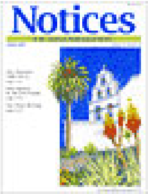
Kazhdan-Lusztig-Vogan Polynomials and Applications

Peter E. Trapa

$$152q^{22}+3472q^{21}+38791q^{20}+293021q^{19}+1370892q^{18}+4067059q^{17}+7964$$

A GOOD REFERENCE (FROM A DIFFERENT POINT OF VIEW)

A GOOD REFERENCE (FROM A DIFFERENT POINT OF VIEW)



David Vogan, “The Character Table of E8,”
Notices, October 2007

what's the big picture?

- An algorithm (due to Lusztig in Vogan in 1983) for computing local intersection cohomology groups of certain singular varieties was implemented in a software package (`atlas`) written by Fokko du Cloux.

- An algorithm (due to Lusztig in Vogan in 1983) for computing local intersection cohomology groups of certain singular varieties was implemented in a software package (`atlas`) written by Fokko du Cloux.
- `atlas` works in a very general setting.

WHAT'S THE BIG PICTURE?

- An algorithm (due to Lusztig in Vogan in 1983) for computing local intersection cohomology groups of certain singular varieties was implemented in a software package (`atlas`) written by Fokko du Cloux.
- `atlas` works in a very general setting. In particular, it computes the "classical" Kazhdan-Lusztig polynomials p_{yw} very efficiently.

What's the big picture?

- An algorithm (due to Lusztig in Vogan in 1983) for computing local intersection cohomology groups of certain singular varieties was implemented in a software package (`atlas`) written by Fokko du Cloux.
- `atlas` works in a very general setting. In particular, it computes the "classical" Kazhdan-Lusztig polynomials p_{yw} very efficiently.
- It transpired that running the algorithm for the "largest" simple exceptional real group was barely possible.

- An algorithm (due to Lusztig in Vogan in 1983) for computing local intersection cohomology groups of certain singular varieties was implemented in a software package (`atlas`) written by Fokko du Cloux.
- `atlas` works in a very general setting. In particular, it computes the "classical" Kazhdan-Lusztig polynomials p_{yw} very efficiently.
- It transpired that running the algorithm for the "largest" simple exceptional real group was barely possible. (This was done on the UW's machine `sage` wih the help of William Stein.)

WHAT'S THE BIG PICTURE?

- An algorithm (due to Lusztig in Vogan in 1983) for computing local intersection cohomology groups of certain singular varieties was implemented in a software package (`atlas`) written by Fokko du Cloux.
- `atlas` works in a very general setting. In particular, it computes the "classical" Kazhdan-Lusztig polynomials p_{yw} very efficiently.
- It transpired that running the algorithm for the "largest" simple exceptional real group was barely possible. (This was done on the UW's machine `sage` wih the help of William Stein.)
- The information encoded in the output lies at the heart of many problems in representation theory and suggests new lines of inquiry.

Let G be a group.

Let G be a group. A representation of G is a (complex) vector space V and a homomorphism

$$G \longrightarrow \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ denotes the group of invertible transformations of V.

Let G be a group. A representation of G is a (complex) vector space V and a homomorphism

$$G \longrightarrow \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ denotes the group of invertible transformations of V. The representation is unitary if its image lands in unitary matrices.

Let G be a group. A representation of G is a (complex) vector space V and a homomorphism

$$G \longrightarrow \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ denotes the group of invertible transformations of V. The representation is unitary if its image lands in unitary matrices. There is an obvious notion of equivalence

Let G be a group. A representation of G is a (complex) vector space V and a homomorphism

$$G \longrightarrow \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ denotes the group of invertible transformations of V. The representation is unitary if its image lands in unitary matrices. There is an obvious notion of equivalence and of irreducibility.

TYPICAL SOURCE OF REPRESENTATIONS

Suppose G acts on a finite set X.

TYPICAL SOURCE OF REPRESENTATIONS

Suppose G acts on a finite set X. Consider

$$\mathcal{F}(X) = \{f \ : \ X \longrightarrow \mathbb{C}\}.$$

TYPICAL SOURCE OF REPRESENTATIONS

Suppose G acts on a finite set X. Consider

$$\mathcal{F}(X) = \{f \ : \ X \longrightarrow \mathbb{C}\}.$$

Then we obtain a representation

$$\begin{aligned} \pi \ : \ G &\longrightarrow \mathrm{GL}(\mathcal{F}(X)) \\ g &\mapsto \pi(g) \end{aligned}$$

where

$$[\pi(g)f](x) = f(g^{-1} \cdot x).$$

Suppose G acts on a finite set X. Consider

$$\mathcal{F}(X) = \{f \ : \ X \longrightarrow \mathbb{C}\}.$$

Then we obtain a representation

$$\begin{aligned} \pi \ : \ G &\longrightarrow \mathrm{GL}(\mathcal{F}(X)) \\ g &\mapsto \pi(g) \end{aligned}$$

where

$$[\pi(g)f](x) = f(g^{-1} \cdot x).$$

For example, suppose G is finite and we consider it acting on itself on the left. Then

$$\mathcal{F}(G) = \mathbb{C}[G] =$$

Suppose G acts on a finite set X. Consider

$$\mathcal{F}(X) = \{f \ : \ X \longrightarrow \mathbb{C}\}.$$

Then we obtain a representation

$$\begin{aligned} \pi \ : \ G &\longrightarrow \mathrm{GL}(\mathcal{F}(X)) \\ g &\mapsto \pi(g) \end{aligned}$$

where

$$[\pi(g)f](x) = f(g^{-1} \cdot x).$$

For example, suppose G is finite and we consider it acting on itself on the left. Then

$$\mathcal{F}(G) = \mathbb{C}[G] = \bigoplus_{(\pi,V) \text{ irred}} V^{\oplus \dim(V)}.$$

Initial setting for today (refined later)

$G_{\mathbb{R}}$ is a real Lie group.

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$ $(n = 1 : \mathbb{R}^\times)$.

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$ $(n = 1 : \mathbb{R}^{\times})$.
- $\mathrm{GL}(n, \mathbb{C})$

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$ $(n = 1 : \mathbb{R}^{\times})$.
- $\mathrm{GL}(n, \mathbb{C})$ $(n = 1 : \mathbb{C}^{\times})$.

