quad (\overbrace{\tfrac{p-q}{2}, \dots, \tfrac{p-q}{2}}^{r}; \overbrace{\tfrac{q-p}{2}, \dots, \tfrac{q-p}{2}}^{s}) + (a_1, a_2, \dots, a_x, 0, \dots, 0, d_1, \dots, d_v; c_1, c_2, \dots, c_z, 0, \dots, 0, b_1, \dots, b_y).$$

(2) *If σ is a K-type for $\widetilde{U}(p,q)$ which occurs in the oscillator representation, given by*

$$(3.5)\quad (\underbrace{\tfrac{r-s}{2}, \dots, \tfrac{r-s}{2}}_{p}; \underbrace{\tfrac{s-r}{2}, \dots, \tfrac{s-r}{2}}_{q}) + (x_1, x_2, \dots, x_p; y_1, \dots, y_q),$$

then the degree of σ (for the dual pair $\big(U(p,q),U(r,s)\big)$) is given by

$$\sum_{i=1}^{p}|x_i|+\sum_{i=1}^{q}|y_i|. \tag{3.6}$$

Remark 3.7. Notice that the degree of a K-type depends on the dual pair under consideration. In particular, for the dual pair $\big(U(p,q),U(r,s)\big)$, the degree of a K-type for $\widetilde{U}(p,q)$ depends on $r-s$.

Using Lemma 3.2, it is now easy to determine when χ occurs in $\mathcal{H}$. Table 1 summarizes the result. Let χ be the character $\det^{\frac{k}{2}}$ of $\widetilde{U}(p,q)$. We consider the dual pair $\big(U(p,q),U(r,s)\big)$, and let $w=r-s$. Recall that we must have $w\equiv k(mod\,2)$ for χ to be genuine. (For $p\geq q$ and $k=0$, Table 1 is summarized in Figure 4.1. For $p\geq q$ and $k>0$, see Figure 4.2.)

k	w	r,s
$k>0$	$w=k$	$s\geq q$
	$w=-k$	$r\geq p$
	$-k<w<k$	$r\geq p, s\geq q$
	$\lvert w\rvert>k$	stable range
$k=0$	$w=0$	all cases
	$w\neq 0$	stable range
$k<0$	$w=k$	$r\geq q$
	$w=-k$	$s\geq p$
	$-k<w<k$	$r\geq q, s\geq p$
	$\lvert w\rvert>k$	stable range

TABLE 1. Occurrence of χ in the space of joint harmonics

4. THE MAIN THEOREM

We now state our main theorem. As in the last section, we look at the theta lifts of the one-dimensional representation $\chi=\det^{\frac{k}{2}}$ of $\widetilde{U}(p,q)$. For $pq=0$, this is just a restatement (and refinement[3]) of some of the results of [A]. For the derived functor module notation, the reader is referred to Section 2.8.

Theorem 4.1. *Let $\chi=\det^{\frac{k}{2}}$ be a one-dimensional representation of $\widetilde{U}(p,q)$, and suppose r and s are such that χ occurs in $\mathcal{H}$ for the dual pair $\big(U(p,q),U(r,s)\big)$ (see Table 1 and Figures 4.1 and 4.2).*

(1) *If $k\geq 0$, $r\geq p$, and $s\geq q$ then*
$$\theta_{r,s}(\chi)=A_{\mathfrak{q}}(\det{}^{\frac{k}{2}}_{p,q}\otimes \mathbb{1}_{r-p,s-q}).$$
(2) *If $k\leq 0$, $r\geq q$, and $s\geq p$, then*
$$\theta_{r,s}(\chi)=A_{\mathfrak{q}}(\mathbb{1}_{r-q,s-p}\otimes \det{}^{\frac{k}{2}}_{q,p}).$$
(3) *If $pq\neq 0$ and we are neither in Case (1) nor in Case (2), then $\theta_{r,s}(\chi)$ is nonzero and nonunitary.*

(4) *Suppose $pq=0$ and we are neither in Case (1) nor (2). (If we define $w:=r-s$, these hypotheses imply that either $w=k$ or $w=-k$.) We have the following computations.*

(a) *If $G=\widetilde{U}(p,0)$ and $w=k\geq 0$, let $t=p-r$. (Notice that $t>0$ by assumption.) Then*
$$\theta_{r,s}(\chi)=A_{\mathfrak{q}}(\det{}^{\frac{k+t}{2}}_{r,0}\otimes \det{}^{\frac{-t}{2}}_{0,s}).$$

[3]Some of the derived functor modules listed in [A] actually vanish.

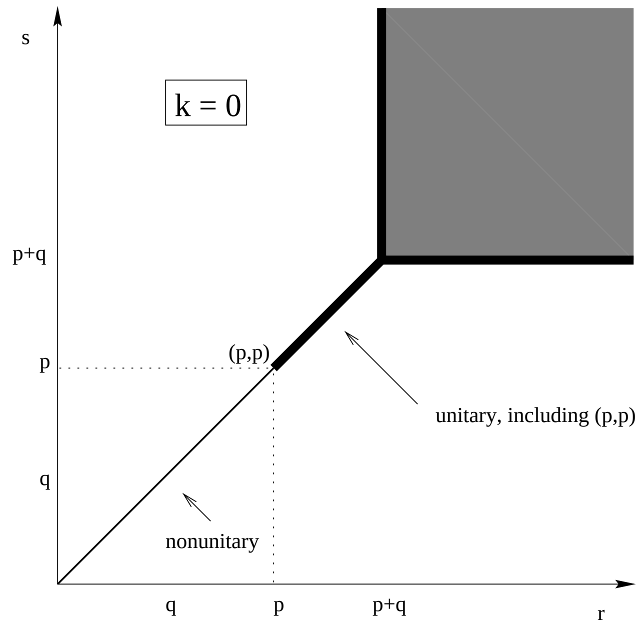

FIGURE 4.1. Occurrence of the trivial representation of $\widetilde{U}(p,q)$. The shaded areas and heavy lines indicate unitary lifts, and the lighter line segment indicates nonunitary lifts. (Theorem 5.1 reduces the general case to the indicated case of $p \geq q$.)

(b) *If $G = \widetilde{U}(p,0)$ and $w = k \leq 0$, let $t = p - s > 0$. Then*

$$\theta_{r,s}(\chi) = A_{\mathfrak{q}}(\det_{r,0}^{\frac{t}{2}} \otimes \det_{0,s}^{\frac{k-t}{2}}).$$

(c) *If $G = \widetilde{U}(0,q)$ and $w = -k \geq 0$, let $t = q - r > 0$. Then*

$$\theta_{r,s}(\chi) = A_{\mathfrak{q}}(\det_{0,s}^{\frac{t}{2}} \otimes \det_{r,0}^{\frac{k-t}{2}}).$$

(d) *If $G = \widetilde{U}(0,q)$ and $w = -k \leq 0$, let $t = q - s > 0$. Then*

$$\theta_{r,s}(\chi) = A_{\mathfrak{q}}(\det_{0,s}^{\frac{k+t}{2}} \otimes \det_{r,0}^{\frac{-t}{2}}).$$

Remark 4.2. It follows from [T, Theorem 7.9] that *all of the derived functor modules indicated in the theorem are nonzero.*

Corollary 4.3. *Let χ be a genuine one-dimensional representation of $\widetilde{U}(p,q)$. Then $\theta_{r,s}(\chi) \neq 0$ if and only if χ occurs in the space of joint harmonics for the dual pair $\big(U(p,q), U(r,s)\big)$ (see Table 1 and Figures 4.1 and 4.2).*

Proof. The "only if" statement is part of Theorem 3.1. The opposite direction follows from the computation in Theorem 4.1. □

For $k \geq 0$, Figures 4.1 and 4.2 indicate the occurrence of $\det^{k/2}$ in the theta correspondence for $\big(U(p,q), U(r,s)\big)$. Note that the general shape of the figures depends only on the relative sizes of $p-q$ and k.

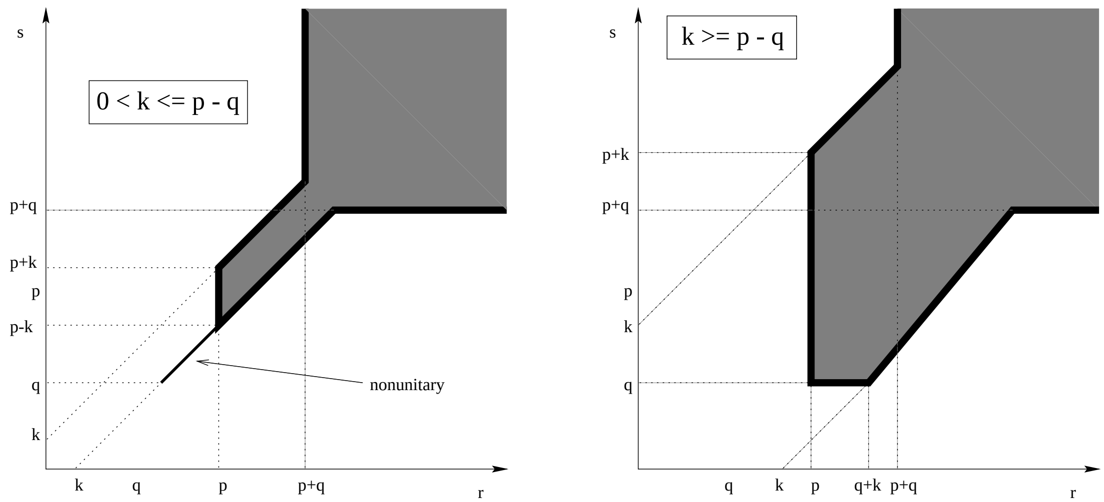

FIGURE 4.2. Occurrence of the $\det^{k/2}$ representation of $\widetilde{U}(p,q)$ for (respectively) $0 < k \leq p - q$ and $k \geq p - q > 0$. Again heavy lines and shaded regions indicate unitary lifts, and Theorem 5.1 reduces the general case to the indicated case of $p \geq q$.

