GENERALIZED ROBINSON-SCHENSTED ALGORITHMS FOR REAL GROUPS

PETER E. TRAPA

ABSTRACT. In the context of $GL(n,\mathbb{C})$, the classical Robinson-Schensted algorithm arises in both the computation of Kazhdan-Lusztig cells and in the parametrization of the irreducible components of the Steinberg variety of triples. Springer has shown how this latter parametrization generalizes to give a generalized Robinson-Schensted algorithm for any real linear reductive group. Here we compute the algorithm for $U(p,q)$, $SU^*(2n)$, and $GL(n,\mathbb{R})$. We then relate the answer to the structure of Kazhdan-Lusztig-Vogan cells for these groups.

1. INTRODUCTION

Young's parameterization of the irreducible representations of the symmetric group S_n, together with the corresponding dimension formula and isotypic decomposition of $\mathbb{C}[S_n]$, implies that the cardinality of S_n coincides with the number of same-shape pairs of standard Young tableaux of size n. The Robinson-Schensted algorithm is a constructive bijection between the two sets which inexplicably turns up in a number of diverse applications. We begin by recalling two of them.

Recall that each $w \in S_n$ indexes a diagonal $GL(n,\mathbb{C})$ orbit, say $Q(w)$, on the product of two flag varieties for $GL(n,\mathbb{C})$. Fix the standard basis e_i of $\mathbb{C}^n$, and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ be the upper triangular Borel in $\mathfrak{gl}(n,\mathbb{C})$. Let F denote the flag whose jth subspace F_j is the span of the e_i, $i \leq j$, and let ${}^wF$ denote the flag with ${}^wF_j$ spanned by $e_{w^{-1}(i)}$, $i \leq j$. (We choose $Q(w)$ so as to contain the pair $(F, {}^wF)$.) Now let N denote a generic (nilpotent) in $Ad(w)\mathfrak{n} \cap \mathfrak{n}$. It is not difficult to see that N restricts to a nilpotent endomorphism of each F_j and ${}^wF_j$, and we obtain two tableau (whose shape coincide with Jordan form of N) as follows. We construct the first tableau by requiring its first j boxes coincide with the Jordan form of the restriction of N to F_j, and obtain a second tableau by replacing F_j with ${}^wF_j$. Steinberg ([St2]) showed that the resulting tableaux are precisely the pair associated to w by the classical Robinson-Schensted algorithm.

On the other hand, the elements of S_n (or, more suggestively, the orbits $Q(w)$) index the irreducible Harish-Chandra modules of $GL(n,\mathbb{C})$ with fixed regular integral infinitesimal character. Because G is complex, we can speak of left and right annihilators (in the enveloping algebra $U(\mathfrak{gl}(n,\mathbb{C}))$) of the Harish-Chandra module associated to $Q(w)$. Using Joseph's tableau classification of the primitive spectrum of $U(\mathfrak{gl}(n,\mathbb{C}))$, we obtain two standard tableaux associated to w; Joseph proved that this construction again agrees with the Robinson-Schensted algorithm. Said differently, the left and right fibers of the Robinson-Schensted algorithm compute the left and right Kazhdan-Lusztig cells for $\mathfrak{gl}(n,\mathbb{C})$.

One is led to ask if the above coincidence applies to other groups outside of $GL(n,\mathbb{C})$. So let G be a linear reductive group with maximal compact subgroup K, let $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ denote the corresponding complexifications, and write X for the flag variety of $G_{\mathbb{C}}$. Using the elementary geometry of the generalized Steinberg variety, Springer has shown how to parametrize $K_{\mathbb{C}}\backslash X$ in terms of certain kinds of tableaux — see Section 3 for details. (The parametrization extends the construction given above for $GL(n,\mathbb{C})$, and because of Steinberg's result, it is called a generalized Robinson-Schensted algorithm.) On the other hand, one can attach representation theoretic invariants to the Harish-Chandra modules for G parametrized by the trivial local systems on orbits in $K_{\mathbb{C}}\backslash X$. Here the relevant invariants are annihilators and associated varieties, and they too can be recast in terms of tableaux. Such data often carries information about the structure of Harish-Chandra cells for G, so the initial question becomes: Does the generalized Robinson-Schensted algorithm have anything to do with the representation theory and structure of Kazhdan-Lusztig-Vogan cells of Harish-Chandra modules for G?

In general, the answer is a negative one. For the complex classical groups outside of type A, it is easy to see that the generalized Robinson-Schensted algorithm will bear no simple relationship to the structure of Kazhdan-Lusztig cells for these groups (see Remark 3.2). This suggests a narrow window of possibilities for an affirmative answer to the above question, namely the type A real groups. Our main result is Theorem 5.6; it gives a positive answer for $U(p,q)$, $SU^*(2n)$, and $GL(n,\mathbb{R})$. Actually, the theorem relates the generalized Robinson-Schensted algorithm to the representation theory of these groups, from which conclusions about cell structure follow.

It is worth mentioning that the generalized Robinson-Schensted algorithms discussed here have origins far deeper than the enumerative geometry of the generalized Steinberg variety. The algorithms refine a notion of geometric cells for G, which are (conjecturally) exactly analogous to Kazhdan-Lusztig cells except that one begins with a topological action of the complex Weyl group, rather than the coherent continuation action. We take the opportunity to give a brief exposition in Section 4.

The paper is organized as follows. We fix notation in Section 2, and recall some preliminary facts about associated varieties and annihilators of Harish-Chandra modules. In Section 3, we describe Springer's parametrization of $K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B$, the generalized Robinson-Schensted algorithm of the title. We relate the parametrization to the study of geometric cells, which we explain in Section 4 by recalling how the elements of a geometric cell index a basis of a representation of W. In Section 5, we restrict our attention to $G = U(p,q)$, and state our main result (Theorem 5.6) relating Springer's parametrization to the the representation theory of $U(p,q)$; analogous results for $SU^*(2n)$ and $GL(n,\mathbb{R})$ are also given. In particular, this computes (real) Richardson orbits for these groups (Remark 5.8). We conclude Section 5 by giving a representation theoretic interpretation of a shape-preserving involution on the set of standard Young tableaux first studied by Spaltenstein (Corollary 5.11). In Section 6, we work out an explicit description of $K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B$ for the relevant groups, and write down Vogan's duality on the level of orbits. We give explicit calculations (mostly due to Garfinkle [G]) of annihilators of Harish-Chandra modules in Section 7. In Section 8, we assemble the results of Sections 6 and 7 to prove Theorem 5.6. The proof is not very conceptual, and there turn out to be rather subtle reasons why this is the case (see Remark 5.9). We conclude by applying our results to give an elementary computation of some associated varieties for the type A groups under consideration.

Acknowledgements. I would like to thank David Vogan for many useful conversations. I was partially supported by NSF grant DMS 97-29995.

2. Notation and Preliminaries

2.1. **General Notation.** Let G to be an arbitrary linear real reductive group, let θ be the Cartan involution, and write K for the maximal compact subgroup consisting of the fixed points of θ. We write $K_{\mathbb{C}}$ and $G_{\mathbb{C}}$ for the corresponding complexifications. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the complexified Cartan decomposition, and let B be a Borel subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Write Δ^+ for the roots of $\mathfrak{h}$ in $\mathfrak{n}$ and let $\rho = \rho(\Delta^+)$. Let W be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$ and write w_o for the long element.

For $\nu \in \mathfrak{h}^*$ define Verma modules by

$$M_{\mathfrak{b}}(w\nu) = \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_{ww_o\nu-\rho}),$$

and denote their unique irreducible quotients by $L_{\mathfrak{b}}(w\nu)$. The definition is arranged so that $L_{\mathfrak{b}}(\nu) = M_{\mathfrak{b}}(\nu)$ and $L_{\mathfrak{b}}(w_o\nu)$ is finite-dimensional (if ν is integral, dominant, and regular).

We let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g}$ and write $\mathcal{N}_\theta$ for $\mathcal{N} \cap \mathfrak{p}$. Given $N \in \mathcal{N}$, we let X^N denote the fixed points of $\exp N$ on the complex flag variety $X = G_{\mathbb{C}}/B$. If we identify X with the set of Borel subalgebras of $\mathfrak{g}$ (by mapping the identity coset eB to $\mathfrak{b}$), then X^N consists of those Borel subalgebras containing N.

Given $v \in K_{\mathbb{C}} \backslash X$, and a $K_{\mathbb{C}}$ equivariant local system ϕ on v, the Beilinson-Bernstein theory produces an irreducible Harish-Chandra module for G (with infinitesimal character ρ) which we denote $L_G(v, \phi)$. When ϕ is trivial, we write $L_G(v)$ instead. Finally we let $\widehat{G}_\rho$ denote the set of irreducible Harish-Chandra modules for G with infinitesimal character ρ.

Next we carefully define some real forms of interest. Let $V \simeq \mathbb{C}^n$ be spanned by vectors $e_1, \ldots, e_n$. For $p+q = n$, define a form $\langle\ ,\ \rangle$ on V via

$$\langle \sum_{i=1}^n a_i e_i, \sum_{j=1}^n b_j e_j \rangle = \sum_{i \le p} \overline{a_i} b_i - \sum_{i \ge p+1} \overline{a_i} b_i.$$

The group $U(p,q)$ is defined to be the subset of $GL(n,\mathbb{C})$ preserving this form. With this definition $K = U(p) \times U(q)$ (embedded block diagonally) and $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$. Next, $SU^*(2n)$ is defined to be $SL(n,\mathbb{H})$ viewed as $2n$ dimensional complex matrices via the identification $\mathbb{H} = \mathbb{C} \oplus k\mathbb{C}$. In this setting, $K = U(n,\mathbb{H})$, and $K_{\mathbb{C}} = Sp(2n,\mathbb{C})$ defined with respect to the symplectic form

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Finally, if $G = GL(n,\mathbb{R})$, then $K = O(n)$ and $K_{\mathbb{C}} = O(n,\mathbb{C})$.

2.2. **Notation for S_n.** Write $\Sigma(n)$ for the set of involutions in S_n and define

$$\Sigma_0(n) = \{\sigma \in \Sigma \mid \sigma(i) \neq i \text{ for all } i\}$$

to be the set of involutions without fixed points. Let

$$\Sigma_\pm(n) = \{(\sigma,\epsilon) \in \Sigma \times \{+,-\}^n \mid \epsilon_i = + \text{ if } \sigma(i) > i \text{ and } \epsilon_i = - \text{ if } \sigma(i) < i\}.$$

We view $\Sigma_\pm(n)$ as the set of involution in S_n with signed fixed points. (The definition arranges a convenient normalization of the signs for the non-fixed points.) Finally we write $\Sigma_\pm(p,q)$ for the subset of $\Sigma_\pm(p+q)$ whose elements have exactly p of the ϵ_i's labeled $+$ (and q labeled $-$). The reader may find the pictures in Example 6.8 useful.

2.3. **Tableau notation.** Given a partition $n = n_1 + \cdots + n_k$ with the n_i decreasing, we may attach a left justified arrangement of n boxes with n_i boxes in the ith row. Such an arrangement is called a Young diagram of size n, and the set of all such is denoted $\mathcal{D}(n)$. We write $\mathcal{D}_e(n)$ for those diagrams whose rows are all even. Given $D \in \mathcal{D}_e(n)$, the transpose D^{tr} is a diagram in which each row length occurs an even number of times. We denote this set $\mathcal{D}_e^{tr}(n)$. A Young tableau of size n is an arrangement of $1, \ldots, n$ in a Young diagram of size n, so that the number increase across rows and down columns. We write $\mathcal{T}(n)$ for the set of Young tableaux of size n, and correspondingly $\mathcal{T}_e(n)$ and $\mathcal{T}_e^{tr}(n)$ for those tableaux of the indicated shape.

A signed Young tableau of signature (p, q) is an equivalence class of Young diagrams whose boxes are filled with p pluses and q minuses so that the signs alternate across rows; two signed Young diagrams are equivalent if they can be made to coincide by interchanging rows of equal length. (Note that the equivalence relation preserves shapes.) We write $\mathcal{T}_{\pm}(p, q)$ for the set of signature (p, q) signed Young tableaux.

Given $w \in S_n$, we let $RS(w)$ denote the 'counting' (or 'right' or 'Q-') tableau described by the Robinson-Schensted algorithm (as in [Sag]).

2.4. **Primitive ideals.** A two-sided ideal in the enveloping algebra $U(\mathfrak{g})$ is called primitive if it is the annihilator of a simple $U(\mathfrak{g})$ module. A primitive ideal is said to have infinitesimal character $\nu \in \mathfrak{h}^*$ if it contains the maximal ideal in the center of $U(\mathfrak{g})$ corresponding to ν. We write $\text{Prim}(U(\mathfrak{g}))_\nu$ for the set of primitive ideals in $U(\mathfrak{g})$ with infinitesimal character ν. The following results are well-known and can be found, for instance, in Sections 7.4 and 5.23 of [Ja].

Theorem 2.1 (Duflo). *The map*

$$W \longrightarrow Prim(U(\mathfrak{g}))_\rho$$

sending w to the primitive ideal $\text{Ann}(L_{\mathfrak{b}}(w\rho))$ is a surjection of W (in fact, the involutions of W) onto the set of primitive ideals with infinitesimal character ρ.