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$ $(n = 1 : \mathbb{R}^\times)$.
- $\mathrm{GL}(n, \mathbb{C})$ $(n = 1 : \mathbb{C}^\times)$.
- $\mathrm{U}(n)$

Initial setting for today (refined later)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$ $(n = 1 : \mathbb{R}^{\times})$.
- $\mathrm{GL}(n, \mathbb{C})$ $(n = 1 : \mathbb{C}^{\times})$.
- $\mathrm{U}(n)$ $(n = 1 : S^1)$,

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$ $(n = 1 : \mathbb{R}^{\times})$.
- $\mathrm{GL}(n, \mathbb{C})$ $(n = 1 : \mathbb{C}^{\times})$.
- $\mathrm{U}(n)$ $(n = 1 : S^1)$, $\mathrm{U}(p, q)$.

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$ $(n = 1 : \mathbb{R}^\times)$.
- $\mathrm{GL}(n, \mathbb{C})$ $(n = 1 : \mathbb{C}^\times)$.
- $\mathrm{U}(n)$ $(n = 1 : S^1)$, $\mathrm{U}(p, q)$.
- $\mathrm{SO}(n), \mathrm{SO}(p, q), \mathrm{SO}(n, \mathbb{C})$, $\mathrm{SO}^*(2n)$.

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n, \mathbb{R})$ $(n = 1 : \mathbb{R}^{\times})$.
- $\mathrm{GL}(n, \mathbb{C})$ $(n = 1 : \mathbb{C}^{\times})$.
- $\mathrm{U}(n)$ $(n = 1 : S^1)$, $\mathrm{U}(p, q)$.
- $\mathrm{SO}(n), \mathrm{SO}(p, q), \mathrm{SO}(n, \mathbb{C})$, $\mathrm{SO}^*(2n)$.
- $\mathrm{Sp}(n), \mathrm{Sp}(p, q), \mathrm{Sp}(\mathfrak{n}, \mathbb{C}), \mathrm{Sp}(n, \mathbb{R})$.

INITIAL SETTING FOR TODAY (REFINED LATER)

$G_{\mathbb{R}}$ is a real Lie group.
For example:

- $\mathrm{GL}(n,\mathbb{R})$ $(n = 1 : \mathbb{R}^{\times})$.
- $\mathrm{GL}(n,\mathbb{C})$ $(n = 1 : \mathbb{C}^{\times})$.
- $\mathrm{U}(n)$ $(n = 1 : S^1)$, $\mathrm{U}(p,q)$.
- $\mathrm{SO}(n), \mathrm{SO}(p,q), \mathrm{SO}(n,\mathbb{C})$, $\mathrm{SO}^*(2n)$.
- $\mathrm{Sp}(n), \mathrm{Sp}(p,q), \mathrm{Sp}(\mathfrak{n},\mathbb{C}), \mathrm{Sp}(n,\mathbb{R})$.
- G2, F4, E6, E7, E8

A representation of $G_{\mathbb{R}}$ is a continuous homomorphism

$$G_{\mathbb{R}} \longrightarrow \mathrm{GL}(\mathcal{H})$$

where $\mathrm{GL}(\mathcal{H})$ denotes the group of bounded linear invertible operators on a Hilbert space $\mathcal{H}$. The representation is unitary if its image is in unitary operators.

A representation of $G_{\mathbb{R}}$ is a continuous homomorphism

$$G_{\mathbb{R}} \longrightarrow \mathrm{GL}(\mathcal{H})$$

where $\mathrm{GL}(\mathcal{H})$ denotes the group of bounded linear invertible operators on a Hilbert space $\mathcal{H}$. The representation is unitary if its image is in unitary operators. There is an obvious notion of equivalence and of irreducibility.

Suppose $G_{\mathbb{R}}$ acts on X "nicely". Consider

$$\mathcal{F}(X) = \{f \;:\; X \longrightarrow \mathbb{C} \;\big|\; \dots\}.$$

Then we obtain a representation

$$\begin{aligned} G &\longrightarrow \mathrm{GL}(\mathcal{F}(X)) \\ g &\mapsto \pi(g) \end{aligned}$$

where

$$[\pi(g)f](x) = f(g^{-1} \cdot x).$$

Suppose $G_{\mathbb{R}}$ acts on X "nicely". Consider

$$\mathcal{F}(X) = \{f \,:\, X \longrightarrow \mathbb{C} \mid \ldots\}.$$

Then we obtain a representation

$$\begin{aligned} G &\longrightarrow \mathrm{GL}(\mathcal{F}(X)) \\ g &\mapsto \pi(g) \end{aligned}$$

where

$$[\pi(g)f](x) = f(g^{-1} \cdot x).$$

For example, if X admits a G-invariant measure, we can consider $L^2(X, \mu)$.

(A) If $G_{\mathbb{R}} = \mathbb{R}^{\times}$ the irreducible representations are all of the form

$$\begin{aligned} \mathbb{R}^{\times} &\longrightarrow \mathrm{GL}(\mathbb{C}) = \mathbb{C}^{\times} \\ x &\mapsto e^{\mu x} \end{aligned}$$

where μ is a fixed complex number.

(A) If $G_{\mathbb{R}} = \mathbb{R}^\times$ the irreducible representations are all of the form

$$\begin{aligned} \mathbb{R}^\times &\longrightarrow \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times \\ x &\mapsto e^{\mu x} \end{aligned}$$

where μ is a fixed complex number. The representation is unitary if and only if μ is purely imaginary.

(A) If $G_{\mathbb{R}} = \mathbb{R}^{\times}$ the irreducible representations are all of the form

$$\begin{aligned} \mathbb{R}^{\times} &\longrightarrow \mathrm{GL}(\mathbb{C}) = \mathbb{C}^{\times} \\ x &\mapsto e^{\mu x} \end{aligned}$$

where μ is a fixed complex number. The representation is unitary if and only if μ is purely imaginary.

(B) If $G_{\mathbb{R}} = S^1$ the irreducible representations are all of the form

$$\begin{aligned} S^1 &\longrightarrow \mathrm{GL}(\mathbb{C}) = \mathbb{C}^{\times} \\ e^{i\theta} &\mapsto e^{in\theta} \end{aligned}$$

where n is a fixed integer.