5. General facts

In this section we recall some, and state and prove a few more, facts about the Howe correspondence which we need for the proof of Theorem 4.1

The groups $U(p,q)$ and $U(q,p)$ are canonically isomorphic, but their Howe correspondences are different. The following theorem relates the two correspondences and is in [P1].

Theorem 5.1. *Let π and π' be genuine irreducible admissible representations or $\widetilde{U}(p,q)$ and $\widetilde{U}(r,s)$ respectively, with π^* and π'^* the contragredient representations. Then*

(1) $\theta_{r,s}(\pi) = \pi'$ *if and only if* $\theta_{s,r}(\pi^*) = \pi'^*$;

(2) $\pi \leftrightarrow \pi'$ *in the correspondence for the dual pair* $\big(U(p,q), U(r,s)\big)$ *if and only if* $\pi \leftrightarrow \pi'$ *in the correspondence for the dual pair* $\big(U(q,p), U(s,r)\big)$.

A crucial part of our argument will rely on the induction principle ([Ku],[AB1],[P1]) which we state next.

Theorem 5.2 (Induction principle for $U(p,q)$)**.** *For $i = 1,2$, let $\pi_i \in \mathrm{Irr}(\widetilde{U}(p_i,q_i))$, $\sigma_i \in \mathrm{Irr}(GL(k_i,\mathbb{C}))$, and suppose that $\pi_1 \leftrightarrow \pi_2$ and $\sigma_1 \leftrightarrow \sigma_2$ in the correspondences for the dual pair $\big(U(p_1,q_1), U(p_2,q_2)\big)$ and $\big(GL(k_1,\mathbb{C}), GL(k_2,\mathbb{C})\big)$ respectively. Let χ_1 and χ_2 be the characters of $GL(k_1,\mathbb{C})$ and $GL(k_2,\mathbb{C})$ given by*

$$\begin{aligned} \chi_1(g_1) &= |\det(g_1)|^{p_2+q_2+k_2-p_1-q_1-k_1}, \quad \textit{and} \\ \chi_2(g_2) &= |\det(g_2)|^{p_1+q_1+k_1-p_2-q_2-k_2}, \quad \textit{for } g_i \in GL(k_i,\mathbb{C}). \end{aligned} \tag{5.3}$$

Let ω be the oscillator representation for the dual pair

$$\big(G_1, G_2\big) = \big(\widetilde{U}(p_1+k_1, q_1+k_1), \widetilde{U}(p_2+k_2, q_2+k_2)\big). \tag{5.4}$$

Then there are parabolic subgroups $P_i = M_iN_i$ of G_i with Levi factors $M_i \cong \widetilde{U}(p_i,q_i) \times GL(k_i,\mathbb{C})$, and a nonzero $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, K_1 \times K_2)$-map of the associated Harish-Chandra modules

$$\Phi : \omega \longrightarrow Ind_{P_1}^{G_1}(\pi_1 \otimes \sigma_1 \otimes \chi_1) \otimes Ind_{P_2}^{G_2}(\pi_2 \otimes \sigma_2 \otimes \chi_2). \tag{5.5}$$

Remark 5.6. If we formally define the oscillator representation of $\widetilde{Sp}(0,\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ to be the non-trivial character of this group, and the correspondences for $\big(U(p,q),U(0,0)\big)$ and $\big(GL(k,\mathbb{C}),GL(0,\mathbb{C})\big)$ (both dual pairs in $\widetilde{Sp}(0,\mathbb{R})$) to be $\mathbb{1} \leftrightarrow \mathbb{1}$, we can allow $p_i = q_i = 0$ or $k_i = 0$ for $i = 1$ or 2 in Theorem 5.2.

In order to be able to apply the induction principle successfully, we need to know something about the correspondence for the dual pairs $\big(GL(k,\mathbb{C}),GL(l,\mathbb{C})\big)$. For our purposes, it will be sufficient to consider the case $k = l$. Recall that the irreducible admissible representations of $GL(k,\mathbb{C})$ may be parametrized by pairs $(\mu,\nu) \in \mathbb{Z}^k \times \mathbb{C}^k$ [D]. Given such a pair, let $\rho_{\mu,\nu}$ be the unique irreducible quotient of

$$Ind_{MN}^{GL(k,\mathbb{C})}(\chi_{\mu,\nu}), \tag{5.7}$$

where MN is a parabolic subgroup of $GL(k,\mathbb{C})$ with Levi factor $M \cong (\mathbb{C}^\times)^k$ and N such that $\mathrm{Re}\langle\nu,\alpha\rangle \geq 0$ for all $\alpha \in \Delta(\mathfrak{n})$. The following Theorem is in [AB1]. We have accounted for the fact that the embedding of the groups there is different from ours (see §4 of [P1] for details).

Theorem 5.8 (Adams-Barbasch). *In the Howe correspondence for the dual pair* $\big(GL(k,\mathbb{C}),GL(k,\mathbb{C})\big)$, *every irreducible admissible representation of* $GL(k,\mathbb{C})$ *occurs. Explicitly, the correspondence is given by*

$$\rho_{\mu,\nu} \longleftrightarrow \rho_{\mu,-\nu}. \tag{5.9}$$

We can use the induction principle to determine what happens to a theta lift under persistence. This will reduce the proof of our theorem to the first occurrence, at least in most cases. But first we single out a large number of cases where the notions of LKT (in the sense of Vogan) and K-type of minimal degree (as in Theorem 3.1) coincide. This will help us pick out the correct constituent of the induced representation that we obtain when applying the induction principle.

Lemma 5.10. *Let π and π' be genuine irreducible admissible representations of $\widetilde{U}(p,q)$ and $\widetilde{U}(r,s)$ respectively, and suppose that $\theta_{r,s}(\pi) = \pi'$. Let σ' be a LKT of π'. If $r+s \geq p+q$ then σ' is of minimal degree in π'.*

Proof. By persistence, there is k such that $\theta_{p+k,q+k}(\pi') \neq 0$ and $p+q+2k = r+s$ or $r+s-1$. By Theorem 6.1 of [P1] and Proposition 3.12 of [P2], σ' is of minimal degree in π' for the dual pair $\big(U(p+k,q+k),U(r,s)\big)$. But the degrees of any K-type for $\widetilde{U}(r,s)$ for this dual pair and the dual pair $\big(U(p,q),U(r,s)\big)$ coincide, so the lemma follows. □

Theorem 5.11. *Let π be a genuine irreducible admissible representation of $\widetilde{U}(p,q)$. Suppose $r+s \geq p+q$ and $\theta_{r,s}(\pi) \neq 0$. Set $d = r+s-p-q$, and let l be a positive integer. Then*

$$\theta_{r,s}(\pi) = \pi(m,\lambda,\Psi,\mu,\nu) \textit{ if and only if } \theta_{r+l,s+l}(\pi) = \pi(m+l,\lambda,\Psi,\mu^l,\nu^l),$$

where $\mu^l = (\mu_1,\mu_2,\ldots,\mu_m,0,\ldots,0)$ *and* $\nu^l = (\nu_1,\ldots,\nu_m,d+1,d+3,\ldots,d+2l-1)$.

Proof. We first prove that $\theta_{r,s}(\pi) = \pi(m,\lambda,\Psi,\mu,\nu)$ implies $\theta_{r+l,s+l}(\pi) = \pi(m+l,\lambda,\Psi,\mu^l,\nu^l)$. Use Theorem 5.2 with $p_1 = p$, $q_1 = q$, $p_2 = r$, $q_2 = s$, $k_1 = 0$, and $k_2 = l$ to get a nonzero map

$$\omega \longrightarrow \pi \otimes I \tag{5.12}$$

where I is an induced representation of $\widetilde{U}(r+l,s+l)$ which has $\pi_l = \pi(m,\lambda,\Psi,\mu^l,\nu^l)$ as its unique LKT constituent, i. e., the LKT's of π_l and those of I coincide and occur with multiplicity one in I. The theta lift $\theta_{r+l,s+l}(\pi)$ is a constituent of I. To show that $\theta_{r+l,s+l}(\pi) = \pi_l$ we only need to show that $\theta_{r+l,s+l}(\pi)$ contains a LKT of π_l.

Let σ' be a LKT of π'. Then by Lemma 5.10, σ' is of minimal degree for the dual pair $\big(U(p,q),U(r,s)\big)$ in π'. By Theorem 3.1, σ' corresponds in $\mathcal{H}$ to a K-type σ of minimal degree in π. Since σ is also of minimal degree for the dual pair $\big(U(p,q),U(r+l,s+l)\big)$, it corresponds in the space of joint harmonics for this dual pair to a K-type σ_l of $\theta_{r+l,s+l}(\pi)$. We will show that σ_l is a LKT of π_l. We compute LKT's from Langlands parameters using the techniques described in [P1], §3. If $\mathfrak{t}_0$ is the diagonal Cartan subalgebra of $\mathfrak{u}(r,s)$, denote the restriction of $d\chi_{\mu,\nu}$ to $\mathfrak{t}$ by μ. Then we can consider $\lambda+\mu \in i\mathfrak{t}_0^*$. Write

$$\begin{aligned}\lambda+\mu = (&\underbrace{a_1,\dots,a_1}_{\alpha_1},\dots,\underbrace{a_x,\dots,a_x}_{\alpha_x},\underbrace{0,\dots,0}_{z},\underbrace{b_1,\dots,b_1}_{\beta_1},\dots,\underbrace{b_y,\dots,b_y}_{\beta_y};\\ &\underbrace{a_1,\dots,a_1}_{\gamma_1},\dots,\underbrace{a_x,\dots,a_x}_{\gamma_x},\underbrace{0,\dots,0}_{w},\underbrace{b_1,\dots,b_1}_{\delta_1},\dots,\underbrace{b_y,\dots,b_y}_{\delta_y}),\end{aligned} \tag{5.13}$$

with $a_1 > a_2 > \cdots > a_x > 0 > b_1 > \cdots > b_y$. Recall that $a_i, b_j \in \frac{1}{2}\mathbb{Z}$ for all i,j and $|z-w| \le 1$. Let $\alpha = \sum_{j=1}^{x} \alpha_j$, and define β, γ, and δ analogously. As in the proof of Lemma 5.10 let k be such that $p+q+2k = r+s$ or $r+s-1$ and recall that $\theta_{p+k,q+k}(\pi') \neq 0$. From [P1], §5, and [P2] we see that $p+k = \alpha+\delta+min\{z,w\}$ and $q+k = \beta+\gamma+\min\{z,w\}$.