Theorem 2.2 (Joseph). *Assume $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Then the set $Prim(U(\mathfrak{g}))_\rho$ is in bijection with the set $\mathcal{T}(n)$, the set of Young tableaux of size n. The bijection sends $\text{Ann}(L_{\mathfrak{b}}(w\rho))$ to the 'counting' tableau $RS(w)$ obtained from the Robinson-Schensted algorithm applied to w. (Equivalently, $\text{Ann}(L_{\mathfrak{b}}(w\rho)) = \text{Ann}(L_{\mathfrak{b}}(w'\rho))$ if and only if $RS(w) = RS(w')$.)*

2.5. **Associated Varieties.** We briefly recall Vogan's construction of the associated variety of a Harish-Chandra module; more details can be found in [V4]. Given a finitely generated $(\mathfrak{g}, K)$ module L, one can always consider a K-invariant good filtration, and form the associated graded object $\text{gr}(L)$. Using the identification of $\mathfrak{g}/\mathfrak{k}$ with $\mathfrak{p}$, $\text{gr}(L)$ becomes an $(S(\mathfrak{p}), K_{\mathbb{C}})$ module, i.e. a $K_{\mathbb{C}}$ equivariant coherent sheaf on $\mathfrak{p}^*$ (or $\mathfrak{p}$). The support of $\text{gr}(L)$ is called the associated variety of X, and is denoted $\text{AV}(L)$. If in addition to being finitely generated, L has finite length (for instance, if L is an irreducible Harish-Chandra module), then $\text{AV}(L)$ is contained in the nullcone $\mathcal{N}_\theta$, and hence is a union of closures of elements in $K_{\mathbb{C}} \backslash \mathcal{N}_\theta$.

As the definition might suggest, the associated variety is difficult to compute in general. We mention one important class of examples for which the computation is known. Fix $v \in K_{\mathbb{C}} \backslash X$, and suppose that there is a parabolic subgroup Q containing B so that the image of v under the projection $G_{\mathbb{C}}/B \xrightarrow{\pi} G_{\mathbb{C}}/Q$ is closed. Assume further that v is dense in $\pi^{-1}(\pi(v))$ and that $\pi(v)$ admits a G invariant volume form. (The latter condition is automatic in the case that $\mathrm{rank}(G) = \mathrm{rank}(K)$ — see [Zi, Lecture 2], for instance). Then $L_G(v)$ (Notation 2.1) is a derived functor module induced from a one dimensional character (see [VZ, Section 6]). The following theorem, which is well-known to experts, can be found in [Tr]. In its statement, $T^*_{\pi(v)}(G_{\mathbb{C}}/P)$ denotes the conormal bundle to the orbit $\pi(v)$, and μ denotes the $G_{\mathbb{C}}$ moment map for $T^*(G_{\mathbb{C}}/Q)$.

Theorem 2.3. *Assume v is of the form described above and retain the notations of the previous paragraph. Then*

$$\overline{\mathrm{AV}(L_G(v))} = \mu(T^*_{\pi(v)}(G_{\mathbb{C}}/Q)).$$

In particular, note that $\overline{\mathrm{AV}(L_G(v))}$ is the closure of a single $K_{\mathbb{C}}$ orbit; see the second paragraph following Proposition 3.1 for more details. This orbit is the $K_{\mathbb{C}}$ orbit corresponding (under the Kostant-Sekiguchi bijection) to a real Richardson orbit as defined, for example, in Section 4 of [Tr].

3. Springer's parametrization of $K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B$

The purpose of this section is to prove an unpublished result of Springer's (Proposition 3.1 below).

Begin by considering the generalized Steinberg variety

$$M = \{(N, gB) \in \mathcal{N}_\theta \times G_{\mathbb{C}}/B \mid N \in Ad(g)\mathfrak{u}\}.$$

(When G is itself a complex Lie group, $G_{\mathbb{C}}$ is diffeomorphic to $G \times G$, $K_{\mathbb{C}}$ is the diagonal copy of G, and M is the familiar Steinberg variety of triples [St1].) Write $\mathrm{Irr}(M)$ for the irreducible components of M. Once we use an invariant bilinear form to identify $\mathfrak{p}$ and $\mathfrak{p}^*$, it is easy to see that M is the union of the conormal bundles of $K_{\mathbb{C}}$ orbits on $X = G_{\mathbb{C}}/B$. Given $v \in K_{\mathbb{C}}\backslash X$, the conormal bundle $T^*_v(X)$ need not be irreducible (since $K_{\mathbb{C}}$ need not be connected), but this is not serious: the irreducible components of $T^*_v(X)$ form a single orbit under the component group of $K_{\mathbb{C}}$. Hence it is clear that $T^*_v(X)$ is pure of dimension equal to the dimension of X. It is also clear that $K_{\mathbb{C}}$ acts on $\mathrm{Irr}(M)$, and that each $T^*_v(X)$ is a single orbit in $K_{\mathbb{C}}\backslash\mathrm{Irr}(M)$. Hence we conclude that M has pure dimension $\dim(X)$ and that $K_{\mathbb{C}}\backslash\mathrm{Irr}(M)$ is parametrized by $K_{\mathbb{C}}\backslash X$.

On the other hand, we can consider the subset $M_{N,C}$ of M consisting of the closure of the $K_{\mathbb{C}}$ saturation of $N \times C \subset M$; here C is an irreducible component of the Springer fiber X^N. Now all such C have dimension equal to $\frac{1}{2}[\dim(Z_{G_{\mathbb{C}}}(N)) - \mathrm{rank}(G_{\mathbb{C}})]$, and hence we conclude that closure of the $K_{\mathbb{C}}$ orbit $M_{N,C}$ is a single $K_{\mathbb{C}}$ orbit on $\mathrm{Irr}(M)$ of pure dimension

$$\dim(K_{\mathbb{C}}) - \dim(Z_{K_{\mathbb{C}}}(N)) + \frac{1}{2}[\dim(Z_{G_{\mathbb{C}}}(N)) - \mathrm{rank}(G_{\mathbb{C}})].$$

A result of Kostant-Rallis ([KoR, Proposition 5]) insures that $\frac{1}{2}\dim(Z_{G_{\mathbb{C}}}(N)) - \dim(Z_{K_{\mathbb{C}}}(N))$ is equal, independent of N, to $\frac{1}{2}\dim(G_{\mathbb{C}}) - \dim(K_{\mathbb{C}})$. Applying this to the formula for $\dim(M_{N,C})$ we see that the $M_{N,C}$ each have dimension $\dim(X)$, and hence exhaust $K_{\mathbb{C}}\backslash\mathrm{Irr}(M)$.

A final point to consider is that the closures $M_{N,C}$ need not be distinct. To take this into account we need to consider the component group $A_{K_{\mathbb{C}}}(N)$ of the centralizer $Z_{K_{\mathbb{C}}}(N)$. We then obtain the following result.

Proposition 3.1 (Springer). *The set of $K_{\mathbb{C}}$ orbits on* $\mathrm{Irr}(M)$ *is parametrized by* $K_{\mathbb{C}}\backslash X$ *and by pairs consisting of a* $K_{\mathbb{C}}$*-orbit* $K_{\mathbb{C}} \cdot N$ *in* $\mathcal{N}_\theta$ *and an orbit of* $A_{K_{\mathbb{C}}}(N)$ *on the set of irreducible components of* X^N.

Hence we conclude that there is a bijection between $K_{\mathbb{C}}\backslash X$ and pairs consisting of a $K_{\mathbb{C}}$-orbit $K_{\mathbb{C}} \cdot N$ in $\mathcal{N}_\theta$ and an orbit of $A_{K_{\mathbb{C}}}(N)$ on the set of irreducible components of X^N. Write μ_{orb} for the map which takes $K_{\mathbb{C}}\backslash X$ to $K_{\mathbb{C}}\backslash\mathcal{N}_\theta$; the notation is meant to suggest a moment map image, which we now describe.

From the discussion preceding the proposition, we can define μ_{orb} as follows. Recall the moment map $\mu : T^*(X) \longrightarrow \mathfrak{g}^*$. Given an orbit $v \in K_{\mathbb{C}}\backslash X$, we can consider its conormal bundle $T^*_v(X)$ inside $T^*(X)$, and from the definition of μ it is not difficult to see that the moment map image $\mu(T^*_v(X))$ actually lives in $\mathcal{N}_\theta$. Since μ is proper and $T^*_v(X)$ is irreducible up to the action of the component group of $K_{\mathbb{C}}$, $\mu(T^*_v(X))$ is a $K_{\mathbb{C}}$ equivariant subvariety of $\mathcal{N}_\theta$ whose irreducible components form a single orbit under the action of the component group of $K_{\mathbb{C}}$. Since the number of $K_{\mathbb{C}}$ orbits on $\mathcal{N}_\theta$ is finite, $\mu(T^*_v(X))$ is the closure of a single $K_{\mathbb{C}}$ orbit. In this way we obtain the element $\mu_{\mathrm{orb}}(v) \in K_{\mathbb{C}}\backslash\mathcal{N}_\theta$.

The fibers of μ_{orb} thus give an interesting partition of $K_{\mathbb{C}}\backslash X$ into disjoint subsets. For a fixed $\mathcal{O} \in K_{\mathbb{C}}\backslash\mathcal{N}_\theta$, we call $\mu^{-1}_{\mathrm{orb}}(\mathcal{O})$ a geometric cell of $K_{\mathbb{C}}$ orbits for G. The terminology is suggestive, and will be explained in the next section.

Remark 3.2. In the complex case, X is a product of two flag varieties for G, and there is a natural refinement of geometric cells into left and right cells. McGovern [Mc2] (following van Leeuwen [vL]) has given a complete description of left and right geometric cells in the complex classical case. Note that it is clear that geometric cells will look quite different from Kazhdan-Lusztig cells: the image of μ_{orb} may be a non-special nilpotent orbit, while it is only the special ones which are relevant to the structure of Kazhdan-Lusztig cells.

4. Geometric cells and Weyl group representations

In this section, we take the opportunity to recall a few of the remarkable properties of geometric cells and, in particular, describe their relation with Weyl group representations. (The only potential novelty of this section is the conjectural part of Theorem/Conjecture 4.2 below.) Though not absolutely essential, in this section we assume that G is connected.

In [Ro], Rossmann gives an action of the Weyl group on the top Borel-Moore homology group $H^{\mathrm{top}}(M, \mathbb{C})$ of the Steinberg variety M. (When G is complex, Rossmann shows that the construction coincides with an earlier one given by Kazhdan-Lusztig in [KazL].) Now the fundamental classes of the conormal bundles $T^*_v(X)$, $v \in K_{\mathbb{C}}\backslash X$, give a basis for $H^{\mathrm{top}}(M, \mathbb{C})$. In analogy with the case of the coherent continuation representation, one considers subquotients of the Rossmann action which are minimal with respect to being spanned by fundamental classes of conormal bundles. In this way one obtains a partition of $K_{\mathbb{C}}\backslash X$ into disjoint subsets, which one would like to call some sort of cells. Joseph conjectured that that the cells obtained in this way are precisely the geometric cells defined at the end of the previous section[1].

In any case, Tanasaki ([Ta, Lemma 2.10]) did show that the elements of a fixed geometric cell index a basis of a representation of W. (The gap in that paper concerned the assertion

[1] Lemma 2.11 of [Ta] gives a proof, but Joseph has pointed out a gap in it.

that this representation is the minimal subquotient described above.) Since the action is not particularly easy to describe, we give a (partly conjectural) alternate description based on ideas of Hotta [Ho], Joseph [Jo], and D. King.

To begin, fix $\mathcal{O}^{K_{\mathbb{C}}} = K \cdot N \in K_{\mathbb{C}} \backslash \mathcal{N}_\theta$, and write $\mathcal{O}^{G_{\mathbb{C}}}$ for the $G_{\mathbb{C}}$ orbit through N. Now the natural inclusion of centralizers

$$Z_{K_{\mathbb{C}}}(N) \longrightarrow Z_{G_{\mathbb{C}}}(N)$$

induces a map on the level of component groups

$$A_{K_{\mathbb{C}}}(N) \longrightarrow A_{G_{\mathbb{C}}}(N).$$

Write $A(N)$ for the image of this map; it corresponds to a subgroup $H(N) \subset Z_{G_{\mathbb{C}}}(N)$ which contains the connected component of the identity $Z^{\circ}_{G_{\mathbb{C}}}(N)$. Hence we may consider the orbit cover

$$\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} = G_{\mathbb{C}}/H(N) \xrightarrow{\pi} \mathcal{O}^{G_{\mathbb{C}}}.$$

The next lemma identifies the elements of the geometric cell $\mu^{-1}_{\mathrm{orb}}(\mathcal{O}^{K_{\mathbb{C}}})$ with the intrinsic geometry of $\widetilde{\mathcal{O}}^{G_{\mathbb{C}}}$. To state the lemma, we need some notation. Recall the fixed Borel B with nilradical $\mathfrak{n}$, write $\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} \cap \mathfrak{n}$ for $\pi^{-1}(\mathcal{O}^{G_{\mathbb{C}}} \cap \mathfrak{n})$, and let $\mathrm{Irr}(\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} \cap \mathfrak{n})$ denote the set of its irreducible components. (More intrinsically, such irreducible components exhaust the B-stable Lagrangian subvarieties of $\widetilde{\mathcal{O}}^{G_{\mathbb{C}}}$ — these have been studied recently in the context of the orbit method (see [GrV], for instance) — but we will not need any of this here.) The following lemma goes back to Spaltenstein [Spa1].