(A) If $G_{\mathbb{R}} = \mathbb{R}^{\times}$ the irreducible representations are all of the form

$$\begin{aligned} \mathbb{R}^{\times} &\longrightarrow \mathrm{GL}(\mathbb{C}) = \mathbb{C}^{\times} \\ x &\mapsto e^{\mu x} \end{aligned}$$

where μ is a fixed complex number. The representation is unitary if and only if μ is purely imaginary.

(B) If $G_{\mathbb{R}} = S^1$ the irreducible representations are all of the form

$$\begin{aligned} S^1 &\longrightarrow \mathrm{GL}(\mathbb{C}) = \mathbb{C}^{\times} \\ e^{i\theta} &\mapsto e^{in\theta} \end{aligned}$$

where n is a fixed integer. Each of these representations is unitary.

Notice that the irreducible unitary representations on the previous slide are exactly the ones needed for Fourier analysis:

$$L^2(\mathbb{R}) = \widehat{\int_{\mu \in i\mathbb{R}}} e^{\mu x} d\mu \qquad\qquad L^2(S^1) = \widehat{\bigoplus_{n \in \mathbb{Z}}} \mathbb{C}e^{in\theta}.$$

Problem. Understand the (equivalence classes of) irreducible unitary representations of $G_{\mathbb{R}}$, the so called unitary dual $\widehat{G}^u_{\mathbb{R}}$.

Problem. Understand the (equivalence classes of) irreducible unitary representations of $G_{\mathbb{R}}$, the so called unitary dual $\widehat{G}^u_{\mathbb{R}}$.

Subproblem. Understand what “understand” should mean in the previous problem.

REVISED SETTING

Without additional hypothesis, this is hopeless.

Without additional hypothesis, this is hopeless. Let G be a connected complex algebraic group. Suppose σ is an antiholomorphic involution of G.

Without additional hypothesis, this is hopeless. Let G be a connected complex algebraic group. Suppose σ is an antiholomorphic involution of G. (So if f is algebraic, then so is $g \mapsto \overline{f(\sigma(g))}$.)

Without additional hypothesis, this is hopeless. Let G be a connected complex algebraic group. Suppose σ is an antiholomorphic involution of G. (So if f is algebraic, then so is $g \mapsto \overline{f(\sigma(g))}$.) Consider only groups of the form

$$G_{\mathbb{R}} = \{g \mid \sigma(g) = g\}.$$

Without additional hypothesis, this is hopeless. Let G be a connected complex algebraic group. Suppose σ is an antiholomorphic involution of G. (So if f is algebraic, then so is $g \mapsto \overline{f(\sigma(g))}$.) Consider only groups of the form

$$G_{\mathbb{R}} = \{g \mid \sigma(g) = g\}.$$

Duflo showed that the problems on the previous page essentially reduce to the case where $G_{\mathbb{R}}$ is very close to being simple. (The list of examples given above is close to being complete.)

The algebraic setting *is* restrictive, but for deep reasons it is often the “right” setting.

The algebraic setting *is* restrictive, but for deep reasons it is often the "right" setting.

Some very natural nonalgebraic examples:

(A) $\widetilde{\mathrm{GL}}(n, \mathbb{R})$,

The algebraic setting *is* restrictive, but for deep reasons it is often the "right" setting.

Some very natural nonalgebraic examples:

(A) $\widetilde{\mathrm{GL}}(n,\mathbb{R})$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$,

The algebraic setting *is* restrictive, but for deep reasons it is often the "right" setting.

Some very natural nonalgebraic examples:

(A) $\widetilde{\mathrm{GL}}(n,\mathbb{R})$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, $\widetilde{Sp}(2n,\mathbb{R})$.

The algebraic setting *is* restrictive, but for deep reasons it is often the "right" setting.

Some very natural nonalgebraic examples:

(A) $\widetilde{\mathrm{GL}}(n,\mathbb{R})$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, $\widetilde{Sp}(2n,\mathbb{R})$.

(B) $\mathrm{SO}_e(p,q)$.

Problem. Understand the (equivalence classes of) irreducible unitary representations of $G_{\mathbb{R}}$, the so called unitary dual $\widehat{G}^u_{\mathbb{R}}$.

Subproblem. Understand what "understand" should mean in the previous problem.

Suppose G acts on X.

Suppose G acts on X.
Question. How can we use this “symmetry” to learn something about X?

Suppose G acts on X.
Question. How can we use this "symmetry" to learn something about X?

Answer (Gelfand). Decompose Hilbert spaces of functions on X, and reassemble this information to make deductions about X.

Suppose G acts on X.
Question. How can we use this "symmetry" to learn something about X?

Answer (Gelfand). Decompose Hilbert spaces of functions on X, and reassemble this information to make deductions about X.

(Stupid example: S^1 is compact, $\mathbb{R}^\times$ is noncompact.)

Suppose G acts on X.
Question. How can we use this "symmetry" to learn something about X?

Answer (Gelfand). Decompose Hilbert spaces of functions on X, and reassemble this information to make deductions about X.

(Stupid example: S^1 is compact, $\mathbb{R}^\times$ is noncompact.)

At least this gives a utilitarian interpretation of "understand" on the previous slide: one needs to understand enough for the application at hand.

Harish-Chandra succeeded in decomposing

$$L^2(G_{\mathbb{R}})$$

explicitly.

Harish-Chandra succeeded in decomposing

$$L^2(G_{\mathbb{R}})$$

explicitly. The representation he needed are called tempered:

$$\widehat{G}_{\mathbb{R}}^{\text{temp}}$$

Harish-Chandra succeeded in decomposing

$$L^2(G_{\mathbb{R}})$$

explicitly. The representation he needed are called tempered:

$$\widehat{G}_{\mathbb{R}}^{\text{temp}} \subset \widehat{G}_{\mathbb{R}}^{u}.$$

Example: $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$

$$\Gamma_N = \{g \in G_{\mathbb{R}} \mid g \cong \mathrm{Id} \bmod N\}.$$

Example: $G_{\mathbb{R}} = \mathrm{SL}(2,\mathbb{R})$

$$\Gamma_N = \{g \in G_{\mathbb{R}} \mid g \cong \mathrm{Id} \bmod N\}.$$

Decompose

$$L^2(\Gamma_N \backslash G_{\mathbb{R}}).$$

These are spaces of modular forms.