Let $\mathfrak{q}(\lambda+\mu) = \mathfrak{l}(\lambda+\mu) \oplus \mathfrak{u}(\lambda+\mu)$ be the τ stable parabolic subalgebra of $\mathfrak{u}(r,s)_{\mathbb{C}}$ determined by $\lambda+\mu$. Then the LKT's of I_s are of the form $\Lambda = \lambda+\mu+\rho(\mathfrak{u}\cap\mathfrak{p}) - \rho(\mathfrak{u}\cap\mathfrak{k}) + \delta_L$, where $\rho(\mathfrak{u}\cap\mathfrak{p})$ and $\rho(\mathfrak{u}\cap\mathfrak{k})$ are one half the sums of the noncompact and compact roots in $\mathfrak{u} = \mathfrak{u}(\lambda+\mu)$ respectively, and δ_L is a fine weight of L. n Any LKT of π' will therefore be of the form

$$\begin{aligned}\sigma' = &\left(\frac{p-q}{2},\dots,\frac{p-q}{2};\frac{q-p}{2},\dots,\frac{q-p}{2}\right)\\ &+(\underbrace{A_1,\dots,A_1}_{\alpha_1},\dots,\underbrace{A_x,\dots,A_x}_{\alpha_x},\underbrace{0,\dots,0}_{z},\underbrace{B_1,\dots,B_1}_{\beta_1},\dots,\underbrace{B_y,\dots,B_y}_{\beta_y};\\ &\underbrace{C_1,\dots,C_1}_{\gamma_1},\dots,\underbrace{C_x,\dots,C_x}_{\gamma_x},\underbrace{0,\dots,0}_{w},\underbrace{D_1,\dots,D_1}_{\delta_1},\dots,\underbrace{D_y,\dots,D_y}_{\delta_y}),\end{aligned} \tag{5.14}$$

where

$$\begin{aligned}
A_i &= a_i + \tfrac{1}{2}\left(\sum_{j=1}^{i-1}(\alpha_j-\gamma_j) + \sum_{j=i+1}^{x}(\gamma_j-\alpha_j) - \alpha+\gamma-z+w\right) + \epsilon_i,\\
B_i &= b_i + \tfrac{1}{2}\left(\sum_{j=1}^{i-1}(\beta_j-\delta_j) + \sum_{j=i+1}^{y}(\delta_j-\beta_j) - \delta+\beta+z-w\right) + \eta_i,\\
C_i &= a_i + \tfrac{1}{2}\left(\sum_{j=1}^{i-1}(\gamma_j-\alpha_j) + \sum_{j=i+1}^{x}(\alpha_j-\gamma_j) + \alpha-\gamma+z-w\right) - \epsilon_i,\\
D_i &= b_i + \tfrac{1}{2}\left(\sum_{j=1}^{i-1}(\delta_j-\beta_j) + \sum_{j=i+1}^{y}(\beta_j-\delta_j) + \delta-\beta-z+w\right) - \eta_i,
\end{aligned} \tag{5.15}$$

and the ϵ_i and η_i are either 0 or $\pm\frac{1}{2}$, chosen so that A_i, B_i, C_i, and D_i are integers for all i. Using Lemma 3.2, we get that

$$\begin{aligned}\sigma = &\left(\frac{r-s}{2},\dots,\frac{r-s}{2};\frac{s-r}{2},\dots,\frac{s-r}{2}\right)\\ &+(\underbrace{A_1,\dots,A_1}_{\alpha_1},\dots,\underbrace{A_x,\dots,A_x}_{\alpha_x},\underbrace{0,\dots,0}_{p-\alpha-\delta},\underbrace{D_1,\dots,D_1}_{\delta_1},\dots,\underbrace{D_y,\dots,D_y}_{\delta_y};\\ &\underbrace{C_1,\dots,C_1}_{\gamma_1},\dots,\underbrace{C_x,\dots,C_x}_{\gamma_x},\underbrace{0,\dots,0}_{q-\beta-\gamma},\underbrace{B_1,\dots,B_1}_{\beta_1},\dots,\underbrace{B_y,\dots,B_y}_{\beta_y}),\end{aligned} \tag{5.16}$$

and

$$(5.17)\quad \sigma_l = \left(\frac{p-q}{2},\dots,\frac{p-q}{2};\frac{q-p}{2},\dots,\frac{q-p}{2}\right)$$
$$+\,(\underbrace{A_1,\dots,A_1}_{\alpha_1},\dots,\underbrace{A_x,\dots,A_x}_{\alpha_x},\underbrace{0,\dots,0}_{z+l},\underbrace{B_1,\dots,B_1}_{\beta_1},\dots,\underbrace{B_y,\dots,B_y}_{\beta_y};$$
$$\underbrace{C_1,\dots,C_1}_{\gamma_1},\dots,\underbrace{C_x,\dots,C_x}_{\gamma_x},\underbrace{0,\dots,0}_{w+l},\underbrace{D_1,\dots,D_1}_{\delta_1},\dots,\underbrace{D_y,\dots,D_y}_{\delta_y}).$$

Now

$$(5.18)\quad \lambda+\mu^l = (\underbrace{a_1,\dots,a_1}_{\alpha_1},\dots,\underbrace{a_x,\dots,a_x}_{\alpha_x},\underbrace{0,\dots,0}_{z+l},\underbrace{b_1,\dots,b_1}_{\beta_1},\dots,\underbrace{b_y,\dots,b_y}_{\beta_y};$$
$$\underbrace{a_1,\dots,a_1}_{\gamma_1},\dots,\underbrace{a_x,\dots,a_x}_{\gamma_x},\underbrace{0,\dots,0}_{w+l},\underbrace{b_1,\dots,b_1}_{\delta_1},\dots,\underbrace{b_y,\dots,b_y}_{\delta_y}),$$

so a straightforward computation yields that any LKT of π_l differs from σ_l by at most the choices for the ϵ_i and η_i. Since all such K-types have the same Vogan norm, and σ_l is a K-type occurring in I, it must be a LKT of I, hence of π_l. So we conclude that $\theta_{r,s}(\pi) = \pi(m,\lambda,\Psi,\mu,\nu)$ implies $\theta_{r+l,s+l}(\pi) = \pi(m+l,\lambda,\Psi,\mu^l,\nu^l)$.

To prove the other direction in the theorem, suppose $\theta_{r+l,s+l}(\pi) = \pi(m+l,\lambda,\Psi,\mu^l,\nu^l)$ and $\theta_{r,s}(\pi) \neq \pi(m,\lambda,\Psi,\mu,\nu)$. We can apply the above argument to $\theta_{r,s}(\pi)$ to compute $\theta_{r+l,s+l}(\pi)$ and explicitly check that $\theta_{r+l,s+l}(\pi) \neq \pi(m+l,\lambda,\Psi,\mu^l,\nu^l)$, a contradiction. □

Corollary 5.19. *Retain the notation and hypothesis of Theorem 4.1, but further assume that* $\theta_{r,s}(\chi) \neq 0$. *For a fixed a positive integer* l,

(1) *If* $k \geq 0$, $r \geq p$, *and* $s \geq q$ *then*
$$\theta_{r,s}(\chi) = A_{\mathfrak{q}}(\det_{p,q}^{\frac{k}{2}} \otimes \mathbb{1}_{r-p,s-q}).$$
if and only if
$$\theta_{r+l,s+l}(\chi) = A_{\mathfrak{q}}(\det_{p,q}^{\frac{k}{2}} \otimes \mathbb{1}_{r+l-p,s+l-q}).$$
(2) *If* $k \leq 0$, $r \geq q$, *and* $s \geq p$, *then*
$$\theta_{r,s}(\chi) = A_{\mathfrak{q}}(\mathbb{1}_{r-q,s-p} \otimes \det_{q,p}^{\frac{k}{2}}).$$
if and only if
$$\theta_{r+l,s+l}(\chi) = A_{\mathfrak{q}}(\mathbb{1}_{r+l-q,s+l-p} \otimes \det_{q,p}^{\frac{k}{2}}).$$

Proof. To deduce the present statement from Theorem 5.11, one needs to compute the Langlands parameters of the indicated derived functor modules and compare them with the ones appearing in the theorem. The Langlands parameters of all weakly fair derived functor modules for $U(p,q)$ are known as a consequence of [T, Theorem 7.9] (see also Remark 7.11 of that paper), and so the corollary amounts to a very complicated bookkeeping exercise. We omit the details. □

6. Proof of Theorem 4.1: stable range

Fix $\big(U(p,q),U(r,s)\big)$ such that $\min\{r,s\} \geq p{+}q$. Under this hypothesis we now prove Theorem 4.1. The main idea is as follows. Let X denote one of the representations appearing in Theorem 4.1(1–2) (assuming the strict inequality $\min\{r,s\} > p{+}q$). We first compute $\mathrm{AV}(X)$, and conclude X is of low rank. Using Theorem 2.6, we conclude $X = \theta(Y)$ for some representation Y of some $U(p',q')$ with $\min\{r,s\} \geq p'{+}q'$. Now the associated variety calculation shows that X is very small, and Theorem 2.5 (and the computation of Example 2.4) shows that Y must be very small — one dimensional in fact. A lowest K type argument shows $p = p'$ and $q = q'$, and infinitesimal character considerations specify Y as the character χ appearing in Theorem 4.1. Finally we deduce the case $\min\{r,s\} = p+q$ from Corollary 5.19.