Lemma 4.1. *Fix notation as in the previous two paragraphs. Then there is a natural bijection from the set* $\mathrm{Irr}(\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} \cap \mathfrak{n})$ *to the set of* $A(N)$ *orbits on* $\mathrm{Irr}(X^N)$. *(By Proposition 3.1, this latter set is in natural correspondence with the elements of the geometric cell* $\mu^{-1}_{orb}(\mathcal{O}^{K_{\mathbb{C}}})$.*)*

Sketch. As indicated, the parenthetical assertion follows from Proposition 3.1, once we observe that the $A(N)$ and $A_{K_{\mathbb{C}}}(N)$ orbits on $\mathrm{Irr}(X^N)$ coincide. For the first assertion, let η_1 denote the projection $G_{\mathbb{C}} \longrightarrow X$, and write $G^N_{\mathbb{C}}$ for $\eta_1^{-1}(X^N)$. Then $\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} \cap \mathfrak{n} \simeq H(N) \backslash G^N_{\mathbb{C}}$; write η_2 for the projection of $G^N_{\mathbb{C}}$ onto $\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} \cap \mathfrak{n}$. If C is an irreducible component of X^N, then $\eta_2(\eta_1^{-1}(C))$ is an irreducible component of $\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} \cap \mathfrak{n}$. It is straightforward to check that this correspondence implements the bijection of the lemma. □

We are going to attach a polynomial on $\mathfrak{h}$ to each element of $\mathrm{Irr}(\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} \cap \mathfrak{n})$. Through the W action on $\mathfrak{h}$, W will act on the span of these polynomials; this will be the representation of W indexed by the cell $\mu^{-1}_{\mathrm{orb}}(\mathcal{O}^{K_{\mathbb{C}}})$. The idea, due to Joseph [Jo] (who attributes the idea to D. King), is to measure the growth of the $\mathfrak{h}$-weight spaces of the ring of functions $R(\overline{U})$ on the closure an element of $U \in \mathrm{Irr}(\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} \cap \mathfrak{n})$. More precisely, since $R(\overline{U}) \subset S(\mathfrak{n}^*)$, $\mathfrak{h}$ acts locally finitely on $R(\overline{U})$ with weights of the form $\sum_{\alpha \in -\Delta^+} \mathbb{N}\alpha$. For a fixed $H \in \mathfrak{h}$ and $j \in \mathbb{N}$, write

$$R(\overline{U})^H_j = \{f \in R(\overline{U}) \mid H \cdot f = -jf\}.$$

If H satisfies $\alpha(H) \leq 0$ for all $\alpha \in \Delta^+$, we can write

$$\sum_{j=0}^{k} \dim(R(\overline{U})^H_j) = \pm \left(\frac{p_U(H)}{\prod_{\alpha \in -\Delta^+} \alpha(H)} \right) k^{\dim(U)} + \text{ terms of lower order in } k;$$

here $p_U \in S(\mathfrak{h}^*)$ is a polynomial on $\mathfrak{h}$. We then have the following theorem. When $\widetilde{\mathcal{O}}^{G_\mathbb{C}} = \mathcal{O}^{G_\mathbb{C}}$, the following statement can be extracted from [Ho] and [Ro]; their methods probably extend to handle the general case.

Theorem/Conjecture 4.2. *Fix $\mathcal{O}^{K_\mathbb{C}} = K_\mathbb{C}{\cdot}N \in K_\mathbb{C}\backslash\mathcal{N}_\theta$, and write $\widetilde{\mathcal{O}}^{G_\mathbb{C}}$ for the orbit cover of $G_\mathbb{C}\cdot N$ defined above. Then the span of the polynomials p_U as U ranges over $\mathrm{Irr}(\widetilde{\mathcal{O}}^{G_\mathbb{C}} \cap \mathfrak{n})$ is the W representation attached to the geometric cell $\mu_{orb}^{-1}(\mathcal{O}^{K_\mathbb{C}})$ by Rossmann's action; moreover, the basis element p_U corresponds to the fundamental class of the conormal bundle to the orbit $v \in \mu_{orb}^{-1}(\mathcal{O}^{K_\mathbb{C}})$ associated to U by Lemma 4.1. Finally, the representation coincides with the $A(N)$ invariants in Springer's W action (twisted, as usual, by the sign character) on $H^{top}(X^N, \mathbb{C})$; the basis element p_U corresponds to the sum of the fundamental classes of elements in the $A(N)$ orbit on $\mathrm{Irr}(X^N)$ attached to U by Lemma 4.1.*

Finally, it is worth noting that in the complex case, Kashiwara and Tanisaki [KaT] have shown that the characteristic cycle functor relates the Kazhdan-Lusztig basis of the coherent continuation representation with the geometric cell basis of the topological W action. In this way, Tanisaki [Ta] (in types B and C) and Kashiwara-Saito (in type A) gave examples of highest weight modules with reducible characteristic cycles. Based on the results of Schmid and Vilonen ([SV]), the same methods should be able to detect reducible characteristic cycles of representations of real groups.

5. Robinson-Schensted algorithms for type A real groups.

We are going to specialize the statement of Proposition 3.1 in the case of $U(p,q)$, $SU^*(2n)$, and $GL(n,\mathbb{R})$ to obtain explicit maps from $K_\mathbb{C}\backslash X$ to certain kinds of tableaux. (The resulting maps are called generalized Robinson-Schensted algorithms, as explained in Remark 5.4.) In order to make everything explicit, we first have to parametrize $K_\mathbb{C}\backslash\mathcal{N}_\theta$ and $A_{K_\mathbb{C}}(N)\backslash\mathrm{Irr}(X^N)$ by tableaux. The tableau parametrizations of $K_\mathbb{C}\backslash\mathcal{N}_\theta$ are well-known.

Lemma 5.1. *We have the following parametrizations of $K_\mathbb{C}\backslash\mathcal{N}_\theta$:*
- (a) *For $G = U(p,q)$, $K_\mathbb{C}\backslash\mathcal{N}_\theta$ is parametrized by $\mathcal{T}_\pm(p,q)$.*
- (b) *For $G = SU^*(2n)$, $K_\mathbb{C}\backslash\mathcal{N}_\theta$ is parametrized by $\mathcal{D}_e^{tr}(2n)$.*
- (c) *For $G = GL(n,\mathbb{R})$, $K_\mathbb{C}\backslash\mathcal{N}_\theta$ is parametrized by $\mathcal{D}(n)$.*

Sketch. If $G = GL(n,\mathbb{R})$, then the Jordan form of $N \in \mathcal{N}_\theta$ is a complete invariant for the action of $K_\mathbb{C}$, and (c) follows. When $G = SU^*(2n)$ the Jordan form (over $\mathbb{H}$) identifies $K_\mathbb{C}\backslash\mathcal{N}_\theta$ with $\mathcal{D}(n)$. We prefer, however, to consider the Jordan form over $\mathbb{C}$; this amounts to duplicating each row to a get a tableau whose row lengths all occur an even number of times, thus parametrizing $K_\mathbb{C}\backslash\mathcal{N}_\theta$ by $\mathcal{D}_e^{tr}(2n)$ as in (b). Finally if $G = U(p,q)$, the set of nilpotent elements with a fixed Jordan form breaks into smaller orbits under the action of $K_\mathbb{C} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$. This is reflected in the parametrization by signed tableaux; see [CMc, Theorem 9.3.3], for instance. □

We turn to a tableau parametrization of $A_{K_\mathbb{C}}(N)\backslash\mathrm{Irr}(X^N)$. In each of the above cases, we claim that $A_{K_\mathbb{C}}(N)$ acts trivially on $\mathrm{Irr}(X^N)$. This is obvious if $G_\mathbb{R} = U(p,q)$ since $A_{K_\mathbb{C}}(N)$ is trivial. For $SU^*(2n)$ or $GL(n,\mathbb{R})$, the component groups need not be trivial, so we argue as follows. Clearly the $A_{K_\mathbb{C}}(N)$ orbits on $\mathrm{Irr}(X^N)$ coincide with those of the image of $A_{K_\mathbb{C}}(N)$ in $A_{G_\mathbb{C}}(N)$. In fact, $A_{G_\mathbb{C}}(N)$ acts trivially on $\mathrm{Irr}(X^N)$. To see this, note that the $A_{G_\mathbb{C}}(N)$ orbits on $\mathrm{Irr}(X^N)$ coincide with the orbits of the component group $A_{G'_\mathbb{C}}(N)$ for any connected $G'_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. If $G'_{\mathbb{C}}$ is adjoint, the component group is trivial, so the claim follows.

If X is the flag variety for $GL(n,\mathbb{C})$, then $\mathrm{Irr}(X^N)$ is parametrized by standard Young tableau of size n; we now describe the parametrization. Given $N \in \mathcal{N}$, choose $F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^n) \in X^N$. We obtain a tableau T of size n whose shape is the Jordan form of N by requiring that the first j boxes of T coincide with the Jordan form of N restricted to F_j; we write $T = \gamma(N,F)$. The tableau $\gamma(N,F)$ (as a function of F) is constant on an open subset of each irreducible component of X^N; moreover, it distinguishes such components (see [St1]). Hence we obtain a map $\mathrm{Irr}(X^N) \longrightarrow \mathcal{T}(n)$, which we also denote by γ. (Actually, there is a twist of γ which gives an equally natural parametrization of $\mathrm{Irr}(X^N)$; we return to this following Theorem 5.6.)

Proposition 3.1 then reduces to the following explicit statements.

Corollary 5.2. *Recall Notations 2.1 and 2.3.*

(a) *If $G = U(p,q)$, Proposition 3.1 gives a bijection $(\mu^a_{orb}, RS^a_{orb})$ between $K_{\mathbb{C}}\backslash X$ and the same-shape subset of $\mathcal{T}_{\pm}(p,q) \times \mathcal{T}(p+q)$.*

(b) *If $G = SU^*(2n)$, Proposition 3.1 gives a bijection $(\mu^b_{orb}, RS^b_{orb})$ between $K_{\mathbb{C}}\backslash X$ and the same-shape subset of $\mathcal{D}^{tr}_e(2n) \times \mathcal{T}^{tr}_e(2n)$.*

(c) *If $G = GL(n,\mathbb{R})$, Proposition 3.1 gives a bijection $(\mu^c_{orb}, RS^c_{orb})$ between $K_{\mathbb{C}}\backslash X$ and the same-shape subset of $\mathcal{D}(n) \times \mathcal{T}(n)$.*

(Clearly μ^b_{orb} and μ^c_{orb} are redundant, but we choose to keep them to preserve the analogy.)

Remark 5.3. To be absolutely explicit, we summarize how to compute the bijections appearing in the corollary. Begin with a fixed $v \in K_{\mathbb{C}}\backslash X$, and choose a flag F in v. Consider the moment map image of the corresponding fiber of the conormal bundle to v in X,

$$\mu(T^*_v(X)|_F).$$

Let N be a generic nilpotent in the image. (The sense in which generic is to be understood is explained in the next paragraph.) The $K_{\mathbb{C}}$ orbit through N is $\mu_{\mathrm{orb}}(v)$, and we define $RS_{\mathrm{orb}}(v)$ to be $\gamma(N,F)$ (in the notation introduced just before the corollary).

When we say that N is generic in $\mu(T^*_v(X)|_F)$, we mean that N is not contained in the boundary of $\mu(T^*_v(X))$. (Recall that $\mu(T^*_v(X)$ is the closure of the $K_{\mathbb{C}}$ orbit $\mu_{\mathrm{orb}}(v)$, so this condition makes sense.) Equivalently, the hypothesis that N is generic in $\mu(T^*_v(X)|_F)$ means that, for each i, the dimension of the nilpotent $GL(F_i)$ orbit through $N|_{F_i}$ is as large as possible.

Remark 5.4. If we take $G = GL(n,\mathbb{C})$, then Proposition 3.1 reduces to a bijection from S_n to same-shape pairs of standard Young tableau of size n. As we mentioned in the introduction, Steinberg ([St2]) discovered that the bijection is the Robinson-Schensted algorithm and, motivated by this fact, Springer calls the bijections appearing in parts (a)–(c) (or, more generally, in Proposition 3.1) generalized Robinson-Schensted algorithms.

Remark 5.5. To understand the *existence* of the bijections appearing in the corollary, one need not make reference to Proposition 3.1. Consider part (b) for example. In Proposition 6.1 below, we will see that $K_{\mathbb{C}}\backslash X$ is parametrized by $\Sigma_0(2n)$. Now S_{2n} acts on $\Sigma_0(2n)$ by conjugation, and the isotropy group at a fixed $\sigma \in \Sigma_0(2n)$ is isomorphic to $W(C_n) \simeq (\mathbb{Z}/2)^n \rtimes S_n$, the Weyl group of type C_n. (To get the standard realization of $W(C_n)$, let σ

interchange 1 with $2n$, 2 with $2n-1$, and so on.) The induced representation $\mathrm{ind}_{W(C_r)}^{S_{2n}}(\mathbb{C}_{\mathrm{triv}})$ decomposes as

$$\mathbb{C}[S_{2n}/W(C_n)] = \bigoplus_{\pi \in \mathcal{D}_e^{tr}(2n)} E_\pi .$$

(We are using Young's parametrization of $\widehat{S}_{2n}$ in terms of $\mathcal{D}(2n)$.) Since $\Sigma_0(2n)$ parametrizes $K_{\mathbb{C}}\backslash X$, we can conclude that $K_{\mathbb{C}}\backslash X$ indexes a basis for $\mathbb{C}[S_{2n}/W(C_n)]$. Young's dimension formula for E_π then gives the existence of the bijection appearing in part (b) of the corollary. We leave it to the reader to use Proposition 6.1 to give similar abstract interpretations of the bijections appearing in the corollary. (See [BV], for instance.)