More generally, ask the same question for congruence subgroups of general groups $G_{\mathbb{R}}$

More generally, ask the same question for congruence subgroups of general groups $G_{\mathbb{R}}$, i.e decompose

$$L^2(\Gamma_N \backslash G_{\mathbb{R}}).$$

These are spaces of automorphic forms.

Define the “automorphic dual” of $G_{\mathbb{R}}$ to be those unitary representations that appear in some $L^2(\Gamma \backslash G_{\mathbb{R}})$:

More generally, ask the same question for congruence subgroups of general groups $G_{\mathbb{R}}$, i.e decompose

$$L^2(\Gamma_N \backslash G_{\mathbb{R}}).$$

These are spaces of automorphic forms.

Define the "automorphic dual" of $G_{\mathbb{R}}$ to be those unitary representations that appear in some $L^2(\Gamma \backslash G_{\mathbb{R}})$:

$$\widehat{G}_{\mathbb{R}}^{\text{temp}} \subset \widehat{G}_{\mathbb{R}}^{A} \subset \widehat{G}_{\mathbb{R}}^{u}.$$

- The full unitary dual $\widehat{G}^u_{\mathbb{R}}$ appears more tractable than $\widehat{G}^A_{\mathbb{R}}$.

- The full unitary dual $\widehat{G}^u_{\mathbb{R}}$ appears more tractable than $\widehat{G}^A_{\mathbb{R}}$.
- For instance, if $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$ the full unitary dual is known, but the autormorphic dual is the subject of the (still open) Selberg 1/4 Conjecture.

- The full unitary dual $\widehat{G}^u_{\mathbb{R}}$ appears more tractable than $\widehat{G}^A_{\mathbb{R}}$.
- For instance, if $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$ the full unitary dual is known, but the autormorphic dual is the subject of the (still open) Selberg 1/4 Conjecture.
- But, in general, $\widehat{G}^u_{\mathbb{R}}$ is poorly organized, while deep conjectures of Langlands and Arthur predict a spectacularly well-organized structure on $\widehat{G}^A_{\mathbb{R}}$.

- The full unitary dual $\widehat{G}^u_{\mathbb{R}}$ appears more tractable than $\widehat{G}^A_{\mathbb{R}}$.
- For instance, if $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$ the full unitary dual is known, but the autormorphic dual is the subject of the (still open) Selberg 1/4 Conjecture.
- But, in general, $\widehat{G}^u_{\mathbb{R}}$ is poorly organized, while deep conjectures of Langlands and Arthur predict a spectacularly well-organized structure on $\widehat{G}^A_{\mathbb{R}}$. Roughly:
    - The automorphic dual of $\mathrm{SL}(2, \mathbb{R})$ fits inside $\widehat{G}^A_{\mathbb{R}}$ in comprehensible ways.

- The full unitary dual $\widehat{G}^u_{\mathbb{R}}$ appears more tractable than $\widehat{G}^A_{\mathbb{R}}$.
- For instance, if $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$ the full unitary dual is known, but the autormorphic dual is the subject of the (still open) Selberg 1/4 Conjecture.
- But, in general, $\widehat{G}^u_{\mathbb{R}}$ is poorly organized, while deep conjectures of Langlands and Arthur predict a spectacularly well-organized structure on $\widehat{G}^A_{\mathbb{R}}$. Roughly:
  - The automorphic dual of $\mathrm{SL}(2, \mathbb{R})$ fits inside $\widehat{G}^A_{\mathbb{R}}$ in comprehensible ways.
  - No comparable statement can be true for the full unitary dual.

Instead of considering only unitary representations, Harish-Chandra discovered a larger, more tractable category (of so-called Harish-Chandra modules).

Instead of considering only unitary representations, Harish-Chandra discovered a larger, more tractable category (of so-called Harish-Chandra modules).

Given an irreducible unitary representation of $G_{\mathbb{R}}$, one can differentiate at the identity and complexify to obtain a representation of the complex Lie algebra $\mathfrak{g}$.

Instead of considering only unitary representations, Harish-Chandra discovered a larger, more tractable category (of so-called Harish-Chandra modules).

Given an irreducible unitary representation of $G_{\mathbb{R}}$, one can differentiate at the identity and complexify to obtain a representation of the complex Lie algebra $\mathfrak{g}$. The category of such representations are, in fact, equivalent to the category of modules for the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$.

Instead of considering only unitary representations, Harish-Chandra discovered a larger, more tractable category (of so-called Harish-Chandra modules).

Given an irreducible unitary representation of $G_{\mathbb{R}}$, one can differentiate at the identity and complexify to obtain a representation of the complex Lie algebra $\mathfrak{g}$. The category of such representations are, in fact, equivalent to the category of modules for the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$.

So from π, one obtains a $\mathrm{U}(\mathfrak{g})$ module X_π.

Instead of considering only unitary representations, Harish-Chandra discovered a larger, more tractable category (of so-called Harish-Chandra modules).

Given an irreducible unitary representation of $G_{\mathbb{R}}$, one can differentiate at the identity and complexify to obtain a representation of the complex Lie algebra $\mathfrak{g}$. The category of such representations are, in fact, equivalent to the category of modules for the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$.

So from π, one obtains a $\mathrm{U}(\mathfrak{g})$ module X_π.

But some information is lost in this procedure.

Harish-Chandra's early work (continued)

To keep track of the lost information, one must keep track of the restriction of π to a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$,

$$\pi|_{K_{\mathbb{R}}} = \widehat{\bigoplus_{\mu}} n_\mu E_\mu.$$

To keep track of the lost information, one must keep track of the restriction of π to a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$,

$$\pi|_{K_{\mathbb{R}}} = \widehat{\bigoplus_{\mu}} n_\mu E_\mu.$$

The structure of the resulting $\mathrm{U}(\mathfrak{g})$ module and representation of $K_{\mathbb{R}}$ can be axiomatized. The relevant definition is that of a Harish-Chandra module.

Harish-Chandra's early work (continued)

Every irreducible unitary representation gives rise to an irreducible Harish-Chandra module.