Lemma 6.1. *Fix $\big(U(p,q), U(r,s)\big)$ such that $\min\{r,s\} \geq p+q$. Let X be one of the derived functor modules for $\widetilde{U}(r,s)$ appearing in Theorem 4.1(1–2). Then X is nonzero, and (in the notation of Example 2.4), $\mathrm{AV}(X)$ is the closure of the orbit parameterized by*

$$(1^+)^{r-(p+q)}(1^-)^{s-(p+q)}(2^+)^p(2^-)^q.$$

In particular, $\mathrm{AV}(\mathrm{Ann}(X))$ is the closure of the nilpotent orbit with $p+q$ Jordan blocks of size 2 and $r+s-2(p+q)$ blocks of size 1. Moreover,

$$\mathrm{rk}(X) = p+q,$$

and hence X is of low rank if and only if $\min\{r,s\} > p+q$.

Proof. As indicated in Remark 4.2, the nonvanishing follows from [T, Theorem 7.9]. The computation of $\mathrm{AV}(X)$ is given in Proposition 5.4 and Lemma 5.6 of [T], and the assertion about $\mathrm{AV}(\mathrm{Ann}(X))$ then follows trivially. The final assertion follows from the definition of low rank and the hypothesis that $\mathrm{rk}(G) = \min\{r,s\} > p+q$. $\square$

Proof of Theorem 4.1 in the stable range. First assume that we have the strict inequality $\min\{r,s\} > p+q$. Since X has low rank (by Lemma 6.1), Theorem 2.6 implies that there exists $\big(U(p',q'), U(r,s)\big)$ with $\min\{r,s\} \geq p'+q'$, a unitary representation Y of $\widetilde{U}(p',q')$, and a character δ of $\widetilde{U}(r,s)$ such that $\theta(Y) = X \otimes \delta$. Theorem 2.1(1) explicitly computes the infinitesimal character of X, and a simple argument with the infinitesimal character correspondence (see Proposition 9.1 below), shows that in fact δ must be trivial. So $X = \theta(Y)$. From Theorem 2.5 and Example 2.4, we conclude that $\mathrm{AV}(\mathrm{Ann}(Y))$ is the zero orbit. (If $\mathrm{Ann}(Y)$ had a different associated variety, Theorem 2.5 would imply that $\mathrm{AV}(\mathrm{Ann}(X))$ differed from the computation in Lemma 6.1, a contradiction.) Hence Y is a one dimensional representation of $\widetilde{U}(p',q')$. The infinitesimal character correspondence dictates that $Y = \det^{k/2}$, and so Theorem 4.1 (for $\min\{r,s\} > p+q$) is reduced to showing $p = p'$ and $q = q'$. Lemma 5.10 (together with Theorem 3.1 and Lemma 3.2) computes the LKT of X explicitly in terms of p' and q'. On the other hand, we can compute the LKT of X directly using [T]. A detailed (but elementary) check shows that for these two computations to match, we must have $p = p'$ and $q = q'$. So the theorem is proved for $\min\{r,s\} > p+q$.

The case $\min\{r,s\} = p+q$ now follows from Corollary 5.19. (The crucial hypothesis that $\theta_{r,s}(\det^{k/2}) \neq 0$ is automatically satisfied since we are in the stable range.) $\square$

As a consequence of Lemma 6.1 and Example 2.4, we deduce that the associated varieties of θ lifts in the stable range behave nicely with respect to the orbit correspondence (as predicted by a conjecture of Howe's).

Corollary 6.2. *Recall the notation of Section 2.4. Suppose $\min\{r,s\} \geq p+q$, and Y is a one dimensional genuine representation of $\widetilde{U}(p,q)$. Then*

$$\mathrm{AV}(\theta(Y)) = \mu'' \circ (\mu')^{-1}(\mathrm{AV}(Y)).$$

We now strengthen the conclusion of Corollary 6.2 by considering multiplicities in the associated cycle.

Proposition 6.3. *Let X be one of the derived functor modules for $\widetilde{U}(r,s)$ appearing in Theorem 4.1(1–2). Then $\mathrm{AV}(X)$ appears with multiplicity one in the associated cycle of X.*

Proof. Write $X = A_{\mathfrak{q}}(\lambda)$, and assume first that the λ is in the good range for $\mathfrak{q}$, so that the infinitesimal character $\lambda + \rho(\mathfrak{l})$ is dominant. Then X is the space of global sections of a $\mathcal{D}_{-(\lambda+\rho(\mathfrak{l}))}$ module, say $\mathcal{X}$, on the partial flag variety G/Q, whose support is a closed K orbit $\mathbb{O}_{\mathfrak{q}}$. The characteristic cycle of $\mathcal{X}$ consists of the single conormal bundle $T^*_{\mathbb{O}_{\mathfrak{q}}}(G/Q)$ with multiplicity one. Observe that

$$G \cdot (\mathfrak{u} \cap \mathfrak{p}) = G \cdot \mathfrak{u}. \tag{6.4}$$

To see this, note that Lemma 6.1 shows that $G \cdot (\mathfrak{u} \cap \mathfrak{p})$ is the closure of the orbit with $p+q$ Jordan blocks of size 2 and the remaining blocks all of size 1; this agrees with the well-known recipe for $G \cdot \mathfrak{u}$ (e.g. [CM, Chapter 6]). We thus conclude that the moment map μ for $T^*_{\mathbb{O}_{\mathfrak{q}}}(G/Q)$ is birational onto its image. Since μ is birational, the multiplicity in the associated cycle coincides with that in the characteristic cycle, and the proposition follows. (More details of this kind of argument can be found in [Ch].)

Still assuming λ is in the good range, place X in its coherent family Θ. Let λ' be any one dimensional $(\mathfrak{l}, L \cap K)$ module in the weakly fair range for $\mathfrak{q}$; so $\Theta(\lambda' + \rho(\mathfrak{l})) = A_{\mathfrak{q}}(\lambda')$ (e.g. [KnV, Lemma 8.29]). Let $p(\gamma)$ denote the multiplicity of $\mathrm{AV}(X)$ in the associated cycle of $\Theta(\gamma)$. Then p extends to a harmonic polynomial ([V1]). Whenever λ is in the good range, the preceding paragraph implies $p(\lambda+\rho(\mathfrak{l})) = 1$. But

$$\{\lambda + \rho(\mathfrak{l}) \mid \lambda \text{ is in the good range for } \mathfrak{q}\}$$

is Zariski dense in

$$S = \{\lambda + \rho(\mathfrak{l}) \mid \lambda \text{ is a one dimensional } (\mathfrak{l}, L \cap K) \text{ module}\}.$$

Hence p is identically one on S. In particular $p(\lambda'+\rho(\mathfrak{l})) = 1$, for any λ' in the weakly fair range for $\mathfrak{q}$, and the proposition follows. □

Corollary 6.5. *Suppose* $\min\{r, s\} \geq p+q$, *and* Y *is a one dimensional genuine representation of* $\widetilde{U}(p,q)$. *Then the multiplicity of* $\mathrm{AV}(Y)$ *in the associated cycle of* Y *coincides with the multiplicity of* $\mu'' \circ (\mu')^{-1}(\mathrm{AV}(Y))$ *in the associated cycle of* $\theta(Y)$. *(This common value is one.)*

Remark 6.6. The argument given in the proof of Proposition 6.3 is quite general, but it hinges on the key assumption in Equation (6.4). In practice, this condition is satisfied when considering stable range lifts. For instance consider the stable range lift of a representation, say F, of a compact group. These lifts have been tabulated as derived functor modules in [A], and their associated varieties are easily seen to be compatible with the orbit correspondence. On the other hand, the argument in the proof of Proposition 6.3 applies (with a few straight-forward modifications) to compute the multiplicity in the associated cycle of the lift, which turns out to be just the dimension of F. This provides a short proof of Theorem C in [NOT].

7. Compact dual pairs

We now consider the compact case of Theorem 4.1. In light of Theorem 5.1, we may restrict our attention to the dual pairs of the form $\big(U(p,0), U(r,s)\big)$. Table 2 lists the Langlands parameters of $\theta_{r,s}(\chi)$ for χ any one-dimensional representation of $U(p,0)$. Write $\chi = \det^{\frac{k}{2}}$ with $k \in \mathbb{Z}$, and let $w = r - s$. Again by Theorem 5.1, it is sufficient to determine the θ-lift for $k \geq 0$. For a given choice of w, we list only the first occurrence if it is at rank p or higher. If χ occurs at rank $p-1$ or lower (this happens in cases 1 and 2a), we give all lifts with $r + s \leq p + 1$. The Langlands parameters for lifts higher up in the Witt tower may then be obtained using Theorem 5.11. Suppose $\theta_{r,s}(\chi) = \pi(m, \lambda, \Psi, \mu, \nu)$. Then $\rho(\lambda, \Psi)$ is always a holomorphic discrete series or limit of holomorphic discrete series with Harish-Chandra parameter of the form

$$\lambda = (\alpha, \alpha - 1, \ldots, \alpha - r + m + 1; \beta, \beta - 1, \ldots, \beta - s + m + 1), \tag{7.1}$$

hence is determined by the values of α and β. The parameter μ is always of the form $\mu = (\mu_1, \mu_1, \ldots, \mu_1)$, and ν may be written $\nu = (\nu_1, \nu_1 + 2, \ldots, \nu_1 + 2m - 2)$. Consequently, the Langlands parameters of $\theta_{r,s}(\chi)$ are uniquely determined by the values of m, α, β, μ_1, and ν_1, which are given in the table. If $w < 0$, we write $d = -w$.