Recall that the classical Robinson-Schensted algorithm computes left and right annihilators of Harish-Chandra modules for $GL(n,\mathbb{C})$. Here is the analog for type A real groups.

Theorem 5.6. *For $G = U(p,q)$, $SU^*(2n)$, or $GL(n,\mathbb{R})$, the generalized Robinson-Schensted algorithms (μ_{orb}, RS_{orb}) (of Corollary 5.2 and Remark 5.3) compute annihilators and associated varieties of Harish-Chandra modules for G. More precisely, we have:*

(a) *Let $G = U(p,q)$, fix $v \in K_{\mathbb{C}}\backslash X$, and let $L_G(v)$ be the irreducible Harish-Chandra module associated to the trivial local system on v. Recall the tableau parametrizations of Theorem 2.2 and Lemma 5.1. We have*

$$(\mu^a_{orb}(v), RS^a_{orb}(v)) = (\mathrm{AV}(L_G(v)), \mathrm{Ann}(L_G(v))).$$

(b) *Let $G = SU^*(2n)$, and consider $v \in K_{\mathbb{C}}\backslash X$. Then*

$$(\mu^b_{orb}(v), RS^b_{orb}(v)) = (\mathrm{AV}(L_G(v)), \mathrm{Ann}(L_G(v))).$$

(c) *For $G = GL(n,\mathbb{R})$, there are either one or two $K_{\mathbb{C}}$ equivariant local systems ϕ on any given orbit $v \in K_{\mathbb{C}}\backslash X$ such that*

$$(\mu^c_{orb}(v), RS^c_{orb}(v)) = (\mathrm{AV}(L_G(v,\phi)), \mathrm{Ann}(L_G(v,\phi))).$$

Remark 5.7. Explicit computations of $(\mu_{\mathrm{orb}}, RS_{\mathrm{orb}})$ (or, more precisely, the right-hand sides of the above equalities) are given in Theorems 7.1, 7.2, and 8.7. In particular, it is possible to describe the local system(s) ϕ appearing in part (c) very explicitly. (We give enough details to do this in the comments following Theorem 8.7.)

Remark 5.8. If v is of the form described in Theorem 2.3, then $\mu_{\mathrm{orb}}(v)$ is (the $K_{\mathbb{C}}$ orbit corresponding to) a real Richardson orbit. To locate which v are of the required form, one must refer to the the Langlands parameter computations in [VZ, Section 6]. This requires some reasonably involved bookkeeping, but is quite tractable in practice.

Remark 5.9. The proof of Theorem 5.6, which we defer until Section 8, is entirely empirical: we simply work out both sides of the equality in Theorem 5.6, and show they coincide. In the course of the proof, we give an explicit identification of $K_{\mathbb{C}}\backslash X$, and work out Vogan's duality on the level of orbits (Section 6). (None of this is new, but none of it is written down anywhere.) In Section 7, we also make some results of Garfinkle [G] a little more explicit.

It is natural to ask for a more conceptual proof than the computational one we give below. Here is what one may hope to be true. By combining Proposition 3.1 and Lemma 4.1, we can associate to each $v \in K\backslash X$ a component of $\widetilde{\mathcal{O}}^{\mathbb{C}} \cap \mathfrak{n}$; here $\widetilde{\mathcal{O}}^{\mathbb{C}}$ is the cover of the complex nilpotent orbit described in Section 4. By projection, we get a component of $\mathcal{O}^{\mathbb{C}} \cap \mathfrak{n}$, and such components are all of the form $\overline{B(Ad(w)\mathfrak{n} \cap \mathfrak{n})}$, for some (generally nonunique) $w \in W$ (see [Spa1]). From w, we can build the primitive ideal $I(w) = \mathrm{Ann}(L_{\mathfrak{b}}(w\rho))$, and it is natural to ask if this ideal is related to the annihilator of the irreducible Harish-Chandra module parametrized by the trivial local system on v.

The hope expressed in the previous paragraph is unreasonable because the primitive ideal $I(w)$ is not well defined — i.e. there can be two elements w_1 and w_2 such that

$$\overline{B(Ad(w_1)\mathfrak{n} \cap \mathfrak{n})} = \overline{B(Ad(\rho)\mathfrak{n} \cap \mathfrak{n})},$$

but for which $\mathrm{Ann}(L_{\mathfrak{b}}(w_1\rho)) \neq \mathrm{Ann}(L_{\mathfrak{b}}(w_2\rho))$. This kind of complication does not arise in our type A setting, but the existence of counterexamples in other types (the smallest of which is in C_4 — see [Mc2]) indicates that no easy conceptual principle is at work here.

To conclude this section we dispense with a slight ambiguity concerning the parametrization $\gamma : \mathrm{Irr}(X^N) \longrightarrow \mathcal{T}(n)$. There is another equally natural choice of this parametrization obtained as follows. Given $N \in \mathcal{N}$, consider a generic $F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^n) \in U \in \mathrm{Irr}(X^N)$. (By analogy with the discussion in the definition of γ, we leave it to the reader to formulate the precise meaning of 'generic' here.) We can build a tableau T' (whose shape is the Jordan form of N) by requiring the first j boxes coincide with the Jordan form of N viewed as a nilpotent endomorphism of F_n/F_{n-j}. Write γ' for the resulting map $\mathrm{Irr}(X) \longrightarrow \mathcal{T}(n)$.

Actually, it is better to think of γ' as follows. Given a flag $F = (F_i)$, define $F_i^\vee = (F_n/F_{n-i})^*$; here $*$ denotes vector space dual. Clearly $F_i^\vee \hookrightarrow F_{i+1}^\vee$, so we have constructed a dual flag $F^\vee = (F_i^\vee)$. (Despite the notation, this has nothing to do with Vogan's duality described below.) Now if $\exp N$ fixes F, and $N^\vee$ denotes the transpose endomorphism of F_n^*, then it is clear that $\exp N^\vee$ fixes $F^\vee$. From the definitions, it's easy to verify that $\gamma'(N, F) = \gamma(N^\vee, F^\vee)$.

In any case, it was Spaltenstein who apparently first noticed that the map $\gamma'\gamma^{-1}$ gives an interesting shape-preserving involution on $\mathcal{T}(n)$; Douglass has recently computed it [Do]. His computation reduces to the following result, which will be crucial in the course of our proof of Theorem 5.6.

Proposition 5.10. *For $T \in \mathcal{T}(n)$ write $T = RS(\sigma)$ for a unique $\sigma \in \Sigma(n)$. Then*

$$\gamma'\gamma^{-1}(T) = RS(w_o \sigma w_o).$$

In particular, both $RS(\sigma)$ and $RS(w_o\sigma w_o)$ have the same shape.

We can interpret Spaltenstein's involution in a more representation theoretic setting as follows. Write τ for the type A diagram automorphism. It induces an involution, say $I \mapsto {}^{\tau}I$, on the set of primitive ideals in the enveloping algebra of $\mathfrak{g}$. Explicitly we can write ${}^{\tau}(\mathrm{Ann}(L))) = \mathrm{Ann}({}^{\tilde{\tau}}L)$; here L is an irreducible $\mathfrak{g}$ module, $\tilde{\tau}$ is any automorphism coming from the outer diagram automorphism, and ${}^{\tilde{\tau}}L$ is the irreducible $\mathfrak{g}$ obtained by composing the action on L with $\tilde{\tau}$.

Corollary 5.11. *Let L be an irreducible $\mathfrak{g}$ module with trivial infinitesimal character. Then, in terms of Theorem 2.2,*

$${}^{\tau}(\mathrm{Ann}(L)) = \gamma'\gamma^{-1}(\mathrm{Ann}(L)).$$

Pf. Theorem 2.1 reduces the proof to the $L = L_{\mathfrak{b}}(w\rho)$ case. Since $\mathrm{Ann}({}^{\bar{\tau}}L_{\mathfrak{b}}(w\rho)) = \mathrm{Ann}(L_{\mathfrak{b}}(w_o w w_o \rho))$, the corollary now follows immediately from Proposition 5.10. □

Hence the ambiguity concerning γ and γ' amounts to a choice of orientation of the type A Dynkin diagram.

6. $K_{\mathbb{C}}$-orbits on X, Vogan duality

In order to prove Theorem 5.6, we will need a precise description of the $K_{\mathbb{C}}$-orbits on X for certain type A real forms.

Proposition 6.1. *Recall Notations 2.1 and 2.2, and fix B to be the upper triangular Borel in $GL(n,\mathbb{C})$ (or $GL(2n,\mathbb{C})$ in part (b) below).*

(a) *For $G = U(p,q)$, $K_{\mathbb{C}}\backslash X$ is parametrized by $\Sigma_{\pm}(p,q)$. The correspondence takes an involution with signed fixed points (σ,ϵ) to the $K_{\mathbb{C}}$ orbit $v_{(\sigma,\epsilon)}$ through gB where $g \in GL(n,\mathbb{C})$ is defined as follows.*

(i) *If $\sigma(l) = l$ and $\epsilon_l = +$, then*

$$g_{kl} = \begin{cases} 1 & \text{if } k = \#\{j \mid j \leq l, \epsilon_j = +\} \\ 0 & \text{else.} \end{cases}$$

(ii) *If $\sigma(l) = l$ and $\epsilon_l = -$, then*

$$g_{kl} = \begin{cases} 1 & \text{if } k = p + \#\{j \mid j \leq l, \epsilon_j = -\} \\ 0 & \text{else.} \end{cases}$$

(iii) *If $\sigma(l) > l$, then*

$$g_{kl} = \begin{cases} 1 & \text{if } k = \#\{j \mid j \leq l, \epsilon_j = +\} \\ 1 & \text{if } k = p + \#\{j \mid j \leq \sigma(l), \epsilon_j = -\} \\ 0 & \text{else.} \end{cases}$$

(iv) *If $\sigma(l) < l$, then*

$$g_{kl} = \begin{cases} -1 & \text{if } k = \#\{j \mid j \leq \sigma(l), \epsilon_j = +\} \\ 1 & \text{if } k = p + \#\{j \mid j \leq l, \epsilon_j = -\} \\ 0 & \text{else.} \end{cases}$$

(b) *For $G = SU^*(2n)$, $K_{\mathbb{C}}\backslash X$ is parametrized by $\Sigma_0(2n)$. The correspondence takes a fixed-point free involution σ to the $K_{\mathbb{C}}$ orbit v_σ through gB where $g \in GL(2n,\mathbb{C})$ is defined by*

(i) *If $l < \sigma(l)$,*

$$g_{kl} = \begin{cases} 1 & \text{if } k = \#\{j \mid j \leq l, j < \sigma(j)\} \\ 0 & \text{else.} \end{cases}$$

(ii) *If $l > \sigma(l)$,*

$$g_{kl} = \begin{cases} 1 & \text{if } k = n + \#\{j \mid j \leq \sigma(l), j < \sigma(j)\} \\ 0 & \text{else.} \end{cases}$$

(c) *For $G = GL(n,\mathbb{R})$, $K_{\mathbb{C}}\backslash X$ is parametrized by $\Sigma(n)$. The correspondence takes an involution σ to the orbit v_σ though gB with g defined by*

(i) *If* $l = \sigma(l)$,

$$g_{kl} = \begin{cases} 1 & \text{if } j = l \\ 0 & \text{else.} \end{cases}$$

(ii) *If* $l \neq \sigma(l)$,

$$g_{kl} = \begin{cases} \sqrt{\frac{1}{2i}} & \text{if } k = l \\ i\sqrt{\frac{1}{2i}} & \text{if } k = \sigma(l) \\ 0 & \text{else.} \end{cases}$$

(The factor $\sqrt{\frac{1}{2i}}$ is just a convenient normalization arranged so that for g as defined in part (c), g^2 is the permutation matrix corresponding to σ.)

Sketch. Consider part (c). It is possible to give a very explicit discussion of the formulas appearing above, but in the proofs of Theorems 7.1–7.3, we will need to relate $K_{\mathbb{C}}\backslash X$ to Langlands parameters, and for this purpose it is useful to give a slightly more abstract description of $K_{\mathbb{C}}\backslash X$. (In the proofs of Theorems 7.1–7.3 we omit the details of the correspondence with Langlands parameters. We include the discussion below, so that the reader may have an easier task of supplying the omitted details.) A convenient reference for the argument below is [RSp, Section 10], but the main ideas go back much further, essentially to Matsuki [Ma]. Write $g \cdot_\theta (F, F') = (gF, \theta(g)F')$ for the θ-twisted action of G on $X \times X$. The orbits consist of those flags in relative θ-position w,

$$S_w^\theta = \{(F, F') \mid (F, \theta(F')) \text{ are in position } w\}.$$

Write Δ for the diagonal in $X \times X$. Now $S_w^\theta \cap \Delta$ is transparently a union of orbits for the diagonal action of $K_{\mathbb{C}}$ on Δ, which we will think of as $K_{\mathbb{C}}$ orbits on X. In fact, simple dimension considerations imply that each irreducible component of the intersection is a single $K_{\mathbb{C}}$ orbit. In the case of $GL(n, \mathbb{R})$, the intersections are already irreducible, and they are non-empty exactly when $w \in \Sigma(n)$. In this way, we obtain a bijection $\Sigma(n) \longrightarrow K_{\mathbb{C}}\backslash X$. In terms of the parametrization above, the reader can check that $w \mapsto v_{w_o w}$, the orbit that the formulas of part (c) attach to $w_o w$.