Every irreducible unitary representation gives rise to an irreducible Harish-Chandra module. But the converse is not true:

$$\widehat{G}^{u}_{\mathbb{R}} \subset \widehat{G}^{HC}_{\mathbb{R}}.$$

Harish-Chandra's early work (continued)

Every irreducible unitary representation gives rise to an irreducible Harish-Chandra module. But the converse is not true:

$$\widehat{G}_{\mathbb{R}}^{u} \subset \widehat{G}_{\mathbb{R}}^{HC}.$$

The payoff, however, is that the larger set is more tractable.

Harish-Chandra's early work (continued)

Every irreducible unitary representation gives rise to an irreducible Harish-Chandra module. But the converse is not true:

$$\widehat{G}_{\mathbb{R}}^{u} \subset \widehat{G}_{\mathbb{R}}^{HC}.$$

The payoff, however, is that the larger set is more tractable. The (still open) strategy then becomes to classify the larger set and identify the smaller subset of unitary representations.

Langlands suggested the following classification of $\widehat{G}_{\mathbb{R}}^{HC}$.

Langlands suggested the following classification of $\widehat{G}_{\mathbb{R}}^{HC}$.

Langlands suggested the following classification of $\widehat{G}_{\mathbb{R}}^{HC}$. To each group $G_{\mathbb{R}}$, Langlands associated a complex algebraic group ${}^{L}G$ (on the *dual* side).

Langlands suggested the following classification of $\widehat{G}_{\mathbb{R}}^{HC}$. To each group $G_{\mathbb{R}}$, Langlands associated a complex algebraic group ${}^{L}G$ (on the *dual* side). Langlands proved that to each (suitable equivalence class of) homorphisms

$$\phi \,:\, W_{\mathbb{R}} \longrightarrow {}^{L}G \ldots$$

one can attach a packet of representations $\mathrm{L}(\phi)$ so that

Theorem (Langlands)

$$\widehat{G}_{\mathbb{R}}^{HC} \;=\; \coprod_{\phi} \mathrm{L}(\phi).$$

Here $W_{\mathbb{R}} = \langle \mathbb{C}^{\times}, j\rangle$ is the Weil group of $\mathbb{R}$ where

$$j^2 = -1 \text{ and } jzj^{-1} = \bar{z}.$$

$G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$.

$G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then ${}^L G = \mathrm{GL}(n, \mathbb{C})$ and each $\mathrm{L}(\phi)$ is a single representation.

So irreducible representations of $\mathrm{GL}(n, \mathbb{R})$ are parametrized by equivalence classes of n-dimensional representations of

$$W_{\mathbb{R}} = \langle \mathbb{C}, j \rangle.$$

$G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then ${}^LG = \mathrm{GL}(n, \mathbb{C})$ and each $\mathrm{L}(\phi)$ is a single representation.

So irreducible representations of $\mathrm{GL}(n, \mathbb{R})$ are parametrized by equivalence classes of n-dimensional representations of

$$W_{\mathbb{R}} = \langle \mathbb{C}, j \rangle.$$

Such a representation is a direct sum of one and two dimensional irreducible representations.

$G_{\mathbb{R}} = \mathrm{GL}(n,\mathbb{R})$. Then ${}^LG = \mathrm{GL}(n,\mathbb{C})$ and each $\mathrm{L}(\phi)$ is a single representation.

So irreducible representations of $\mathrm{GL}(n,\mathbb{R})$ are parametrized by equivalence classes of n-dimensional representations of

$$W_{\mathbb{R}} = \langle \mathbb{C}, j \rangle.$$

Such a representation is a direct sum of one and two dimensional irreducible representations.

Other examples: $G_{\mathbb{R}} = \mathrm{Sp}(n,\mathbb{R})$

$G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then ${}^L G = \mathrm{GL}(n, \mathbb{C})$ and each $\mathrm{L}(\phi)$ is a single representation.

So irreducible representations of $\mathrm{GL}(n, \mathbb{R})$ are parametrized by equivalence classes of n-dimensional representations of

$$W_{\mathbb{R}} = \langle \mathbb{C}, j \rangle.$$

Such a representation is a direct sum of one and two dimensional irreducible representations.

Other examples: $G_{\mathbb{R}} = \mathrm{Sp}(n, \mathbb{R})$, then ${}^L G = \mathrm{SO}(2n+1, \mathbb{C})$;

$G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then ${}^L G = \mathrm{GL}(n, \mathbb{C})$ and each $\mathrm{L}(\phi)$ is a single representation.

So irreducible representations of $\mathrm{GL}(n, \mathbb{R})$ are parametrized by equivalence classes of n-dimensional representations of

$$W_{\mathbb{R}} = \langle \mathbb{C}, j \rangle.$$

Such a representation is a direct sum of one and two dimensional irreducible representations.

Other examples: $G_{\mathbb{R}} = \mathrm{Sp}(n, \mathbb{R})$, then ${}^L G = \mathrm{SO}(2n+1, \mathbb{C})$; $G_{\mathbb{R}} = \mathrm{SO}(n+1, n)$, then ${}^L G = \mathrm{Sp}(2n, \mathbb{C})$.

Arthur refined the notation of Langlands parameter.

Arthur refined the notation of Langlands parameter. Roughly, he considered homorphisms

$$\psi \; : \; W_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow {}^L G \ldots$$

so that the map

$$\phi_\psi(z) = \psi\left(z, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right)$$

is a Langlands parameter.

Arthur refined the notation of Langlands parameter. Roughly, he considered homorphisms

$$\psi \ : \ W_{\mathbb{R}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^L G \ldots$$

so that the map

$$\phi_\psi(z) = \psi\left(z, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right)$$

is a Langlands parameter. Then one one can attach a packet of representations $\mathrm{A}(\psi)$ so that

$$\mathrm{L}(\phi_\psi) \subset \mathrm{A}(\psi)$$

and so that

Conjecture (Arthur)

$$\widehat{G}^A_{\mathbb{R}} = \bigcup_{\psi} \mathrm{A}(\psi).$$

Notice that it is relationships on the L-group side that suggest extra structure.