Case	r,s	m	α	β	μ_1	ν_1
1. $k=0, w=0$	$r=s\le\frac{p+1}{2}$	0	$\frac{p-1}{2}$	$\frac{-p+2s-1}{2}$	—	—
2. $k>0, w=k$						
a. $w\le p$	$r=s+k, 0\le s\le\frac{p-k+1}{2}$	0	$\frac{k+p-1}{2}$	$\frac{k-p+2s-1}{2}$	—	—
b. $w\ge p$	$r=k, s=0$	0	$\frac{k+p-1}{2}$	—	—	—
3. $k>0, -k\le w<k$						
a. $w\ge p$	$r=w, s=0$	0	$\frac{k+p-1}{2}$	—	—	—
b. $0\le w\le p\le\frac{k+w}{2}+1$	$r=p, s=p-w$	0	$\frac{k+p-1}{2}$	$\frac{p-w-1}{2}$	—	—
c. $0\le w<\frac{k+w}{2}+1\le p$	$r=p, s=p-w$					
(i) $p-\frac{k+w}{2}$ even		$\frac{2p-k-w}{4}$	$\frac{k+p-1}{2}$	$\frac{k-w-2}{4}$	$\frac{k-w}{2}$	1
(ii) $p-\frac{k+w}{2}$ odd		$\frac{2p-k-w-2}{2}$	$\frac{k+p-1}{2}$	$\frac{k-w}{4}$	$\frac{k-w}{2}$	2
d. $w<0, k-d\ge 2p$	$r=p, s=p+d$	0	$\frac{k+p-1}{2}$	$\frac{p+d-1}{2}$	—	—
e. $w<0, k-d\le 2p$	$r=p, s=p+d$					
(i) $p-\frac{k-d}{2}$ even		$\frac{2p+d-k}{4}$	$\frac{k+p-1}{2}$	$\frac{k+d-2}{4}$	$\frac{k+d}{2}$	1
(ii) $p-\frac{k-d}{2}$ odd		$\frac{2p+d-k-2}{4}$	$\frac{k+p-1}{2}$	$\frac{k+d}{4}$	$\frac{k+d}{2}$	2
4. $k\ge 0, \|w\|>k$						
a. $w>k, \frac{w+k}{2}\ge p$	$r=p+w, s=p$	p	$\frac{p+w-1}{2}$	—	$\frac{k-w}{2}$	$\frac{k+w}{2}-p+1$
b. $w>k, \frac{w+k}{2}\le p$	$r=p+w, s=p$					
(i) $p-\frac{w+k}{2}$ even		$\frac{2p+w+k}{4}$	$\frac{p+w-1}{2}$	$\frac{k-w-2}{4}$	$\frac{k-w}{2}$	1
(ii) $p-\frac{w+k}{2}$ odd		$\frac{2p+w+k-2}{4}$	$\frac{p+w-1}{2}$	$\frac{k-w}{4}$	$\frac{k-w}{2}$	2
c. $w<-k, d-k\le 2p$	$r=p, s=p+d$					
(i) $p-\frac{k-d}{2}$ even		$\frac{2p+d-k}{4}$	$\frac{k+p-1}{2}$	$\frac{k+d-2}{4}$	$\frac{k+d}{2}$	1
(ii) $p-\frac{k-d}{2}$ odd		$\frac{2p+d-k-2}{4}$	$\frac{k+p-1}{2}$	$\frac{k+d}{4}$	$\frac{k+d}{2}$	2
d. $w<-k, d-k\ge 2p$	$r=p, s=p+d$	p	—	$\frac{-p+d-1}{2}$	$\frac{k+d}{2}$	$\frac{d-k}{2}-p+1$

TABLE 2. Theta lifts of characters of $U(p,0)$ (Langlands parameters)

8. Proof of Theorem 4.1 (outside the stable range) and Langlands parameters

Now we are ready to tackle the proof of Theorem 4.1 outside the stable range. Let $pq\neq 0$, and suppose that k, r and s are such that $\det_{p,q}^{\frac{k}{2}}$ occurs in $\mathcal{H}$. As before, because of Theorem 5.1, we may assume that $p\ge q$ and $k\ge 0$. Write $l=p-q$ so that $p=q+l$, and $w=r-s$. If $w\le 0$, write $d=-w$. Table 3 below gives the first occurrence.

Case	First Occurrence
1. $w=k=0$	$r=s=0$
2. $w=k>0$	$r=q+k,\ s=q$
3. $0\le w<k,\ w\le l$	$r=q+l,\ s=q+l-w$
4. $0\le w<k,\ w\ge l$	$r=q+w,\ s=q$
5. $k>0,\ -k\le w\le 0$	$r=q+l,\ s=q+l+d$
6. $\|w\|>k$	stable range

TABLE 3. First occurrence for $\det_{p,q}^{\frac{k}{2}}$

For $|w|>k$ the first occurrence is in the stable range, so we can now restrict our attention to the cases $-k\le w\le k$. First we are going to determine the Langlands parameters of $\theta_{r,s}(\det_{p,q}^{\frac{k}{2}})$ for these cases. Notice that when $k>0$, we have that $\theta_{r,s}(\det_{p,q}^{\frac{k}{2}})\neq 0$ if and only if $\theta_{r-q,s-q}(\det_{l}^{\frac{k}{2}})\neq 0$.

For these cases, as well as for $k = 0$ and $r = s \geq q$, we will in fact express the Langlands parameters of the former in terms of the Langlands parameters of the latter, which are given by Table 2 and Theorem 5.11. The remaining cases ($k = 0$, $r = s < q$) are covered by the second part of the theorem.

Theorem 8.1. *Let $p \geq q$, $k \geq 0$, and r, s be such that $\det_{p,q}^{\frac{k}{2}}$ occurs in $\mathcal{H}$ for the dual pair $\big(U(p,q), U(r,s)\big)$. Suppose $|w| \leq k$. Then the Langlands parameters of $\pi = \theta_{r,s}(\det_{p,q}^{\frac{k}{2}})$ are given as follows.*

(1) *If $k \geq 0$ let $\pi_0 = \pi(m, \lambda, \Psi, \mu, \nu)$ be the theta lift of $\det_{l,0}^{\frac{k}{2}}$ for the dual pair $\big(U(l,0), U(r-q, s-q)\big)$. Then $\pi = \pi(m+q, \lambda, \Psi, \mu', \nu')$, where $\mu' = (\mu_1, \ldots, \mu_m, k, \ldots, k)$ and $\nu' = (\nu_1, \ldots, \nu_m, l+1, l+3, \ldots, l+2q-1)$.*
(2) *If $w = k = 0$ and $r \leq q$ then $\pi = \pi(r, \lambda_{sgn}, \Psi_{sgn}, \mu, \nu)$, where $\mu = (0, \ldots, 0)$ and $\nu = (p+q-2r+1, p+q-2r+3, \ldots, p+q-1)$.*

Proof. Part (2) follows from Proposition 5.4(i) of [LZ]. So we only need to prove Part (1). As before, because of Theorem 5.11 we may restrict our attention to the first occurrence if it is at rank $\geq p+q$, and to the cases $r+s \leq p+q+1$ otherwise. In this second situation, we may further restrict to the cases $r+s < p+q-1$ since the complete correspondence for the cases $p+q = r+s$ and $p+q = r+s \pm 1$ are in [P1] and [P2].

Recall that $\det_{p,q}^{\frac{k}{2}}$ is the unique LKT-constituent of $I_1 = Ind_{P_1}^{\widetilde{U}(p,q)}(\det_{l,0}^{\frac{k}{2}} \otimes \rho_{\zeta,\xi})$, where $P_1 = M_1 N_1$ is a maximal parabolic subgroup of $\widetilde{U}(p,q)$ with Levi factor $M_1 \cong \widetilde{U}(l,0) \times GL(q,\mathbb{C})$, $\zeta = (k, \ldots, k)$, and $\xi = (l+1, l+3, \ldots, l+2q-1)$. Using Theorem 5.2 with $p_1 = l$, $q_1 = 0$, $p_2 = r-q$, $q_2 = s-q$, $k_1 = k_2 = q$, $\pi_1 = \det_{l,0}^{\frac{k}{2}}$, $\pi_2 = \pi_0$, $\chi_1 = |\det|^{r+s-p-q}$, and $\sigma_1 = \rho_{\zeta,\xi} \otimes \chi_1^{-1}$, we get a nonzero $\Big(\widetilde{U}(p,q) \times \widetilde{U}(r,s)\Big)$-map

$$\Phi : \omega \longrightarrow I_1 \otimes I_2 \tag{8.2}$$

Here $I_2 = Ind_{P_2}^{\widetilde{U}(r,s)}(\pi_0 \otimes \rho_{\zeta,-\xi})$, where $P_2 = M_2 N_2$ is a maximal parabolic subgroup of $\widetilde{U}(r,s)$ with Levi factor $M_2 \cong \widetilde{U}(r-q, s-q) \times GL(q,\mathbb{C})$. Notice that π is the unique LKT-constituent of I_2. By replacing some of the entries of ξ by their negatives we can arrange for $\det_{p,q}^{\frac{k}{2}}$ to be a quotient of I_1 (see §6 of [P1]). Let η be the K-type for $\widetilde{U}(r,s)$ which corresponds to $\det_{p,q}^{\frac{k}{2}}$ in the space of joint harmonics.