For parts (a) and (b), we give a concrete sketch of the formulas appearing in the proposition. (The interested reader can adapt the above discussion for $GL(n, \mathbb{R})$ to the cases at hand.) Consider part (b). To each $2n$ dimensional flag $F = (F_0 \subset \cdots F_{2n}) \in X$, we will attach an element $\sigma \in \Sigma_0(2n)$ that depends only on the $K_{\mathbb{C}}$ orbit of F. Let k be the unique index such that the dimension of $F_1^\perp$ in F_{k-1} is equal to the dimension of $F_1^\perp$ in F_k; here $\perp$ is taken with respect to the symplectic form defining $K_{\mathbb{C}} = Sp(2n, \mathbb{C})$. (Generically $k = 2$, so k is a measure of the degeneracy of the orbit through F.) We define $\sigma(1) = k$. Now take any vector $u \in F_k$ but not in F_{k-1}, and let U be the subspace spanned by F_1 and u. Then the symplectic form on F_{2n} descends to a nondegenerate one on F_{2n}/U, and we can define $\sigma(2) \in \{3, \ldots, k-1, k+1, \ldots, 2n\}$ by the analogous procedure applied to the $2n-2$ dimensional flag $(F_i/(F_i \cap U))$ (whose nontrivial subspaces we think of as indexed by $\{2, 3, \ldots, k-1, k+1, \ldots, 2n\}$). Inductively, we obtain $\sigma \in \Sigma_0(2n)$. This data characterizes the $K_{\mathbb{C}}$ orbit of (F_i). The formulas that appear in part (b) are now clear from the above discussion. (In fact, using this description, the closure ordering on the orbits follows easily.)

We argue similarly for part (a) by first attaching $(\sigma, \epsilon) \in \Sigma_\pm(p, q)$ to a fixed flag $(F_i) \in X$. Let V^+ be the span of the first p coordinates of $\mathbb{C}^n$, and V^- the span of the last q. If $\dim(V^+ \cap F_j) = \dim(V^+ \cap F_{j-1})$ then $\sigma(j) = j$ and $\epsilon_j = +$. Similarly if $\dim(V^- \cap F_j) =$

$\dim(V^- \cap F_{j-1})$ then $\sigma(j) = j$ and $\epsilon_j = -$. One the other hand, let j be the smallest index such that $\dim(V^\pm \cap F_j) = \dim(V^\pm \cap F_{j-1})$. We describe $\sigma(j) > j$ and then ϵ_j and $\epsilon_{\sigma(j)}$ are fixed by the convention in the definition of $\Sigma_\pm(p,q)$. So take $u_j \in F_j - F_{j-1}$, and let k be the smallest index (greater than j) so that there is a vector $u_k \in F_k - F_{k-1}$ with the property that the form defining $U(p,q)$ restricts to a signature $(1,1)$ form on U, the span of u_j and u_k. This defines $\sigma(j) = k$, and we can proceed inductively to complete the description of (σ, ϵ). By an argument along the lines of those given in [Ya], one can show that this data characterizes the $K_{\mathbb{C}}$ orbit of (F_i). The explicit formulas of part (a) follow easily. □

Remark 6.2. The statement in part (a) remains valid if we replace $G = U(p,q)$ by $SU(p,q)$. When n is odd, part (c) remains unchanged if $GL(n,\mathbb{R})$ is replaced by $SL(n,\mathbb{R})$. For n even, the $O(n,\mathbb{C})$ orbit parametrized by $\sigma \in \Sigma_0(n) \subset \Sigma(n)$ is a union of two $SO(n,\mathbb{C})$ orbits, as can be see already in the $n = 2$ case.

Next we describe the $K_{\mathbb{C}}$ equivariant local systems on each $v \in K_{\mathbb{C}}\backslash X$, which amounts to computing the centralizer component group $A(v) = Z_{K_{\mathbb{C}}}(v)/Z^\circ_{K_{\mathbb{C}}}(v)$. This group identifies with the component group of the Cartan subgroup of G corresponding to the Cartan subalgebra in a θ-stable Borel contained in v (see [V2] for example). The Cartan subgroups in our real forms of interest are well-known: for $U(p,q)$ and $SU^*(2n)$, they are all connected; and for $GL(n,\mathbb{R})$, the number of their connected components is a power of 2. Parts (a) and (b) of the next result are now clear; for part (c) one must do a little bookkeeping (which we omit).

Proposition 6.3. *Recall Proposition 6.1.*

(a) *For $G = U(p,q)$, each $A(v)$ is trivial.*
(b) *For $G = SU^*(2n)$, each $A(v)$ is trivial.*
(c) *For $G = GL(n,\mathbb{R})$ and $v_\sigma \in K_{\mathbb{C}}\backslash X$ parametrized by $\sigma \in \Sigma(n)$, $A(v_\sigma) \simeq (\mathbb{Z}/2)^r$ where r is the number of fixed points of σ.*

Corollary 6.4. *For $G = GL(n,\mathbb{R})$, the set of $K_{\mathbb{C}}$ equivariant local systems on $v_\sigma \in K_{\mathbb{C}}\backslash X$ is in bijection with the set*

$$\{(\tau, \epsilon) \in \Sigma_\pm(n) \mid \tau = \sigma\}.$$

In particular, the set of pairs consisting of an orbit $v \in K_{\mathbb{C}}\backslash X$ and a $K_{\mathbb{C}}$ equivariant local system on v is in bijection with $\Sigma_\pm(n)$.

Remark 6.5. When $G = SU(p,q)$, the groups $A(v)$ are either trivial or isomorphic to $\mathbb{Z}/2$. The latter case happens precisely when $p = q$ and v corresponds to (σ, ϵ) with $\sigma \in \Sigma_0(2p)$ being fixed-point free. Any orbit for $SL(n,\mathbb{R})$ parametrized by $\sigma \in \Sigma(n)$ with r fixed points has $A(v)$ equal to the subgroup of $(\mathbb{Z}/2)^r$ consisting of those r tuples with an even number of nontrivial elements.

We now turn to representation theory. In [V3], Vogan defines a duality on irreducible Harish-Chandra modules of different real forms (of a complex group and its Langlands dual) that behaves nicely with respect to composition series. The duality is not unique; nonetheless, we will write $L^\vee$ for any choice of the dual of L. The key formal property that we will need is that, up to tensoring with the sign representation, the duality intertwines the coherent continuation representation. This may be found in [V3, Corollary 14.9c] which, in our context, simplifies to the following result.

Proposition 6.6. *Recall Theorem 2.2 and suppose G is a real form of $GL(n,\mathbb{C})$. Then the tableau parametrizing $Ann(L^\vee)$ is the transpose of the one parametrizing $Ann(L)$.*

We now write down (a choice of) the duality on the level of orbits. The next two propositions follow from carefully applying the recipes of [V3].

Proposition 6.7. *Fix $\sigma \in \Sigma_0(2n)$; then (by definition—see Notation 2.2) there is a unique ϵ with $(\sigma,\epsilon) \in \Sigma_\pm(2n)$. Let v_σ and $\check{v} = v_{(\sigma,\epsilon)}$ denote the corresponding orbits (Proposition 6.1). We have*

$$(L_{SU^*(2n)}(v_\sigma))^\vee = L_{SU(n,n)}(\check{v},\check{\phi}),$$

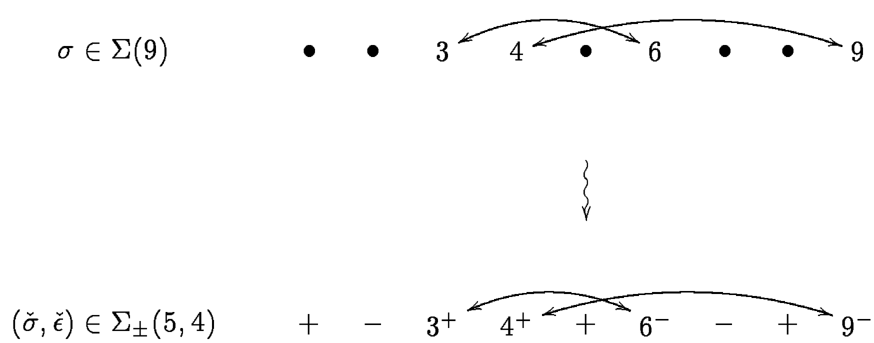
where $\check{\phi}$ is the unique nontrivial local system on the indicated orbit (Remarks 6.2 and 6.5).

In order to describe the duality for $GL(n,\mathbb{R})$ we need to give a combinatorial construction of an involution with signed fixed points $(\check{\sigma},\check{\epsilon}) \in \Sigma_\pm(n)$ from an involution $\sigma \in \Sigma(n)$. So fix $\sigma \in \Sigma(n)$, take $\check{\sigma} = \sigma$, and define

$$\begin{aligned} \epsilon_i &= + \text{ if } i < \sigma(i), \\ \epsilon_i &= - \text{ if } i > \sigma(i); \end{aligned}$$

these definition are required by the normalization in the definition of $\Sigma_\pm(n)$. We assign the first fixed point of σ a $+$ sign, and then require the signs to alternate along the remaining fixed points; more precisely, list the fixed points of σ in increasing order as $r_1,\ldots,r_l$, and set $\epsilon_{r_i} = (-1)^{i+1}$. (We could just as well have chosen $\epsilon_{r_i} = (-1)^i$, reflecting the nonuniqueness of the duality.)

Example 6.8. Given $\sigma = (36)(49) \in \Sigma(9)$, we apply the algorithm as follows.



The picture means that $\check{\sigma} = (36)(49)$ while

$$\check{\epsilon} = (+,-,+,+,+,-,-,+,-).$$

Proposition 6.9. *Let $\sigma \in \Sigma(n)$ and let $(\check{\sigma},\check{\epsilon}) \in \Sigma_\pm(p,q)$ be as defined in the previous paragraph, and let v and $\check{v}$ denote the corresponding orbits under Proposition 6.1. Then*

$$(L_{GL(n,\mathbb{R})}(v))^\vee = L_{U(p,q)}(\check{v}).$$

Remark 6.10. Using Corollary 6.4, we see that the full duality

$$\widehat{GL(n,\mathbb{R})}_\rho \longrightarrow \coprod_{p+q=n} \widehat{U(p,q)}_\rho$$

amounts to a bijection

$$\Sigma_{\pm}(n) \longrightarrow \coprod_{p+q=n} \Sigma_{\pm}(p,q).$$

The algorithm given before the statement of the proposition can be naturally extended to this larger domain giving an explicit formulation of the duality on all of $\widehat{GL(n,\mathbb{R})}_{\rho}$ — we leave the precise formulation to the reader. Below, however, we will make use of one qualitative feature of the answer, namely

$$(L_{GL(n,\mathbb{R})}(v_\sigma,\phi))^\vee = L_{U(p,q)}(v_{(\sigma,\epsilon')}),$$

for some ϵ' depending on σ and ϵ. (In words: the full duality, like the algorithm given in Proposition 6.9, doesn't alter the underlying involution.)

7. Annihilators for $U(p,q)$, $GL(n,\mathbb{R})$, and $SU^*(2n)$

In this section we recall Garfinkle's algorithms to compute $\mathrm{Ann}(L(v))$ for $v \in K_{\mathbb{C}}\backslash X$. The first group we treat is $U(p,q)$. (The following holds verbatim for $SU(p,q)$.) Given $(\sigma,\epsilon) \in \Sigma_{\pm}(p,q)$, form a sequence of pairs of the form

$$(i,\epsilon_i) \text{ if } \sigma(i)=i; \text{ and}$$
$$(i,\sigma(i)) \text{ if } i<\sigma(i).$$

Arrange the pairs in order by their largest entry, with the convention that a sign has numerical size zero. Write $\pi_1,\dots,\pi_r$ for the resulting ordered sequence. (For instance,

$$(1,+),(2,-),(5,+),(3,6),(7,+),(8,-),(4,9)$$

is the sequence corresponding to $(\check{\sigma},\check{\epsilon})$ in Example 6.8).