Notice that it is relationships on the L-group side that suggest extra structure. For instance suppose there is a map

$$\epsilon \ : \ {}^L H \longrightarrow {}^L G;$$

(Such a map need not correspond to any map from $H_{\mathbb{R}}$ to $G_{\mathbb{R}}$.)

Notice that it is relationships on the L-group side that suggest extra structure. For instance suppose there is a map

$$\epsilon \ : \ {}^L H \longrightarrow {}^L G;$$

(Such a map need not correspond to any map from $H_{\mathbb{R}}$ to $G_{\mathbb{R}}$.) Then we can compose any Langlands or Arthur parameter for $H_{\mathbb{R}}$ to get one for $G_{\mathbb{R}}$:

$$W_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow {}^L H \longrightarrow {}^L G.$$

Notice that it is relationships on the L-group side that suggest extra structure. For instance suppose there is a map

$$\epsilon \ : \ {}^LH \longrightarrow {}^LG;$$

(Such a map need not correspond to any map from $H_{\mathbb{R}}$ to $G_{\mathbb{R}}$.) Then we can compose any Langlands or Arthur parameter for $H_{\mathbb{R}}$ to get one for $G_{\mathbb{R}}$:

$$W_{\mathbb{R}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^LH \longrightarrow {}^LG.$$

These relationships go very far toward "understanding" the conjectural description of $\widehat{G}^A_{\mathbb{R}}$.

The sets $L(\phi)$ are explicitly computible.

The sets $\mathrm{L}(\phi)$ are explicitly computible. In fact, the computation has been automated in a software package written by Fokko du Cloux called `atlas`.

The sets $\mathrm{L}(\phi)$ are explicitly computible. In fact, the computation has been automated in a software package written by Fokko du Cloux called `atlas`.

But there is not at present an algorithm to compute individual Arthur packets.

The sets $\mathrm{L}(\phi)$ are explicitly computible. In fact, the computation has been automated in a software package written by Fokko du Cloux called `atlas`.

But there is not at present an algorithm to compute individual Arthur packets. Nonetheless, their union is computable:

Theorem (Barbasch-Vogan)

There is a finite algorithm (computible from the output of `atlas`*) to enumerate the set*

$$\bigcup_{\psi} \mathrm{A}(\psi)$$

which, recall, conjecturally exhausts $\widehat{G}^{A}_{\mathbb{R}}$.

MORE ON THE ALGORITHM

Theorem (Barbasch-Vogan)

There is a finite algorithm (computible from the output of `atlas`*) to enumerate the set*

$$\bigcup_{\psi} \mathrm{A}(\psi)$$

which, recall, conjecturally exhausts $\widehat{G}^{A}_{\mathbb{R}}$.

The algorithm relies on a purely geometric computation encoded in the computation of the KLV polynomials of the title.

Theorem (Barbasch-Vogan)

There is a finite algorithm (computible from the output of `atlas`*) to enumerate the set*

$$\bigcup_{\psi} \mathrm{A}(\psi)$$

which, recall, conjecturally exhausts $\widehat{G}_{\mathbb{R}}^{A}$.

The algorithm relies on a purely geometric computation encoded in the computation of the KLV polynomials of the title. (If one could make finer geometric calculations, then one could also enumerate the individual packets $A(\psi)$. But this is still open.)

Recall that

$$\widehat{G}^A_{\mathbb{R}} \subset \widehat{G}^u_{\mathbb{R}}.$$

Recall that

$$\widehat{G}^{A}_{\mathbb{R}} \subset \widehat{G}^{u}_{\mathbb{R}}.$$

Since there is a conjectural description of $G^{A}_{\mathbb{R}}$ we have a weaker conjecture:

Conjecture

$$\bigcup_{\psi} \mathrm{A}(\psi) \subset G^{u}_{\mathbb{R}}.$$

REALITY CHECK: BACK TO UNITARITY

Conjecture

$$\bigcup_{\psi} \mathrm{A}(\psi) \subset G^u_{\mathbb{R}}.$$

CONJECTURE

$$\bigcup_{\psi} \mathrm{A}(\psi) \subset G_{\mathbb{R}}^{u}.$$

Progress:

- Low rank ("small") groups, spherical representations: lots of partial progress (most notably by Barbasch (2001)).

Conjecture

$$\bigcup_{\psi} \mathrm{A}(\psi) \subset G_{\mathbb{R}}^{u}.$$

Progress:

- Low rank ("small") groups, spherical representations: lots of partial progress (most notably by Barbasch (2001)).
- $\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C}), \mathrm{GL}(n, \mathbb{H}), \mathrm{U}(p, q)$: Vogan (1984).

Conjecture

$$\bigcup_{\psi} \mathrm{A}(\psi) \subset G_{\mathbb{R}}^{u}.$$

Progress:

- Low rank ("small") groups, spherical representations: lots of partial progress (most notably by Barbasch (2001)).
- $\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C}), \mathrm{GL}(n, \mathbb{H}), \mathrm{U}(p, q)$: Vogan (1984).
- $\mathrm{Sp}(2n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$: Barbasch (1985)

CONJECTURE

$$\bigcup_{\psi} \mathrm{A}(\psi) \subset G^u_{\mathbb{R}}.$$

Progress:

- Low rank ("small") groups, spherical representations: lots of partial progress (most notably by Barbasch (2001)).
- $\mathrm{GL}(n,\mathbb{R}), \mathrm{GL}(n,\mathbb{C}), \mathrm{GL}(n,\mathbb{H}), \mathrm{U}(p,q)$: Vogan (1984).
- $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$: Barbasch (1985)
- $\mathrm{Sp}(p,q), \mathrm{SO}^*(2n)$: T (2007).

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{O}(n)$ and $K = \mathrm{O}(n, \mathbb{C})$.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{O}(n)$ and $K = \mathrm{O}(n, \mathbb{C})$.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$.