Claim 1. $\det_{p,q}^{\frac{k}{2}} \otimes \eta$ is in the image of the map Φ in 8.2.

This will imply that the theta lift of $\det_{p,q}^{\frac{k}{2}}$ is a constituent of I_2 which contains the K-type η.

Consider the generalized principal series representation (in the sense of [SV]) $I_s = Ind_{P_s}^{\widetilde{U}(r,s)}(\rho(\lambda, \Psi) \otimes \chi)$, where $P_s = M_s N_s$ is a cuspidal parabolic subgroup of $\widetilde{U}(r,s)$ with Levi factor $M_s \cong \widetilde{U}(r-q-m, s-q-m) \times (\mathbb{C}^\times)^{m+q}$ and χ is the character of $(\mathbb{C}^\times)^{m+q}$ associated to μ' and ν' as in Section 2.7. By induction in stages, every constituent of I_2 is a constituent of I_s. Consequently, if X is a constituent of I_2 which occurs in I_s as a LKT-constituent, then X must be a LKT-constituent of I_2, i.e., $X \cong \pi$. It is easy to see that I_s has a unique LKT-constituent. So we are done if we prove the following

Claim 2. Only the LKT-constituent of I_s contains the K-type η.

Notice that this is immediate if η is a LKT of I_s.

We start with the proof of Claim 2. First consider the following case: Assume $0 < k = w < l$ so that we are looking at the theta lift of $\det_{l+q,q}^{\frac{k}{2}}$ to $\widetilde{U}(r,s) = \widetilde{U}(q+k+t, q+t)$ for some $0 \leq t < \frac{l-k-1}{2}$ (see

Table 2, Case 2a.). This is the case of early occurrence; the first occurrence is at rank $2q+k<p+q$. In this case, $P_s = M_s N_s$ with $M_s \cong \widetilde{U}(k+t,t)\times(\mathbb{C}^\times)^q$,

$$\eta = \Big(\underbrace{\tfrac{l}{2},\dots,\tfrac{l}{2}}_{q+k+t};\underbrace{k-\tfrac{l}{2},\dots,k-\tfrac{l}{2}}_{q},\underbrace{-\tfrac{l}{2},\dots,-\tfrac{l}{2}}_{t}\Big) \qquad \text{(see Lemma 3.2)}, \tag{8.3}$$

$$\lambda = \left(\tfrac{k+l-1}{2},\tfrac{k+l-3}{2},\dots,\tfrac{l-k-2t+1}{2};\tfrac{k-l+2t-1}{2},\tfrac{k-l+2t-3}{2},\dots,\tfrac{k-l+1}{2}\right), \tag{8.4}$$

$\mu' = (k,\dots,k)$, and $\nu' = (l+1,l+3,\dots,l+2q-1)$. One can check that if $k+t \le \frac{l}{2}-1$ then η is not a LKT of I_s. However, I_s has nonsingular infinitesimal character, so Theorem 4.23 of [SV] allows us to determine the Langlands parameters of all the constituents of I_s. We will show that if X is a constituent of I_s which is not the LKT-constituent, then X does not contain the K-type η. Such an X is associated to a more compact Cartan subgroup, i.e., $X \cong \pi(\tilde{m},\tilde{\lambda},\tilde{\Psi},\tilde{\mu},\tilde{\nu})$ with $\tilde{m}<q$. The parameters $(\tilde{\lambda},\tilde{\mu},\tilde{\nu})$ are obtained from (λ,μ',ν') as follows: Remove $q-\tilde{m}$ pairs of coordinates (μ'_i,ν'_i) from (μ',ν'), and for each such pair, add a pair of coordinates with entries $(\frac{1}{2}(\mu'_i+\nu'_i);\frac{1}{2}(\mu'_i-\nu'_i))$ to the discrete series parameter λ, one coordinate on each side of the semicolon. Since $\mu'_i = k$ and $\nu'_i \ge l+1$ for all i, the resulting discrete series parameter

$$\tilde{\lambda} = (\alpha_1,\alpha_2,\dots,\alpha_{r-\tilde{m}};\beta_1,\dots,\beta_{s-\tilde{m}}) \tag{8.5}$$

will have the property that $max\{\alpha_1,\beta_1\} \ge \frac{k+l+1}{2}$. So if $\tilde{\lambda}+\tilde{\mu} = (a_1,\dots,a_r;b_1,\dots,b_s)$ then $a_1 \ne b_1$ and $max\{a_1,b_1\} \ge \frac{k+l+1}{2}$. Let $\Lambda = (A_1,\dots,A_r;B_1,\dots,B_s)$ be the highest weight of a LKT of X. Then if $a_1 > b_1$ we have that $A_1 = a_1 + \frac{q+t}{2} - \frac{q+k+t-1}{2} \ge \frac{k+l+1}{2} - \frac{k-1}{2} = \frac{l}{2}+1 > \frac{l}{2}$. Similarly, if $b_1 > a_1$ then $B_1 \ge k+\frac{l}{2}+1 > k-\frac{l}{2}$. Recall that if η is a K-type of X then its highest weight must be of the form $\Lambda+\sum_i \alpha_i$, where the α_i are roots with the property that $(\alpha_i,\tilde{\lambda}+\tilde{\mu}) \ge 0$. (Here $(,)$ is the dualized trace form as in Section 2.2.) But if $A_1 > \frac{l}{2}$ then this sum will have to contain at least one root of the form $-e_1+e_i$ for some $i>1$, and if $a_1 > b_1$ then $(-e_1+e_i,\tilde{\lambda}+\tilde{\mu}) = -a_1+a_i$ (or $-a_1+b_{i-r}$) <0. Similarly if $a_1 < b_1$ and $B_1 > k-\frac{l}{2}$. Therefore, η is not a K-type of X, so it must be a K-type which occurs in the LKT-constituent only.

In all other cases, it will turn out that η is a LKT of I_s. The calculation is straightforward, but we have to consider a number of different cases, depending on the relative sizes of l, k, and w, and the parity of l (as in Table 2). We demonstrate the calculation for one of the cases, the others will be similar. Assume that $0 \le w < k$, $l \ge \frac{k+w}{2}+1$, and that $l-\frac{k+w}{2}$ is even. (This corresponds to Case 3c(i) of Table 2.) Notice that in this case, the first occurrence of $\det_{p,q}^{\frac{k}{2}}$ is with $\widetilde{U}(r,s) = \widetilde{U}(p,p-w) = \widetilde{U}(q+l,q+l-w)$, and

$$\eta = \Big(\underbrace{\tfrac{l+k-w}{2},\dots,\tfrac{l+k-w}{2}}_{p};\underbrace{\tfrac{k+w-l}{2},\dots,\tfrac{k+w-l}{2}}_{q},\underbrace{-\tfrac{l}{2},\dots,-\tfrac{l}{2}}_{l-w}\Big), \tag{8.6}$$

$m = \frac{2l-k-w}{4}$, $\mu' = (\underbrace{\frac{k-w}{2},\dots,\frac{k-w}{2}}_{m},\underbrace{k,\dots,k}_{q})$, $\nu' = (1,3,\dots,2m-1,l+1,l+3,\dots,l+2q-1)$, and

$$\lambda = \left(\tfrac{k+l-1}{2},\tfrac{k+l-3}{2},\dots,\tfrac{k-w+2}{4};\tfrac{k-w-2}{4},\tfrac{k-w-6}{4},\dots,\tfrac{w-l+1}{2}\right). \tag{8.7}$$

Now assume in addition that l is even (the case l odd being similar). As in Section 5, we compute the LKT's of I_s using the techniques described in §3 of [P1]. We have

$$\begin{aligned}(8.8)\quad \lambda+\mu' = \Bigg(& \underbrace{\tfrac{k+l-1}{2}, \tfrac{k+l-3}{2}, \dots, \tfrac{k+1}{2}}_{\frac{l}{2}}, \underbrace{\tfrac{k}{2}, \dots, \tfrac{k}{2}}_{q}, \underbrace{\tfrac{k-1}{2}, \tfrac{k-3}{2}, \dots, \tfrac{k-w+2}{4}}_{\frac{k+w}{4}}, \underbrace{\tfrac{k-w}{4}, \tfrac{k-w}{4}, \dots, \tfrac{k-w}{4}}_{m}; \\ & \underbrace{\tfrac{k}{2}, \tfrac{k}{2}, \dots, \tfrac{k}{2}}_{q}, \underbrace{\tfrac{k-w}{4}, \tfrac{k-w}{4}, \dots, \tfrac{k-w}{4}}_{m}, \underbrace{\tfrac{k-w-2}{4}, \tfrac{k-w-6}{4}, \dots, \tfrac{w-l+1}{2}}_{l-w-m} \Bigg).\end{aligned}$$