We now give Garfinkle's algorithm describing a same-shape pair of tableaux

$$(\Psi_1^a(v),\Psi_2^a(v)) \in \mathcal{T}_{\pm}(p+q)\times\mathcal{T}(p+q).$$

Each tableau is constructed by inductively adding the pairs π_j. So suppose that we have added $\pi_1,\dots,\pi_{j-1}$ to get a (smaller) same-shape pair of tableau $(T_{\pm},T)$. If $\pi_j=(k,\epsilon_k)$, then we first add the sign ϵ_k to the topmost row of (a signed tableau in the equivalence class of) $T_{\pm}$ so that the resulting tableau has signs alternating across rows. Then add the index j to T in the unique position so that the two new tableaux have the have the same shape. If $\pi_j=(k,\sigma(k))$ we first add k to T using the Robinson-Schensted bumping algorithm to get a a new tableau T', and then add a sign ε (either $+$ or $-$ as needed) to $T_{\pm}$ so that the result is a signed tableau $T'_{\pm}$ of the same shape as T'. We then add the pair $(\sigma(k),-\varepsilon)$ (by the recipe of the first case) to the first row strictly below the row to which ε was added. We continue inductively to get $(\Psi_1^a(v),\Psi_2^a(v)) \in \mathcal{T}_{\pm}(p+q)\times\mathcal{T}(p+q)$. (For a more formal definition, the reader is referred to [G].)

Theorem 7.1. *Let $G=U(p,q)$, take $(\sigma,\epsilon)\in\Sigma_{\pm}(p,q)$, and let $v\in K_{\mathbb{C}}\backslash X$ be the corresponding $K_{\mathbb{C}}$ orbit of Proposition 6.1. Then, given the tableau parametrizations of Theorem 2.2 and Lemma 5.1, $(\Psi_1^a(v),\Psi_2^a(v))$ is the associated variety and annihilator of $L_{U(p,q)}(v)$. The subsets*

$$\{L(v) \mid \Psi_1^a(v)=\mathcal{O}\}$$

of $\widehat{U(p,q)}_\rho$ *exhaust the cells of Harish-Chandra modules as* $\mathcal{O}$ *ranges over the nilpotent* $K_{\mathbb{C}}$ *orbits in* $\mathfrak{p}$.

Pf. We have arranged our parametrization of $K_{\mathbb{C}}\backslash X$ to coincide with the $\mathbb{Z}/2$ data (or, equivalently, Langlands parameters — see Corollary 2.2 of [V2] for the details of the correspondence) that Garfinkle uses in [G]. So the annihilator part follows. The associated variety statement was observed independently by a number of people; see [Tr, Section 4], for example. Barbasch and Vogan [BV] give a counting argument to show how the assertion about cell structure follows from the associated variety statement. □

Next we turn to $SU^*(2n)$. Given $\sigma \in \Sigma_0(2n)$, or the corresponding orbit $v \in K_{\mathbb{C}}\backslash X$, we describe an element $\Psi_2^b(v) \in \mathcal{T}_e^{tr}(n)$. As above σ gives rise to an ordered sequence of pairs of integers $\pi_1, \dots, \pi_n$, by ordering the pairs

$$(i, \sigma(i)) \text{ for } i < \sigma(i)$$

by their maximal entry. We construct the *transpose* of $\Psi_2^b(v)$ inductively by adding the pairs π_j. So suppose the pairs $\pi_1, \dots, \pi_{j-1}$ have been added to produce a tableau T. To add the jth pair $(k, \sigma(k))$, we first add k to T using the Robinson-Schensted procedure to get a tableau T', and then add $\sigma(k)$ to the end of the (unique) row of T' which is longer than the corresponding row of T. Inductively we obtain an element of $\mathcal{T}_e(2n)$ whose transpose (which lives in $\mathcal{T}_e^{tr}(2n)$) we define to be $\Psi_2^b(v)$. We then define $\Psi_1^b(v) \in \mathcal{D}_e^{tr}(2n)$ to be the shape of $\Psi_2^b(v)$. (Again this is redundant, but we elect to preserve the analogy.)

Theorem 7.2. *Let* $G = SU^*(2n)$, *take* $\sigma \in \Sigma_0(2n)$, *and let* v *denote the corresponding* $K_{\mathbb{C}}$*-orbit (Proposition 6.1(b)). Then the pair* $(\Psi_1^b(v), \Psi_2^b(v))$ *is the the associated variety and annihilator of* $L_{SU^*(2n)}(v)$. *The fibers*

$$\{L(v) \mid \Psi_1^b(v) = \mathcal{O}\}$$

exhaust the cells of Harish-Chandra modules for $SU^*(2n)$ *as* $\mathcal{O}$ *ranges over the nilpotent* $K_{\mathbb{C}}$ *orbits on* $\mathfrak{p}$.

Pf. Again, we have arranged our parametrization of $K_{\mathbb{C}}\backslash X$ to coincide with the $\mathbb{Z}/2$ data Garfinkle uses in [G]. So the annihilator part follows. The associated variety part is trivial, since $\Psi_1^b(v)$ is the unique nilpotent orbit whose shape coincides with the shape of $\Psi_2^b(v)$. In the same way as $U(p,q)$, the cell structure follows from the associated variety statement. □

Finally we treat $GL(n,\mathbb{R})$. Given $v \in K_{\mathbb{C}}\backslash X$, let $\check{v}$ denote the orbit for $U(p,q)$ described in Proposition 6.9. We define $\Psi_2^c(v) \in \mathcal{T}(n)$ denote to be the transpose of $\Psi_2^a(\check{v})$, and let $\Psi_1^c(v) \in \mathcal{D}(n)$ (redundantly) denote its shape.

Theorem 7.3. *Let* $G = GL(n,\mathbb{R})$, *take* $\sigma \in \Sigma(n)$, *and let* v *denote the corresponding* $K_{\mathbb{C}}$ *orbit (Proposition 6.1(c)). The pair* $(\Psi_1^c(v), \Psi_2^c(v))$ *is the associated variety and annihilator of* $L_{GL(n,\mathbb{R})}(v)$.

Pf. The annihilator statement follows from definition of Ψ_2^c, together with Proposition 6.6 and the corresponding computation of annihilators for $U(p,q)$ (Theorem 7.1). The associated variety statement follows from same-shape considerations. □

Remark 7.4. Except in special cases, the fibers of Ψ_1^c do *not* parametrize cells of Harish-Chandra modules for $GL(n,\mathbb{R})$. (The fibers of Ψ_1^c will be subsets of orbits, while cells of Harish-Chandra modules will correspond to subsets of local systems on orbits.)

We now note a mildy interesting combinatorial consequence of Corollary 5.11. If σ is an element of $\Sigma(n)$, write ${}^{\tau}\sigma$ for $w_o \sigma w_o$. Similarly if (σ, ϵ) is in $\Sigma_{\pm}(p,q)$ write $({}^{\tau}\sigma, {}^{\tau}\epsilon) \in \Sigma_{\pm}(p,q)$ for the pair

$$\begin{aligned} {}^{\tau}\sigma &= w_o w w_o; \\ {}^{\tau}\epsilon_i &= \epsilon_{w_o i w_o} \quad \text{if } {}^{\tau}\sigma(i) = i; \\ {}^{\tau}\epsilon_i &= -\epsilon_{w_o i w_o} \quad \text{if } {}^{\tau}\sigma(i) \neq i. \end{aligned}$$

(The different condition on the signs arises from the normalizations in the definition of $\Sigma_{\pm}(p,q)$.) Write v and ${}^{\tau}v$ for the corresponding orbits in the case of $G = U(p,q)$, $SU^*(2n)$, or $GL(n,\mathbb{R})$. One can check that (in the notation around Corollary 5.11) we have

$$ {}^{\tilde{\tau}}L_G(v) \cong L_G({}^{\tau}v). $$

Using this isomorphism, we can thus compute the annihilator of ${}^{\tilde{\tau}}L_G(v)$ by applying the appropriate Ψ_2 to ${}^{\tau}v$. On the other hand, Corollary 5.11 and Proposition 5.10 give another means to compute the annihilator by applying the tableau involution $\gamma'\gamma^{-1}$ to the tableau obtained by applying the appropriate Ψ_2 to v. Note that it is not at all clear from the definitions of the various Ψ_2 (especially Ψ_2^a) that we get the same answer; that we do is to be interpreted as a latent symmetry of their definitions.

8. Proof of Theorem 5.6

In this section we prove Theorem 5.6. We first treat the case of $U(p,q)$. In terms of the notation established in Corollary 5.2, we are to prove

$$ (\mu^a_{\mathrm{orb}}(v), RS^a_{\mathrm{orb}}(v)) = (\mathrm{AV}(L_{U(p,q)}(v)), \mathrm{Ann}(L_{U(p,q)}(v))). $$

Half of this is straightforward.

Lemma 8.1. $\mu^a_{orb}(v) = \mathrm{AV}(L_{U(p,q)}(v))$.

Remark 8.2. Because it is clear from the context, we will abbreviate $\mathrm{AV}(L_{U(p,q)}(v))$ by $\mathrm{AV}(v)$; similar notation will apply to $\mathrm{Ann}(v)$. When it is also clear from the context we will drop the superscripts in our notation, abreviating μ^a_{orb} by just μ_{orb} for instance. As usual, we will not distinguish between a tableau and the orbit (or primitive ideal) it parametrizes.

Pf. Yamamoto [Ya] has given an algorithm to compute $\mu_{\mathrm{orb}}(v)$; so to prove the lemma, we have only to compare her algorithm with Garfinkle's. This is possible, but very complicated (mainly because Yamamoto's algorithm itself is complicated), so we give an alternate argument based on the counting considerations of [BV]. From general principles, one knows that $\mu_{\mathrm{orb}}(v)$ is contained in the associated variety $\mathrm{AV}(v)$ (see [BoBr, Propositions 2.6,2.8]). Thus the statement of the lemma is equivalent to the equality of the dimensions of $\mu_{\mathrm{orb}}(v)$ and $\mathrm{AV}(v)$. We prove the equality of dimensions (and hence the lemma) by an induction on the dimension $\mu_{\mathrm{orb}}(v)$. The lemma is clearly true when v is open (i.e. when $\mu_{\mathrm{orb}}(v)$ and $\mathrm{AV}(v)$ are zero). Suppose we can find v such that

$$ \dim(\mu_{\mathrm{orb}}(v)) < \dim(\mathrm{AV}(v)). $$

Choose v so that $\dim(\mathrm{AV}(v))$ is minimal subject to this inequality. Consider the subsets $A, B \subset K\backslash X$,

$$\begin{aligned} A &= \{w \in K\backslash X \mid \mu_{\mathrm{orb}}(w) = \mu_{\mathrm{orb}}(v)\}; \\ B &= \{w \in K\backslash X \mid \mathrm{AV}(w) = \mu_{\mathrm{orb}}(v)\}. \end{aligned}$$

The induction hypothesis says that $B \subset A$, and the inclusion is proper since v is contained in A but not B by hypothesis. But Corollary 5.2(a) and Theorem 7.1 imply (respectively) that the cardinality of A and the cardinality of B are both equal to the number of standard Young tableaux whose shape matches that of $\mu_{\mathrm{orb}}(v)$. This contradicts the proper inclusion $A \subset B$, and hence the dimension inequality above, so the proof is complete. □

The next lemma, which follows directly from [St2, Lemma 1.2], will be crucial for the other half of the theorem.

Lemma 8.3. *Let $F = (0 = F_0 \subset \cdots F_n = \mathbb{C}^n)$ be an n dimensional flag fixed by* $\exp N$. *Let $U \simeq \mathbb{C}^{n-2}$ be an N-stable hyperplane in F_{n-1} and write $F' = F \cap U$ for the flag*

$$(F_0 \cap U) \subset \cdots \subset (F_n \cap U).$$

Then for some index k we can write F' as

$$0 = F_0 \subset \cdots F_{k-1} \subset (F_{k+1} \cap U) \subset \cdots \subset (F_{n-1} \cap U) = U.$$

Let N' denote the restriction of N to U, so that F' is fixed by $\exp N'$. *Assume that the restriction $N'' = N|_{F_{n-1}}$ is a generic extension of N' to F_{n-1} in the sense that* $\dim(G_{\mathbb{C}} \cdot N'')$ *is maximal subject to the condition that*

(a) *$N''|_U = N'$; and*

(b) $\exp N''$ *fixes $F_0 \subset F_1 \subset \cdots \subset F_{n-1}$.*

Let T' denote the tableau obtained from $\gamma(N, F')$ by switching the entries from $1, \ldots, n-2$ to $1, \ldots, k-1, k+1, \ldots, n-1$. Then the first $n-1$ boxes of $\gamma(N, F)$ are obtained by adding k to T' using Robinson-Schensted insertion.

(Of course the statement is really about $n-1$ dimensional flags. We have phrased it in this slightly confusing way in order to make the applications below a little more transparent.)

Before proceeding to the proof, we set aside several characteristics of Garfinkle's algorithm; the easy verification is left to the reader.

Lemma 8.4. *Fix $\sigma \in \Sigma_{\pm}(p, q)$, write v for the corresponding orbit, and let $\pi_1, \ldots, \pi_r$ be the sequence of pairs as described above. Let T denote the tableau constructed from the first $r-1$ pairs $\pi_1, \ldots, \pi_{r-1}$. Then either $\pi_r = (n, \epsilon_n)$ or $\pi_r = (k, n)$. If $\pi_r = (k, n)$, then the first $n-1$ boxes of* $\mathrm{Ann}(L_{U(p,q)}(v))$ *are obtained by adding k to T using the Robinson-Schensted insertion procedure.*

We now give a detailed argument that (in the simplified notation of Remark 8.2) $RS_{\mathrm{orb}}(v) = \mathrm{Ann}(v)$. Take $\sigma \in \Sigma_{\pm}(p, q)$, write v for the corresponding orbit (Proposition 6.1), and write

$$F = (F_0 \subset F_1 \subset \cdots \subset F_n)$$

for the representative given in the proposition. Write $\pi_1, \ldots \pi_r$ for the sequence of pairs attached to σ by the procedure given before Theorem 7.1.