Fix $G_\mathbb{R}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_\mathbb{R}$. Let $K_\mathbb{R}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_\mathbb{R} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_\mathbb{R} \simeq \mathrm{O}(n)$ and $K = \mathrm{O}(n, \mathbb{C})$.
- $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{C})$. Then $\mathfrak{g}_\mathbb{R} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{O}(n)$ and $K = \mathrm{O}(n, \mathbb{C})$.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{U}(n)$ and $K = \mathrm{GL}(n, \mathbb{C})$.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{O}(n)$ and $K = \mathrm{O}(n, \mathbb{C})$.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{U}(n)$ and $K = \mathrm{GL}(n, \mathbb{C})$. Explicitly the inclusion of $\mathfrak{k}$ in $\mathfrak{g}$ is given by the diagonal map

$$\mathfrak{k} \simeq \mathrm{diag}\,[\mathfrak{gl}(n, \mathbb{C})] \subset \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C}) \simeq \mathfrak{g}.$$

Fix $G_\mathbb{R}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_\mathbb{R}$. Let $K_\mathbb{R}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_\mathbb{R} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_\mathbb{R} \simeq \mathrm{O}(n)$ and $K = \mathrm{O}(n, \mathbb{C})$.
- $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{C})$. Then $\mathfrak{g}_\mathbb{R} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_\mathbb{R} \simeq \mathrm{U}(n)$ and $K = \mathrm{GL}(n, \mathbb{C})$. Explicitly the inclusion of $\mathfrak{k}$ in $\mathfrak{g}$ is given by the diagonal map

$$\mathfrak{k} \simeq \mathrm{diag}\left[\mathfrak{gl}(n, \mathbb{C})\right] \subset \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C}) \simeq \mathfrak{g}.$$

The same hold for *any* complex Lie group.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{O}(n)$ and $K = \mathrm{O}(n, \mathbb{C})$.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{U}(n)$ and $K = \mathrm{GL}(n, \mathbb{C})$. Explicitly the inclusion of $\mathfrak{k}$ in $\mathfrak{g}$ is given by the diagonal map

$$\mathfrak{k} \simeq \mathrm{diag}\left[\mathfrak{gl}(n, \mathbb{C})\right] \subset \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C}) \simeq \mathfrak{g}.$$

  The same hold for *any* complex Lie group.
- $G_{\mathbb{R}}$ is the split form of E8.

Fix $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the *complexified* Lie algebra of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup and K its complexification. Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{O}(n)$ and $K = \mathrm{O}(n, \mathbb{C})$.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Meanwhile $K_{\mathbb{R}} \simeq \mathrm{U}(n)$ and $K = \mathrm{GL}(n, \mathbb{C})$. Explicitly the inclusion of $\mathfrak{k}$ in $\mathfrak{g}$ is given by the diagonal map

$$\mathfrak{k} \simeq \mathrm{diag}\left[\mathfrak{gl}(n, \mathbb{C})\right] \subset \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C}) \simeq \mathfrak{g}.$$

  The same hold for *any* complex Lie group.
- $G_{\mathbb{R}}$ is the split form of E8. Then $K_{\mathbb{R}} \simeq \mathrm{Spin}(16)/(\mathbb{Z}/2)$ (but not $\mathrm{SO}(16)$) and and $K = \mathrm{Spin}(16, \mathbb{C})/\pm$.

Introducing the geometry (continued)

In the setting above, let X denote the variety of maximal solvable subalgebras of $\mathfrak{g}$. (This is the largest connected compact Kähler manifold on which K acts with an open orbit.) Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$.

In the setting above, let X denote the variety of maximal solvable subalgebras of $\mathfrak{g}$. (This is the largest connected compact Kähler manifold on which K acts with an open orbit.)

Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Then

$$X \simeq \{(0) = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim(V_i) = i\}$$

In the setting above, let X denote the variety of maximal solvable subalgebras of $\mathfrak{g}$. (This is the largest connected compact Kähler manifold on which K acts with an open orbit.) Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Then

$$X \simeq \{(0) = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim(V_i) = i\}$$

  and $K = \mathrm{O}(n, \mathbb{C})$ acts in the obvious way.

INTRODUCING THE GEOMETRY (CONTINUED)

In the setting above, let X denote the variety of maximal solvable subalgebras of $\mathfrak{g}$. (This is the largest connected compact Kähler manifold on which K acts with an open orbit.) Examples:

- $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Then

  $$X \simeq \{(0) = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim(V_i) = i\}$$

  and $K = \mathrm{O}(n, \mathbb{C})$ acts in the obvious way.
- $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$.

In the setting above, let X denote the variety of maximal solvable subalgebras of $\mathfrak{g}$. (This is the largest connected compact Kähler manifold on which K acts with an open orbit.) Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Then

$$X \simeq \{(0) = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim(V_i) = i\}$$

  and $K = \mathrm{O}(n, \mathbb{C})$ acts in the obvious way.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Then

$$X \simeq \textit{ pairs } \text{of flags as above}$$

In the setting above, let X denote the variety of maximal solvable subalgebras of $\mathfrak{g}$. (This is the largest connected compact Kähler manifold on which K acts with an open orbit.) Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Then

$$X \simeq \{(0) = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim(V_i) = i\}$$

  and $K = \mathrm{O}(n, \mathbb{C})$ acts in the obvious way.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Then

$$X \simeq \textit{ pairs} \text{ of flags as above}$$

  and $K = \mathrm{GL}(n, \mathbb{C})$ acts *diagonally* in the obvious way.

In the setting above, let X denote the variety of maximal solvable subalgebras of $\mathfrak{g}$. (This is the largest connected compact Kähler manifold on which K acts with an open orbit.) Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Then

$$X \simeq \{(0) = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim(V_i) = i\}$$

  and $K = \mathrm{O}(n, \mathbb{C})$ acts in the obvious way.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$. Then

$$X \simeq \textit{ pairs} \text{ of flags as above}$$

  and $K = \mathrm{GL}(n, \mathbb{C})$ acts *diagonally* in the obvious way.
- $G_{\mathbb{R}}$ is the split form of E8 and $K = \mathrm{Spin}(16, \mathbb{C})/\pm$. Then X is 120 dimensional.

Theorem

K acts with finitely many orbits on X.

Theorem

K acts with finitely many orbits on X.

Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. The orbits of $K = \mathrm{O}(n, \mathbb{C})$ on the space of complete flags are paremetrized by *involutions* in S_n.

INTRODUCING THE GEOMETRY (CONTINUED)

THEOREM

K acts with finitely many orbits on X.

Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. The orbits of $K = \mathrm{O}(n, \mathbb{C})$ on the space of complete flags are paremetrized by *involutions* in S_n.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$. The diagonal $\mathrm{GL}(n, C)$ orbits on pairs of flags are parametrized by elements of the symmetric group.