Then

$$\begin{aligned}(8.9)\quad \rho(\mathfrak{u}\cap\mathfrak{p}) = \Bigg(& \underbrace{\tfrac{q+l-w}{2}, \tfrac{q+l-w}{2}, \dots, \tfrac{q+l-w}{2}}_{\frac{l}{2}}, \underbrace{\tfrac{l-w}{2}, \tfrac{l-w}{2}, \dots, \tfrac{l-w}{2}}_{q}, \underbrace{\tfrac{l-w-q}{2}, \tfrac{l-w-q}{2}, \dots, \tfrac{l-w-q}{2}}_{\frac{k+w}{4}}, \\ & \underbrace{\tfrac{2l+k-3w-4q}{8}, \tfrac{2l+k-3w-4q}{8}, \dots, \tfrac{2l+k-3w-4q}{8}}_{m}; \\ & \underbrace{0, 0, \dots, 0}_{q}, \underbrace{\tfrac{-2l-4q-k-w}{8}, \tfrac{-2l-4q-k-w}{8}, \dots, \tfrac{-2l-4q-k-w}{8}}_{m}, \underbrace{\tfrac{-l-q}{2}, \tfrac{-l-q}{2}, \dots, \tfrac{-l-q}{2}}_{l-w-m} \Bigg),\end{aligned}$$

$$\begin{aligned}(8.10)\quad \rho(\mathfrak{u}\cap\mathfrak{k}) = \Bigg(& \underbrace{\tfrac{q+l-1}{2}, \tfrac{q+l-3}{2}, \dots, \tfrac{q+1}{2}}_{\frac{l}{2}}, \underbrace{0, 0, \dots, 0}_{q}, \underbrace{\tfrac{-q-1}{2}, \tfrac{-q-3}{2}, \dots, \tfrac{-2q+2-k-w}{4}}_{\frac{k+w}{4}}, \\ & \underbrace{\tfrac{-2l-4q-k-w}{8}, \tfrac{-2l-4q-k-w}{8}, \dots, \tfrac{-2l-4q-k-w}{8}}_{m}; \\ & \underbrace{\tfrac{l-w}{2}, \tfrac{l-w}{2}, \dots, \tfrac{l-w}{2}}_{q}, \underbrace{\tfrac{-4q+2l+k-3w}{8}, \tfrac{-4q+2l+k-3w}{8}, \dots, \tfrac{-4q+2l+k-3w}{8}}_{m}, \\ & \underbrace{\tfrac{-2q-2+k-w}{4}, \tfrac{-2q-2+k-w}{4} - 1, \dots, \tfrac{-q-l+w+1}{2}}_{l-m-w} \Bigg),\end{aligned}$$

so that $\lambda+\mu'+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k})=\eta$. By integrality considerations, $\delta_L=0$, so the unique LKT of I_s is η, and we are done.

To prove Claim 1, we use the following extended induction principle which is due to Adams and Barbasch ([AB1],[P1]).

Theorem 8.11. *In the setting of Theorem 5.2, let K_1 and K_2 be maximal compact subgroups of G_1 and G_2 respectively. Let ω_M be the oscillator representation for the dual pair $M_1\times M_2$.*

Suppose η_1 is a K_1-type, δ_1 is a $(K_1\cap M_1)$-type, and that η_1 and δ_1 satisfy the following properties:

1. *δ_1 is of minimal degree in $\pi_1\otimes\sigma_1$.*
2. *η_1 is of minimal degree and of multiplicity one in $Ind_{P_1}^{G_1}(\pi_1\otimes\sigma_1\otimes\chi_1)$.*
3. *$deg(\eta_1)=deg(\delta_1)$, and the restriction of η_1 to $(K_1\cap M_1)$ contains δ_1.*
4. *There exist characters α_1 and α_2 of M_1 and M_2 which are trivial on $(K_1\cap M_1)$ and $(K_2\cap M_2)$, and such that $(\pi_1\otimes\sigma_1\otimes\alpha_1)\otimes(\pi_2\otimes\sigma_2\otimes\alpha_2)$ is a quotient of ω_M, and $Ind_{P_1}^{G_1}(\pi_1\otimes\sigma_1\otimes\chi_1\otimes\alpha_1)$ is irreducible.*

Let η_2 be the K_2-type which corresponds to η_1 in the space of joint harmonics, and let Φ be the map in (5.5). Then $\eta_1\otimes\eta_2$ is in the image of Φ.

To show that $\det_{p,q}^{\frac{k}{2}} \otimes \eta$ is in the image of Φ, we apply Theorem 8.11 to our situation. We choose G_1 and G_2 depending on the relative sizes of $p+q$ and $r+s$. If $p+q \geq r+s$, let $G_1 = \widetilde{U}(p,q)$, $M_1 = \widetilde{U}(p-q,0) \times GL(q,\mathbb{C})$, $\pi_1 = \det_{l,0}^{\frac{k}{2}}$, $\chi_1 = |\det|^{r+s-p-q}$, and $\sigma_1 = \rho_{\zeta,\xi} \otimes \chi_1^{-1}$. We take $\eta_1 = \det_{p,q}^{\frac{k}{2}}$ and $\delta_1 = \det_{l,0}^{\frac{k}{2}} \otimes \det^k$, an irreducible representation of $K_1 \cap M_1 = \widetilde{U}(l,0) \times U(q)$ and the unique LKT of $\pi_1 \otimes \sigma_1$. The computations verifying conditions (1) through (3) were done in [P1] and [P2].

If $p+q < r+s$, we let $G_1 = \widetilde{U}(r,s)$, $M_1 = \widetilde{U}(r-q,s-q) \times GL(q,\mathbb{C})$, $\pi_1 = \pi_0$, $\chi_1 = |\det|^{p+q-r-s}$, and $\sigma_1 = \rho_{\zeta,-\xi} \otimes \chi_1^{-1}$. Then we take $\eta_1 = \eta$ which we now know is a LKT of $Ind_{P_1}^{G_1}(\pi_1 \otimes \sigma_1 \otimes \chi_1)$, as well as of the generalized principal series I_s, and δ_1 will be the $(K_1 \cap M_1)$-type obtained from η by restricting the highest weight. The second parts of conditions (2) and (3) are then immediate, and it is straightforward to check that δ_1 is a LKT of $\pi \otimes \sigma_1$. Condition (1) follows using Lemma 5.10, and Lemma 4.1 of [AB1]. The computation verifying that η is of minimal degree in I_s (and hence in $Ind_{P_1}^{G_1}(\pi_1 \otimes \sigma_1 \otimes \chi_1)$) is in [P1] and [P2], and checking that $deg(\eta) = deg(\delta_1)$ is again straightforward (see Lemmas 5.2.8 and 5.3.3 of [P1] for similar calculations).

For condition (4), let $z \in \mathbb{C}$, and let α_1 and α_2 be the characters of $GL(q,\mathbb{C})$ given by $\alpha_1 = |\det|^z$ and $\alpha_2 = |\det|^{-z}$; then extend α_i (for $i = 1,2$) to M_i so that they are trivial on the first factors. Then α_i is trivial on $K_i \cap M_i$, and $\sigma_1 \otimes \alpha_1$ corresponds to $\sigma_2 \otimes \alpha_2$ in the correspondence for the dual pair $\big(GL(q,\mathbb{C}), GL(q,\mathbb{C})\big)$, so that $(\pi_1 \otimes \sigma_1 \otimes \alpha_1) \otimes (\pi_2 \otimes \sigma_2 \otimes \alpha_2)$ is a quotient of ω_M. Now choose z so that $Ind_{P_1}^{G_1}(\pi_1 \otimes \sigma_1 \otimes \chi_1 \otimes \alpha_1)$ is irreducible; that this can be done is well-known, and follows from an argument similar to one used in [SV]. Now all four conditions of Theorem 8.11 are verified, and Claim 1 follows.

This completes the proof of Theorem 8.1. □

Remark 8.12. Notice that for the first occurrence in the good tower, the theta lift of $\det_{p,q}^{\frac{k}{2}}$ is the LKT-constituent of a generalized principal series representation of $\widetilde{U}(q+k,q)$. Lee and Loke [LLo] have determined the composition structure, K-structure, and unitarity of all constituents of these representations. Using this information, one can easily check that the theta lift of a one-dimensional representation of $\widetilde{U}(p,q)$ at first occurrence in the good tower is always finite dimensional, with highest weight η (the highest weight of the K-type which corresponds to $\det^{\frac{k}{2}}$ in the space of joint harmonics).

For the sake of completeness, we list in Table 4 the Langlands parameters for the remaining cases, i. e., for Case 7 of Table 3. In this situation, the first occurrence is in the stable range, so that $(r,s) = (p+q+w, p+q)$ if $w > 0$, and $(r,s) = (p+q, p+q+d)$ if $w = -d < 0$. The Langlands parameters of the theta lifts are of the following form: If $\theta_{r,s}(\det^{\frac{k}{2}}) = \pi(m,\lambda,\Psi,\mu,\nu)$, then $m = 2q + m' \geq 2q$, the representation $\rho(\lambda,\Psi)$ of $\widetilde{U}(r-m, s-m)$ is a holomorphic discrete series or limit of a holomorphic discrete series with λ of the form (7.1), $\mu_1 = \mu_2 = \cdots = \mu_q$, $\mu_{q+1} = \mu_{q+2} = \cdots = \mu_m$, $\nu = (\nu_1, \nu_1+2, \ldots, \nu_1+2q-2, \nu_{q+1}, \nu_{q+1}+2, \ldots, \nu_{q+1}+2(q+m')-2)$. Therefore, we only need to list the data m', α, β, μ_1, μ_{q+1}, ν_1, and ν_{q+1}.

9. Nonunitary lifts

It only remains to prove Part (3) of Theorem 4.1. We will need the correspondence of infinitesimal characters, as well as Parthasarathy's Dirac operator inequality. If $\pi \in \widetilde{U}(p,q)^\frown$ and $\theta_{r,s}(\pi) = \pi'$, let γ and γ' be the infinitesimal characters of π and π' respectively. We refer to the correspondence $\gamma \leftrightarrow \gamma'$ as the correspondence of infinitesimal characters for the dual pair $\big(U(p,q), U(r,s)\big)$. We represent the infinitesimal character of π by an element of $\mathfrak{t}^* \cong \mathbb{C}^{p+q}$. Choose coordinates so that the infinitesimal character of the trivial representation of $\widetilde{U}(p,q)$ is given by $\big(\frac{p+q-1}{2}, \frac{p+q-3}{2}, \ldots, \frac{-p-q+1}{2}\big)$. Similarly for $\widetilde{U}(r,s)$.