First assume that $\pi_r = (n, \epsilon_n)$ and (without loss of generality) that $\epsilon_n = -$. Let U denote the $n-1$ dimensional subspace of V (as in Notation 2.1) spanned by $e_1, \ldots, e_{n-1}$. Let $G' \simeq U(p, q-1)$ denote the subgroup of $GL(U)$ preserving the form $\langle\ ,\ \rangle$ (of Notation 2.1) restricted to U. Then set

$$F' = (F_0 \subset F_1 \subset \cdots \subset F_{n-1}).$$

From Proposition 6.1, one can check that F' is the representative of the orbit v' attached to $\sigma' \in \Sigma_{\pm}(p, q-1)$ determined by the sequence of pairs $\pi_1, \ldots, \pi_{r-1}$ (with π_r omitted).

Now let N be a generic nilpotent (in the sense of Remark 5.3) in the moment map image $\mu(T_v^*(X)|_F)$. By definition,

$$N|_U \text{ is generic in } \mu(T_{v'}^*(X')|_{F'}).$$

Hence, by Remark 5.3, the first $n-1$ boxes of $RS_{\text{orb}}(v)$ coincide with $RS_{\text{orb}}(v')$ which, by induction, we can assume coincides with the tableau T obtained by applying Garfinkle's algorithm to the pairs $\pi_1, \ldots, \pi_{r-1}$. But (from the definition of Garfinkle's algorithm) these are the first $n-1$ boxes of $\text{Ann}(v)$. The last two sentences imply that the first $n-1$ boxes of $RS_{\text{orb}}(v)$ coincide with those of $\text{Ann}(v)$. Lemma 8.1 finishes the proof in this case.

To complete the proof, we must treat the case when $\pi_r = (k, \sigma(k) = n)$. In this case, let U be the subspace of V (as in Notation 2.1) spanned by

$$e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_{n-1},$$

and let $G' \simeq U(p-1, q-1)$ denote the subgroup of $GL(U)$ preserving $\langle\ ,\ \rangle$ restricted to U. Write

$$F' = (F_0 \cap U) \subset \cdots \subset (F_n \cap U).$$

Explicitly from Proposition 6.1, one sees that F' equals

$$0 = F_0 \subset \cdots F_{k-1} \subset (F_{k+1} \cap U) \subset \cdots \subset (F_{n-1} \cap U) = U,$$

and that F' is a representative that the proposition gives for the orbit v' corresponding to $\sigma' \in \Sigma_\pm(p-1, q-1)$ attached to $\pi_1, \ldots, \pi_{r-1}$ (with π_r omitted).

We will show that the first $n-1$ boxes of $RS_{\text{orb}}(v)$ and $\text{Ann}(v)$ coincide. Appealing to Lemma 8.1 then shows that $RS_{\text{orb}}(v) = \text{Ann}(v)$. So let N be a generic nilpotent in $\mu(T_v^*(X))$ (in the sense of Remark 5.3). Appealing to the definitions, once again we find that

$$N' = N|_U \text{ is generic in } \mu(T_{v'}^*(X')|_{F'}).$$

Hence, by induction, we may assume that the tableau T', obtained by relabeling the boxes of $RS_{\text{orb}}(v')$ by $1, \ldots, k-1, k+1, \ldots, n-1$ (instead of $1, \ldots, n-2$), is the tableau that Garfinkle's algorithm attaches to $\pi_1, \ldots, \pi_{r-1}$. Lemma 8.3 implies that the first $n-1$ boxes of $RS_{\text{orb}}(v)$ are obtained by inserting k into T' using Robinson-Schensted. (Actually, there is something subtle to check here; see the discussion in the next paragraph.) In any event, by Lemma 8.4, we see that the first $n-1$ boxes of $RS_{\text{orb}}(v)$ and $\text{Ann}(v)$ coincide. This completes the proof for $U(p,q)$.

As we mentioned above, we must be a little careful about applying Lemma 8.3. The hypothesis of the lemma requires that $N'' = N|_{F_{n-1}}$ be a generic extension of $N' = F|_U$. This would seem to follow immediately from the generic assumption on N, but it is more subtle than that. The generic extension hypothesis of the lemma requires the $GL(F_{n-1})$ orbit through N'' to be maximal, but here we are dealing with $K_{\mathbb{C}}$ orbits. More precisely, the shape of a generic extension of N' to F_{n-1} is obtained by adding some specified corner to the shape of N'; the point is that that the resulting shape may not, a priori, be a subshape of the shape of N (since there are alternating sign conditions to worry about). This never causes problems in our setting because we have two dimensions of freedom: the shape of N' (which is the shape of a signature $(p-1, q-1)$ tableau) plus any corner is a subshape of the shape of a signature (p,q) tableau. So Lemma 8.3 applies, and the argument is complete.

Next we consider the case of $SU^*(2n)$. First we need to record some results analogous to those of Lemma 8.4 and Lemma 8.3.

Lemma 8.5. *Fix $\sigma \in \Sigma_0(2n)$, write $k = \sigma(2n)$, and let v denote the corresponding $K_{\mathbb{C}}$ orbit. Write $\sigma' \in \Sigma_0(2n-2)$ for the involution obtained by viewing σ as a permutation of the letters $1, \ldots, k-1, k+1, \ldots, 2n-1$, and write v' for the corresponding orbit. Let T' denote the tableau obtained by switching the entries of $\mathrm{Ann}(L_{SU^*(2n-1)}(v'))$ from $1, \ldots, 2n-2$ to $1, \ldots, k-1, k+1, \ldots, 2n-1$. Then the first $2n-1$ boxes of $\mathrm{Ann}(L_{SU^*(2n)(v)})$ are obtained by adding k to the transpose of T' using Robinson-Schensted insertion, and then taking the transpose of the resulting diagram.*

The next lemma (especially its proof) explains why the transpose of Robinson-Schensted insertion is appearing.

Lemma 8.6. *Let $F = (0 = F_0 \subset \cdots F_n = \mathbb{C}^n)$ be an n dimensional flag fixed by $\exp N$. Let $U \simeq \mathbb{C}^2$ be an N-stable plane in F_n not contained in F_{n-1}, and write $F' = F/U$ for the flag*

$$0 = F_0/(F_0 \cap U) \subset \cdots \subset F_n/(F_n \cap U).$$

Then for some index k we can write F' as

$$\begin{aligned} 0 = &F_0 \subset \cdots \subset F_{k-1} \subset \\ &F_{k+1}/(F_{k+1} \cap U) \subset \cdots \subset F_{n-1}/(F_{n-1} \cap U). \end{aligned}$$

Let N' denote map induced by N on $F_{n-1}/(F_{n-1} \cap U)$ so that F' is fixed by $\exp N'$. Assume that $N|_{F_{n-1}}$ is a generic lift of N' (in the sense analogous to the condition in Lemma 8.3). Write T' for the tableau obtained from $\gamma(N, F')$ by changing the entries from $1, \ldots, n-2$ to $1, \ldots, k-1, k+1, \ldots, n-1$. Then the first $n-1$ boxes of $\gamma(N, F)$ are obtained by first adding k to the transpose of T' using Robinson-Schensted insertion, and then taking the transpose of the resulting tableau.

Sketch. As in the discussion preceding Proposition 5.10, given any flag $F = (F_i)$, we can form a dual flag $F^\vee = (F_i^\vee)$ defined by $F_i^\vee = (F_n/F_{n-i})^*$. Note that the dual of the flag F/U is of the form $F^\vee \cap V^*$, where V^* is the vector space dual of an $n-2$ dimensional complement to U. Using this observation, together with the explicit form of Proposition 5.10 and the standard interpretation of transpose in terms of the Robinson-Schensted algorithm, one can then deduce the present lemma from Lemma 8.3. We omit the details. ☐

Now we prove $RS^b_{\mathrm{orb}}(v) = \mathrm{Ann}(L_{SU^*(2n)}(v))$, thus completing the proof of Theorem 5.6(b) for $G = SU^*(2n)$. (Since the context is clear, we write this equality as $RS_{\mathrm{orb}}(v) = \mathrm{Ann}(v)$, as in Remark 8.2.) Fix $\sigma \in \Sigma_0$, and write $k = \sigma(n)$. Consider the subspace U of $\mathbb{C}^n$ spanned by the $2n-2$ vectors $e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_{2n-1}$, and let $G' = GL(U)$. As above, we can form the flag $F' = F \cap U$. Then F' is the representative that Proposition 6.1 gives for the orbit v' attached to the involution $\sigma' \in \Sigma_0(2n-2)$ obtained by viewing σ as an involution of the $2n-2$ letters $1, \ldots, k-1, k+1, \ldots, 2n-1$. But now a problem arises: if N is generic in $\mu(T_v^*(X))$, then (except in very special cases) $N|_U$ will not even fix the flag F', let alone be in the moment map image $\mu(T_{v'}^*(X'))$.

Instead we need to define $U = \mathbb{C}e_k \oplus \mathbb{C}e_{2n}$ and form the flag $F' = F/U$ described in Lemma 8.6. Define $G' = GL(F_n/U)$. Then one can check that for this G', we have that F' is again the representative that Proposition 6.1 gives for the orbit v' attached to the involution $\sigma' \in \Sigma_0(2n-2)$ described in the previous paragraph. Moreover, one can check directly that if N is generic in $\mu(T_v^*(X))$, then the projection of N on the quotient F_n/U is indeed generic in $\mu(T_{v'}^*(X'))$. Now the proof proceeds exactly as in the second case of the argument for $U(p,q)$, except that we instead use Lemma 8.5 and Lemma 8.6. (The same parenthetical caveat applies to the application of Lemma 8.6.) We conclude that the first $2n-1$ boxes of $RS_{\mathrm{orb}}(v)$ and $\mathrm{Ann}(v)$ coincide. Since there is a unique shape in $\mathcal{D}_e^{tr}(2n)$ containing the shape of the first $2n-1$ boxes of these tableaux, we conclude that $RS_{\mathrm{orb}}(v) = \mathrm{Ann}(v)$. The $G = SU^*(2n)$ case is complete.

Finally we consider $G = GL(n,\mathbb{R})$. We will deduce Theorem 5.6(c) from the following calculation. In its statement, we let $RS(\sigma)$ denote the standard Young tableau attached to $\sigma \in \Sigma(n)$ by the Robinson-Schensted algorithm.

Theorem 8.7. *Let $G = GL(n,\mathbb{R})$ and fix $v \in K_{\mathbb{C}}\backslash X$ corresponding (under Proposition 6.1) to $\sigma \in \Sigma(n)$. Then the generalized Robinson-Schensted algorithm for G coincides with the transpose of RS,*

$$RS_{orb}(v) = RS(\sigma)^{tr}.$$

Theorem 5.6(c) now follows from explicit computation. In a little more detail, first fix $\sigma \in \Sigma(n)$, assume σ has no fixed points, and write (σ,ϵ) for the corresponding element of $\Sigma_{\pm}(n)$. Write v_σ and $v_{(\sigma,\epsilon)}$ for the orbits described in Proposition 6.1. By Proposition 6.3, the only $K_{\mathbb{C}}$ equivariant local system on v_σ is the trivial one. From Proposition 6.9 and Proposition 6.6, we have

$$\mathrm{Ann}(L_{GL(n,\mathbb{R})}(v_\sigma))^{tr} = \mathrm{Ann}(L_{U(p,q)}(v_{(\sigma,\epsilon)})).$$

Directly from the definitions (and Theorem 7.1), one can verify that the tableau appearing on the right-hand side is $RS(\sigma)$. Hence we have deduced Theorem 5.6(c) from Theorem 8.7 in the fixed-point free case. On the other hand, assume σ has a least one fixed point. Then in view of Remark 6.10 and the definition of Ψ_2^c, Theorem 5.6(c) now follows from Theorem 8.7, Proposition 6.6, and the following observation: given $\sigma \in \Sigma(n)$, there are exactly two elements of the form $(\sigma,\epsilon),(\sigma,\epsilon') \in \Sigma_{\pm}(n)$ with

$$\mathrm{Ann}(L_{U(p,q)}(v_{(\sigma,\epsilon)})) = RS(\sigma) = \mathrm{Ann}(L_{U(p,q)}(v_{(\sigma,\epsilon')})).$$

We leave the (easy) verification of these facts to the reader.

Now we turn to the proof of Theorem 8.7. We will begin by establishing that the first $n-1$ boxes of $RS^c_{\mathrm{orb}}(v_\sigma)$ and $RS(\sigma)^{tr}$ coincide.