Theorem

K acts with finitely many orbits on X.

Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n,\mathbb{R})$. The orbits of $K = \mathrm{O}(n,\mathbb{C})$ on the space of complete flags are paremetrized by *involutions* in S_n.
- $G_{\mathbb{R}} = \mathrm{GL}(n,\mathbb{C})$. The diagonal $\mathrm{GL}(n, C)$ orbits on pairs of flags are parametrized by elements of the symmetric group.
- $G_{\mathbb{R}}$ is the split form of E8 and $K = \mathrm{Spin}(16,\mathbb{C})/\pm$.

Theorem

K acts with finitely many orbits on X.

Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. The orbits of $K = \mathrm{O}(n, \mathbb{C})$ on the space of complete flags are paremetrized by *involutions* in S_n.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$. The diagonal $\mathrm{GL}(n, C)$ orbits on pairs of flags are parametrized by elements of the symmetric group.
- $G_{\mathbb{R}}$ is the split form of E8 and $K = \mathrm{Spin}(16, \mathbb{C})/\pm$. There are about 320,206 orbits

Theorem

K acts with finitely many orbits on X.

Examples:

- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$. The orbits of $K = \mathrm{O}(n, \mathbb{C})$ on the space of complete flags are paremetrized by *involutions* in S_n.
- $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$. The diagonal $\mathrm{GL}(n, C)$ orbits on pairs of flags are parametrized by elements of the symmetric group.
- $G_{\mathbb{R}}$ is the split form of E8 and $K = \mathrm{Spin}(16, \mathbb{C})/\pm$. There are about 320,206 orbits (compared to 696729600 for complex E8).

Basic parameter set: irreducible K-equivariant local systems

$$\mathrm{Loc}_K(X)$$

consisting of a K orbit on X and an irreducible monodromy representation (in this case, a representation of a product of $\mathbb{Z}/2$'s). Examples:

Basic parameter set: irreducible K-equivariant local systems

$$\mathrm{Loc}_K(X)$$

consisting of a K orbit on X and an irreducible monodromy representation (in this case, a representation of a product of $\mathbb{Z}/2$'s). Examples:

- $\mathrm{GL}(n, \mathbb{R})$: an involution in S_n with signed fixed points.

Basic parameter set: irreducible K-equivariant local systems

$$\mathrm{Loc}_K(X)$$

consisting of a K orbit on X and an irreducible monodromy representation (in this case, a representation of a product of $\mathbb{Z}/2$'s). Examples:

- $\mathrm{GL}(n, \mathbb{R})$: an involution in S_n with signed fixed points.
- $\mathrm{GL}(n, \mathbb{C})$: still just an element of S_n.

Basic parameter set: irreducible K-equivariant local systems

$$\mathrm{Loc}_K(X)$$

consisting of a K orbit on X and an irreducible monodromy representation (in this case, a representation of a product of $\mathbb{Z}/2$'s). Examples:

- $\mathrm{GL}(n, \mathbb{R})$: an involution in S_n with signed fixed points.
- $\mathrm{GL}(n, \mathbb{C})$: still just an element of S_n.
- split E8: 453,060 such local systems.

$\mathrm{Loc}_K(X)$ parametrize both irreducible K-equivariant perverse (complexes of) sheaves and irreducible K-equivariant constructable sheaves.

$\mathrm{Loc}_K(X)$ parametrize both irreducible K-equivariant perverse (complexes of) sheaves and irreducible K-equivariant constructable sheaves.

The cohomology sheaves of a perverse sheaf are constructible. So, given $\phi, \psi \in \mathrm{Loc}_K(X)$, it makes sense to consider the multiplicity $m_{\phi,\psi}^{(j)}$ of the irreducible constructible sheave parametrized by ψ in the jth cohomology of the irreducible perverse sheaf parametrized by ϕ.

$\mathrm{Loc}_K(X)$ parametrize both irreducible K-equivariant perverse (complexes of) sheaves and irreducible K-equivariant constructable sheaves.

The cohomology sheaves of a perverse sheaf are constructible. So, given $\phi, \psi \in \mathrm{Loc}_K(X)$, it makes sense to consider the multiplicity $m_{\phi,\psi}^{(j)}$ of the irreducible constructible sheave parametrized by ψ in the jth cohomology of the irreducible perverse sheaf parametrized by ϕ.

$$p_{\phi\psi}(q) = \sum_j m_{\phi\psi}^{(j)} q^j .$$

$\mathrm{Loc}_K(X)$ parametrize both irreducible K-equivariant perverse (complexes of) sheaves and irreducible K-equivariant constructable sheaves.

The cohomology sheaves of a perverse sheaf are constructible. So, given $\phi, \psi \in \mathrm{Loc}_K(X)$, it makes sense to consider the multiplicity $m^{(j)}_{\phi,\psi}$ of the irreducible constructible sheave parametrized by ψ in the jth cohomology of the irreducible perverse sheaf parametrized by ϕ.

$$p_{\phi\psi}(q) = \sum_j m^{(j)}_{\phi\psi} q^j .$$

These are the KLV polnomials of the title.

$\mathrm{Loc}_K(X)$ parametrize both irreducible K-equivariant perverse (complexes of) sheaves and irreducible K-equivariant constructable sheaves.

The cohomology sheaves of a perverse sheaf are constructible. So, given $\phi, \psi \in \mathrm{Loc}_K(X)$, it makes sense to consider the multiplicity $m^{(j)}_{\phi,\psi}$ of the irreducible constructible sheave parametrized by ψ in the jth cohomology of the irreducible perverse sheaf parametrized by ϕ.

$$p_{\phi\psi}(q) = \sum_j m^{(j)}_{\phi\psi} q^j .$$

These are the KLV polnomials of the title. In the case of complex groups, these polynomials are indexed by pairs of Weyl group elements and the p_{yw} were defined by Kazhdan-Lusztig.

Theorem (Lusztig-Vogan)

There is an effective algorithm to compute KLV polynomials.

Theorem (Lusztig-Vogan)

There is an effective algorithm to compute KLV polynomials.

This algorithm is implemented in the `atlas` software.