Case	m'	α	β	μ_1	μ_{q+1}	ν_1	ν_{q+1}
1. $w > k$				$\frac{k+w}{2}$	$\frac{k-w}{2}$	$\frac{-k+w}{2} + l + 1$	
a. $\frac{k+w}{2} \geq l$	l	$\frac{l+w-1}{2}$	—				$\frac{k+w}{2} - l + 1$
b. $\frac{k+w}{2} \leq l$							
(i) $l - \frac{k+w}{2}$ even	$\frac{k+w+2l}{4}$	$\frac{l+w-1}{2}$	$\frac{k-w-2}{4}$				1
(ii) $l - \frac{k+w}{2}$ odd	$\frac{k+w+2l-2}{4}$	$\frac{l+w-1}{2}$	$\frac{k-w}{4}$				2
2. $w < -k$				$\frac{k-d}{2}$	$\frac{k+d}{2}$	$\frac{k+d}{2} + l + 1$	
a. $\frac{d-k}{2} \geq l$	l	—	$\frac{-l+d-1}{2}$				$\frac{d-k}{2} - l + 1$
b. $\frac{d-k}{2} \leq l$							
(i) $l - \frac{d-k}{2}$ even	$\frac{d-k+2l}{4}$	$\frac{l+k-1}{2}$	$\frac{d+k-2}{4}$				1
(ii) $l - \frac{d-k}{2}$ odd	$\frac{d-k+2l-2}{4}$	$\frac{l+k-1}{2}$	$\frac{d+k}{4}$				2

TABLE 4. Theta lifts of characters of $U(p, q)$ in the stable range

Proposition 9.1 ([Pr2]). *Assume that $p + q \leq r + s$, and let $m = r + s - p - q$. The correspondence of infinitesimal characters for the dual pair $\big(U(p, q), U(r, s)\big)$ is given by*

$$\gamma \leftrightarrow (\gamma, \rho_m), \tag{9.2}$$

where $\rho_m = \left(\frac{m-1}{2}, \frac{m-3}{2}, \ldots, \frac{-m+1}{2}\right) \in \mathbb{C}^m$.

Proposition 9.3 (Parthasarathy's Dirac operator inequality, [Pa],[VZ]). *Let π be a unitary representation of a real reductive Lie group G. Let K be a maximal compact subgroup of G, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{k}$, and $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ a theta stable Cartan subalgebra of $\mathfrak{g}$. Fix a K-type occurring in π, of highest weight $\sigma \in \mathfrak{t}^*$, and a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$. Let ρ_c and ρ_n denote one half the sums of the positive compact and non-compact roots respectively, and fix an element w in the Weyl group $W(\mathfrak{t} : \mathfrak{k})$ such that $w(\sigma - \rho_n)$ is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Let $\gamma \in \mathfrak{h}^*$ denote the infinitesimal character of π. Then*

$$(\sigma - \rho_n + w^{-1}\rho_c, \sigma - \rho_n + w^{-1}\rho_c) \geq (\gamma, \gamma). \tag{9.4}$$

We now restate Part (3) of Theorem 4.1 in the form in which we are going to prove it.

Proposition 9.5. *Let $k \geq 0$, $p \geq q \geq 1$, and $r < p$. If $\theta_{r,s}(\det_{p,q}^{\frac{k}{2}}) = \pi \neq 0$ then π is non-unitary.*

Proof. Suppose the conditions of the proposition hold. According to Table 1, we must have that $w = r - s = k$. The Langlands parameters of these lifts are given by Theorem 8.1. If $r + s \leq p + q$, then the infinitesimal character of π is strongly regular in the sense of Salamanca-Riba, hence by Theorem 1.12 of [Sa], π is unitary only if π is of the form $A_{\mathfrak{q}}(\lambda)$. For representations of $\widetilde{U}(r, s)$ with strongly regular infinitesimal character, it is not difficult to check whether given Langlands parameters are those of a representation of this form (see Corollary 11.219 of [KnV] and Proposition 7.4 of [P1]), and we easily determine that in these cases, they are not.

So we may now assume that $r + s > p + q$. Write $p = q + l$. Let σ be the K-type for $\widetilde{U}(r, s)$ which corresponds to $\det^{\frac{k}{2}}$ in the space of joint harmonics. We will show that there is a choice of positive roots such that equation 9.4 is violated.

Notice that $s \geq q$, so let $s = q + m$. Since $r < p$ and $r + s > p + q$, we have

$$m + k < l < 2m + k. \tag{9.6}$$

By Lemma 3.2 we know that

$$\sigma = (\underbrace{\tfrac{l}{2}, \ldots, \tfrac{l}{2}}_{q+m+k}; \underbrace{k - \tfrac{l}{2}, \ldots, k - \tfrac{l}{2}}_{q}, \underbrace{-\tfrac{l}{2}, \ldots, -\tfrac{l}{2}}_{m}). \tag{9.7}$$

Let $\Delta^+(\mathfrak{g},\mathfrak{t}) = \{e_i - e_j | 1 \le i < j \le r+s\}$. Then

$$\rho_n = \left(\tfrac{s}{2},\dots,\tfrac{s}{2};-\tfrac{r}{2},\dots,-\tfrac{r}{2}\right) = \left(\tfrac{q+m}{2},\dots,\tfrac{q+m}{2};\tfrac{-q-m-k}{2},\dots,\tfrac{-q-m-k}{2}\right) \tag{9.8}$$

and $\sigma - \rho_n$ is dominant for $\Delta^+(\mathfrak{k},\mathfrak{t})$, so that we may choose $w=1$ in equation 9.4. We have

$$\begin{aligned}
&\sigma - \rho_n + \rho_c \\
&= \big(\underbrace{\tfrac{l+k-1}{2},\tfrac{l+k-3}{2},\dots,\tfrac{l-k+1}{2}-q-m}_{q+m+k};\underbrace{\tfrac{-l+k-1}{2}+q+m+k,\dots,\tfrac{-l+k+1}{2}+m+k}_{q},\\
&\qquad\qquad \underbrace{\tfrac{-l+k-1}{2}+m,\dots,\tfrac{l-k+1}{2}-m}_{2m-l+k},\underbrace{\tfrac{l-k-1}{2}-m,\dots,\tfrac{-l+k+1}{2}}_{l-k-m}\big)\\
&= \big(a_1,a_2,\dots,a_{q+m+k};b_1,\dots,b_q,c_1,\dots,c_{2m-l+k},d_1,\dots,d_{l-k-m}\big).
\end{aligned} \tag{9.9}$$

The a_j, b_j, c_j, and d_j in (9.9) are each decreasing by steps of 1. Noting that $r+s-p-q = 2m-l+k$ and using Proposition 9.1 we get the infinitesimal character of π

$$\begin{aligned}
\gamma &= \big(\big(\tfrac{k}{2},\dots,\tfrac{k}{2}\big)+\rho_{2q+l},\rho_{2m-l+k}\big)\\
&= \big(\underbrace{\tfrac{l+k-1}{2}+q,\tfrac{l+k-3}{2}+q,\dots,\tfrac{l+k+1}{2}}_{q},\underbrace{\tfrac{l+k-1}{2},\dots,\tfrac{l-k+1}{2}-q-m}_{q+m+k},\\
&\qquad\qquad \underbrace{\tfrac{l-k-1}{2}-q-m,\dots,\tfrac{-l+k+1}{2}-q}_{l-k-m},\underbrace{\tfrac{-l+k-1}{2}+m,\dots,\tfrac{l-k+1}{2}-m}_{2m-l+k}\big)\\
&= \big(\beta_1,\beta_2,\dots,\beta_q,\alpha_1,\dots,\alpha_{q+m+k},\delta_1,\dots,\delta_{l-k-m},\gamma_1,\dots,\gamma_{2m-l+k}\big).
\end{aligned} \tag{9.10}$$

Again, the α_j, β_j, γ_j, and δ_j are each decreasing by steps of 1. To show that

$$\begin{aligned}
(\sigma-\rho_n+\rho_c,\sigma-\rho_n+\rho_c) &= \sum_{j=1}^{q+m+k} a_j^2 + \sum_{j=1}^{q} b_j^2 + \sum_{j=1}^{2m-l+k} c_j^2 + \sum_{j=1}^{l-k-m} d_j^2\\
&< \sum_{j=1}^{q+m+k} \alpha_j^2 + \sum_{j=1}^{q} \beta_j^2 + \sum_{j=1}^{2m-l+k} \gamma_j^2 + \sum_{j=1}^{l-k-m} \delta_j^2 = (\gamma,\gamma)
\end{aligned} \tag{9.11}$$

recall (9.6) and our assumption that $q>0$. Notice also that $a_j = \alpha_j$ and $c_j = \gamma_j$ for all j. Now if $1 \le j \le q$ then $\beta_j - b_j = l-m-k > 0$ by (9.6) and $b_j \ge b_q = \frac{-l+k+1}{2}+m+k > k+\frac{1}{2} > 0$ by (9.6). Consequently, $\beta_j^2 > b_j^2$ for all j. Since $q>0$, $\sum_{j=1}^q \beta_j^2 > \sum_{j=1}^q b_j^2$. Similarly, we have for $1 \le j \le l-k-m$, $d_j - \delta_j = q > 0$ and $d_j \le d_1 = \frac{l-k-1}{2} - m < 0$ by (9.6). So $\delta_j^2 > d_j^2$ for all j, and since (using (9.6) once again) $l-k-m>0$, we have $\sum_{j=1}^{l-k-m}\delta_j^2 > \sum_{j=1}^{l-k-m} d_j^2$. This proves (9.11) and thus the proposition. □

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