So let $G = GL(n,\mathbb{R})$ and take $\sigma \in \Sigma(n)$. Write v for the corresponding orbit and $F = (F_i)$ for the representative given in Proposition 6.1. There are again two cases to consider. First assume that $\sigma(n) = n$. Write $\sigma' \in \Sigma(n-1)$ for the involution obtained by viewing σ as a permutation of n–1 letters $1,\ldots,n$–1. Let v' denote the corresponding orbit for $GL(n\text{–}1,\mathbb{R})$, and let F' denote the representative given in Proposition 6.1. If N is a generic nilpotent in $\mu(T_v^*(X)|_F)$ (in the sense of Remark 5.3), then it is immediate that $N|_{F_{n-1}}$ is generic in $\mu(T_{v'}^*(X'))$. Hence, by Remark 5.3, we see that the first n–1 boxes of $RS^c_{\mathrm{orb}}(v)$ coincide with $RS^c_{\mathrm{orb}}(v')$. By induction we can assume that $RS^c_{\mathrm{orb}}(v') = RS(\sigma')^{tr}$. From the definition of the Robinson-Schensted algorithm (and the fact that $\sigma(n) = n$), we see that the first n–1 boxes of $RS(\sigma)^{tr}$ coincide with $RS(\sigma')^{tr}$. Putting the last three sentences together, we conclude that the first $n-1$ boxes of $RS^c_{\mathrm{orb}}(v_\sigma)$ coincide with those of $RS(\sigma)^{tr}$.

On the other hand, we can make the same conclusion in the case that $\sigma(n) = k \neq n$. The proof proceeds exactly as in the case of $SU^*(2n)$, once we notice that the obvious analog of Lemma 8.5 clearly holds for RS^{tr}. We omit the details.

Hence we conclude that the first $n-1$ boxes of $RS^c_{\mathrm{orb}}(v)$ and $RS(\sigma)^{tr}$ agree. To finish the proof of Theorem 8.7, it is enough to show that the shape of $\mu^c_{\mathrm{orb}}(v_\sigma)$ matches the shape of $RS(\sigma)^{tr}$. One can prove this by duplicating Yamamoto's [Ya] moment map image computations for $GL(n,\mathbb{R})$ and then verifying (as we did in the $U(p,q)$ case) that the shapes coincide. This is elementary, but extremely complicated. With a little sleight of hand, however, we can deduce it from Steinberg's calculation, part of which appears in the following lemma. (See Remark 5.4 and the introduction for more details.)

Lemma 8.8. *Fix $w \in S_n$ and let $A(w) \in GL(n,\mathbb{C})$ denote the corresponding permutation matrix. Let $\mathfrak{n}$ denote the upper-triangular nilradical of $\mathfrak{b}$, and suppose that N is generic in $Ad(A(w))\mathfrak{n} \cap \mathfrak{n}$ (in the sense of Remark 5.3). Then the shape of N coincides with the shape of $RS(w)$, where RS denotes the Robinson-Schensted algorithm.*

Now we prove that $\mu^c_{\mathrm{orb}}(v_\sigma)$ coincides with the shape of $RS(\sigma)^{tr}$. Let $g_\sigma \in GL(n,\mathbb{R})$ denote the element attached to σ by Proposition 6.1. Using an invariant bilinear form to identify the fiber at eB of $T^*(X)$ with $\mathfrak{n}$, we get

$$\mu(T^*_v(X)) = Ad(g_\sigma)\mathfrak{n} \cap \mathfrak{p}.$$

Let N denote a generic element of the image (in the sense of Remark 5.3). Since $\mathfrak{p}$ is the set of symmetric complex matrices, we can find $M \in \mathfrak{n}$ so that

$$(*) \qquad N = g_\sigma M g_\sigma^{-1} = (g_\sigma M g_\sigma^{-1})^{tr}.$$

Clearly N has the same shape as M, so we are to prove that the shape of M is the shape of $RS(\sigma)^{tr}$.

Now one may verify directly that $g_\sigma^2 = A(\sigma)$, the permutation matrix attached to σ. Combined with the fact that g_σ and g_σ^{-1} are symmetric, $(*)$ becomes

$$Ad(A(\sigma))M = M^{tr}.$$

Conjugating by $A(w_o)$, we get

$$Ad(A(w_o\sigma))M = M^{atr},$$

where M^{atr} denotes the anti-transpose of M, i.e. the reflection of M about its antidiagonal. Since $M \in \mathfrak{n}$, so is M^{atr}, and we can apply Lemma 8.8 to conclude that the shape of M coincides with the shape of $RS(w_o\sigma)$. By Proposition 5.10, this is the shape of $RS(\sigma w_o)$. Of course it is well-known that $RS(\tau w_o) = RS(\tau)^{tr}$ for any $\tau \in S_n$. Hence we conclude that the shape of M, and hence of N, coincides with the shape of $RS(\sigma)^{tr}$. The proof is complete. $\square$

Remark 8.9. We conclude by noting that, in some cases, we can give a completely self-contained computation of associated varieties. For any type A group considered above, let μ_{orb} denote the map taking $K_{\mathbb{C}}\backslash X$ to $K_{\mathbb{C}}\backslash \mathcal{N}_\theta$. Of course we always have $\mu_{\mathrm{orb}}(v) \subset \mathrm{AV}(L_G(v,\phi))$; see Propositions 2.6 and 2.8 in [BoBr], for example. Now Proposition 6.6 implies

$$\mathrm{shape}(\mathrm{AV}(L)) = \mathrm{shape}(\mathrm{AV}(L^\vee))^{tr},$$

and so we conclude

$$\mathrm{shape}(\mu_{\mathrm{orb}}(v)) \le \mathrm{shape}(\mathrm{AV}(L_G(v,\phi))) = \mathrm{shape}(\mathrm{AV}(L_{G^\vee}(\check{v},\check{\phi}))^{tr} \le \mathrm{shape}(\mu_{\mathrm{orb}}(\check{v}))^{tr}.$$

(The inequalities are with respect to the standard partial order on partitions.) We have given explicit formulas for μ_{orb} and $\check{v}$, and one can check that in some cases the left and right ends of above chain of inequalities. In these cases, we deduce the shape of $\mathrm{AV}(L_G(v,\phi))$; if $G = GL(n,\mathbb{R})$ or $SU^*(2n)$, this is of course $\mathrm{AV}(L_G(v,\phi))$. When $G = U(p,q)$, we can immediately conclude that $\mu_{\mathrm{orb}}(v)$ is an irreducible component of $\mathrm{AV}(L_G(v))$. If there is only one signature (p,q) tableau of the relevant shape, then we can conclude that indeed $\mu_{\mathrm{orb}}(v) = \mathrm{AV}(L_G(v))$. We can avoid the restrictions on the tableau if we are willing to admit the relatively elementary Barbasch-Vogan [BV] result stating that $\mathrm{AV}(L_G(v))$ is irreducible.

When the two sides of the above chain of inequalities do not coincide, the method gives only partial information. For instance, it is already inconclusive for the trivial representation of $U(p,q)$ when $|p-q| \geq 2$ and $\min(p,q) \geq 1$. Even so, the method does lead to some nontrivial computations. For instance, the method computes all associated varieties of the four modules $L_{GL(3,\mathbb{R})}(v_\sigma)$. (When $\sigma = (12)$ or (23), v_σ is not of the form required by Theorem 2.3, and the proposition does does not apply.) For $GL(4,\mathbb{R})$ it computes the associated varieties of nine of the ten modules $L_{GL(4,\mathbb{R})}(v_\sigma)$, only three of which are handled by Theorem 2.3; it is inconclusive when $\sigma = (23)$.

References

[BV] Barbasch, D. and D. A. Vogan, Jr., Weyl Group Representations and Nilpotent Orbits, *Proceeding of the Park City Conference on Harmonic Analysis*, P. Trombi ed., Progress in Math, Birkhäuser (Boston), 1982, pp 21–33

[BoBr] Borho, W. and J.-L. Brylinski, Differential operators on homogeneous spaces III, *Inv. Math*, **80**(1995), 1–68.

[CMc] Collingwood, D. H. and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Chapman and Hall (London), 1994.

[Do] Douglass, J. M., An involution of the variety of flags fixed by a unipotent linear transformation, *Adv. in Appl. Math.*, **17**(1996), no. 3, 357–379.

[G] Garfinkle, D., The annihilators of irreducible Harish-Chandra modules for $SU(p,q)$ and other type A_{n-1} groups, *Amer. J. Math.*, **115**(1993), 305–369.

[GrV] Graham, W., and D. A. Vogan, Jr., Geometric quantization for nilpotent coadjoint orbits, in *Geometry and representation theory of real and p-adic groups (Cordoba, 1995)*, Progress in Math., Birkhäuser(Boston), **138**(1998), 69–137.

[Ho] Hotta, R., On Joseph's construction of Weyl group representations, *Tohoku Math. J. (2)*, **36**(1984), no. 1, 49–74

[Ja] Jantzen, J.C., *Einhüllende Algebren helbeinfacher Lie-Algebren*, Ergebnisse der Mathematik und ihre Grenzegebeite, Springer(Berlin), **3**(1983).

[Jo] Joseph, A., On the variety of a highest weight module, *J. Algebra*, **88**(1984), no. 1, 238–278.

[KaT] Kashiwara, M., and T. Tanisaki, The characteristic cycles of holonomic systems on a flag manifold related to the Weyl group algebra, *Invent. Math.*, **77**(1984), no. 1, 185–198.

[KazL] Kazhdan, D., and G. Lusztig, A topological approach to Springer's representations, *Adv. Math.*, **38**(1980), no. 2, 222–228.

[KoR] Kostant, B., and S. Rallis, Orbits and representatives associated with symmetric spaces, *Amer. J. Math.*, **93**(1971), 753–809.

[Mc1] McGovern, W. M., Cells of Harish-Chandra modules for real classical Lie groups, *Amer. J. Math.*, **120**(1998), 211–228.

[Mc2] McGovern, W. M., On the Spaltenstein-Steinberg map for complex classical groups, preprint, 1998.

[Ma] Matsuki, T., The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, *J. Math. Soc. Japan*, **31**(1979), no. 2, 331–357.

[MO] Matsuki, T., and T. Oshima, Embeddings of discrete series into principal series, *The orbit method in representation theory*, Progess in Math, Birkhäuser, **82**(1988), 147–175.

[Ro] Rossmann, W., Invariant eigendistributions on a semisimple Lie algebra and homology classes on the conormal variety. II. Representations of Weyl groups, *J. Funct. Anal.*, **96**(1991), no. 1, 155–193.

[RSp] Richardson, R. W., and T. A. Springer, The Bruhat order on symmetric varieties, *Geom. Dedicata*, **35**(1990), no. 1-3, 389–436.

[Sag] Sagan, B. E., *The symmetric group. Representations, combinatorial algorithms, and symmetric functions*, Wadsworth & Brooks/Cole (Pacific Grove, CA), 1991.

[SV] Schmid, W., and K. Vilonen, Characteristic sycles of constructible sheaves, *Inv. Math.*, **124**(1994), 451–502.

[Spa1] Spaltenstein, N., On the fixed point set of a unipotent element on the variety of Borel subgroups. *Topology*, **16**(1977), no. 2, 203–204.

[Spa2] Spaltenstein, N., *Classes unitpotentes et sous-groupes Borel*, Lecture Notes in Math, Springer(Berlin), **946**(1982).

[Sp1] Springer, T. A., lectures at the NATO Summer school, Université de Montréal, 1997.

[Sp2] Springer, T. A., Geometric questions arising in the study of unipotent elements, *Proceedings of Symposia in Pure Mathematics: The Santa Cruz Conference on Finite Groups*, **37**(1980), 255–264.

[St1] Steinberg, R., On the desingularization of the unipotent variety, *Inv. Math.*, **36**(1976), 209–224.

[St2] Steinberg, R., An occurrence of the Robinson-Schensted algorithm, *J. Algebra*, **113**(1988), no. 2, 523–528.

[Ta] Tanisaki, T., Characteristic varieties of highest weight modules and primitive quotients, in *Representations of Lie groups, Kyoto, Hiroshima, 1986*, Adv. Stud. Pure Math., Academic Press (Boston), **14**(1988), 1–30.

[Tr] Trapa, Peter E., Annihilators and associated varieties of $A_{\mathfrak{q}}(\lambda)$ modules for $U(p,q)$, preprint, 1998.

[vL] van Leeuwen, M., A Robinson-Schensted algorithm in the geometry of flags for classical groups, Rijksuniversiteit Thesis, Utrecht, 1989.

[V1] Vogan, D. A., Jr., Gelfand-Kirillov dimension for Harish-Chandra modules, *Inv. Math.*, **48**(1978), 75–78.

[V2] Vogan, D. A., Jr., Irreducible characters of semisimple Lie groups III: Proof of Kazhdan-Lusztig conjecture in the integral case, *Invent. Math.*, **71**(1983), no. 2, 381–417.

[V3] Vogan, D. A., Jr., Irreducible characters of semisimple Lie groups IV: character multiplicity duality, *Duke Math J*, **49**(1982), no. 4, 943–1073

[V4] Vogan, D. A., Jr., Associated varieties and unipotent representations, *Proceeding of the Bowdoin Conference on Harmonic Analysis*, Progress in Math **101**, Birhäuser(Boston), 1991.

[VZ] Vogan, D. A., Jr., and Gregg Zuckerman, Unitary Representations with nonzero cohomology, *Compositio Mathematica*, **120**(1984) 141–187.

[Ya] Yamamoto, Atsuko, Orbits in the flag variety and images of the moment map for classical groups I, *Representation Theory*, **1**(1997), 329–404.

[Zi] Zierau, R., Representation in Dolbeault Cohomology, to appear in the IAS/Park City Mathematics Series.

School of Mathematics, The Institute for Advanced Study, Princeton, NJ 08540

E-mail address: ptrapa@math.ias.edu