

Figure 1. The Plane Rotation Formula

The rotation formula depicted above says that under a plane rotation about the origin, if the image of $(1,0)$ is (X,Y) then the image of (x,y) is $(xX - yY, xY + yX)$, and the relation $X^2 + Y^2 = 1$ is an immediate consequence. The equation of the standard right circular cone is $x^2 + y^2 = z^2$. If the axis of this cone is rotated in space from $(0,0,1)$ to $(-Z,0,X)$ and the vertex is shifted by r along that axis, the rotation formula gives the equation of resulting cone as:

$$((xX - zZ) - rZ)^2 + y^2 = ((xZ + zX) + rX)^2.$$

The intersection of the rotated shifted cone with the $x - y$ -plane $z = 0$ is obtained by dropping the terms involving z :

$$(xX - rZ)^2 + y^2 = (xZ + rX)^2.$$

Expand the squares:

$$X^2x^2 - 2rXZx + r^2Z^2 + y^2 = Z^2x^2 + 2rXZx + r^2X^2.$$

collect like terms:

$$(X^2 - Z^2)x^2 - 4rXZx + y^2 = r^2(X^2 - Z^2).$$

and divide by $X^2 - Z^2$ to arrive at:

$$x^2 - 2rtx + \frac{y^2}{c} = r^2.$$

where

$$c = X^2 - Z^2, \quad s = 2XZ, \quad \text{and } t = \frac{s}{c} = \frac{2XZ}{X^2 - Z^2}.$$

Add $(rt)^2$ to both sides to complete the x -square:

$$(x - rt)^2 + \frac{y^2}{c} = r^2(1 + t^2) = \frac{r^2}{c^2} = a^2.$$

where we have let $a = \sqrt{r/c}$ and used $c^2(1 + t^2) = 1$. Divide to put in standard form:

$$\frac{(x - rt)^2}{a^2} + \frac{y^2}{ca^2} = 1.$$

With $h = rt$, and $b = \sqrt{|c|a^2}$, if $c > 0$, this becomes

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and if $c < 0$,

$$\frac{(x - h)^2}{a^2} - \frac{y^2}{b^2} = 1.$$

In terms of the angle of rotation θ , if $(X, Z) = (\cos \theta, \sin \theta)$ the original rotation formula with $(x, z) = (X, Z)$ shows that

$$(c, s) = (\cos(\theta + \theta), \sin(\theta + \theta)) = (\cos 2\theta, \sin 2\theta) = (XX - ZZ, XZ + ZX) = (X^2 - Z^2, 2XZ)$$

This explains our notation

$$(c, s) = (\cos 2\theta, \sin 2\theta) = (X^2 - Z^2, 2XZ), \quad t = \frac{s}{c} = \tan(2\theta).$$

The identity $c^2(1 + t^2) = 1$ may be viewed as $1 + \tan^2(2\theta) = \sec^2(2\theta)$, but all we need is $X^2 + Z^2 = 1$ from which:

$$1 + t^2 = 1 + \frac{(2XZ)^2}{(X^2 - Z^2)^2} = \frac{(X^2 - Z^2)^2}{(X^2 - Z^2)^2} + \frac{4X^2Z^2}{(X^2 - Z^2)^2} = \frac{(X^2 + Z^2)^2}{(X^2 - Z^2)^2} = \frac{1}{(X^2 - Z^2)^2} = \frac{1}{c^2}.$$

The standard reduction to standard form of the intersection of the fixed cone $z^2 = x^2 + y^2$ with a plane in a form $z = ax + by + c$ or something similar seems significantly less straightforward, and usually neglects the fact that the (x, y) coordinates in space must be transformed to represent standard rectangular coordinates on the tilted plane.

In standard College Algebra curricula, conic sections are often discussed immediately following matrices and determinants without the slightest connection being noted, and only conics that are co-axial with the coordinate axes are considered, missing the most significant aspects of the subject. The rotation formula also makes it possible to include conic sections that are not co-axial with the coordinate axes, at an elementary level, and thereby to highlight the connection with matrices, determinants, and ultimately the classification of the equations of mathematical physics into elliptic and hyperbolic.

$$au^2 + 2buv + cv^2 = 1$$

$$(u, v) = (xX - yY, xY + yX)$$

$$a(xX - yY)^2 + 2b(xX - yY)(xY + yX) + c(xY + yX)^2 = Ax^2 + 2Bxy + cy^2$$

with

$$A = aX^2 + 2bXY + cY^2, \quad B = 2b(X^2 - Y^2) - (a - c)2XY, \quad C = aY^2 - 2bXY + cX^2.$$

So $B = 0$ if

$$t = \frac{2XY}{X^2 - Y^2} = \frac{2b}{a - c},$$

and in rotated coordinates, the conic section $au^2 + 2buv + cv^2 = 1$ takes the form

$$Ax^2 + Cy^2 = 1.$$

Using $X^2 + Y^2 = 1$, we can also show that

$$AC - B^2 = (aX^2 + 2bXY + cY^2)(aY^2 - 2bXY + cX^2) - (2b(X^2 - Y^2) - (a - c)2XY)^2 = ac - b^2.$$

When $B = 0$, $AC - B^2 = AC$. When A and C are both positive, the conic is an ellipse, and when $AC < 0$, the conic section is hyperbolas. So the sign of $ac - b^2$ also distinguishes the two generic cases of conic sections.

In terms of matrices,

$$au^2 + 2buv + cv^2 = [u \quad v] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} xX - yY \\ xY + yX \end{bmatrix} = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}.$$

and the quantities $ac - b^2$ and $AC - B^2$ are seen to be determinants.

Algebraic Equations in Standard Form and Geometric Information

Standard Form of Parabolas with vertex (h, k) and focal length f :

$$y - k = +\frac{(x - h)^2}{4f} \quad (\text{Opening up})$$

$$x - h = +\frac{(y - k)^2}{4f} \quad (\text{Opening right})$$

$$y - k = -\frac{(x - h)^2}{4f} \quad (\text{Opening down})$$

$$x - h = -\frac{(y - k)^2}{4f} \quad (\text{Opening left})$$

Standard Form of Ellipses

Center (h, k) , Vertices: $(h + a, k)$, $(h - a, k)$, $(h, k + b)$, $(h, k - b)$:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \quad a > b > 0 \quad (\text{Horizontal Major Axis})$$

Focal length: $f = \sqrt{a^2 - b^2}$, Foci: $(h + f, k)$, $(h - f, k)$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \quad b > a > 0 \quad (\text{Vertical Major Axis})$$

Focal length: $f = \sqrt{b^2 - a^2}$, Foci: $(h, k + f)$, $(h, k - f)$

Standard Form of Hyperbolas

Center (h, k) , Fundamental Rectangle: $(h + a, k)$, $(h - a, k)$, $(h, k + b)$, $(h, k - b)$,

Focal length: $f = \sqrt{a^2 + b^2}$:

$$+\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1, \quad (\text{Horizontal Major Axis})$$

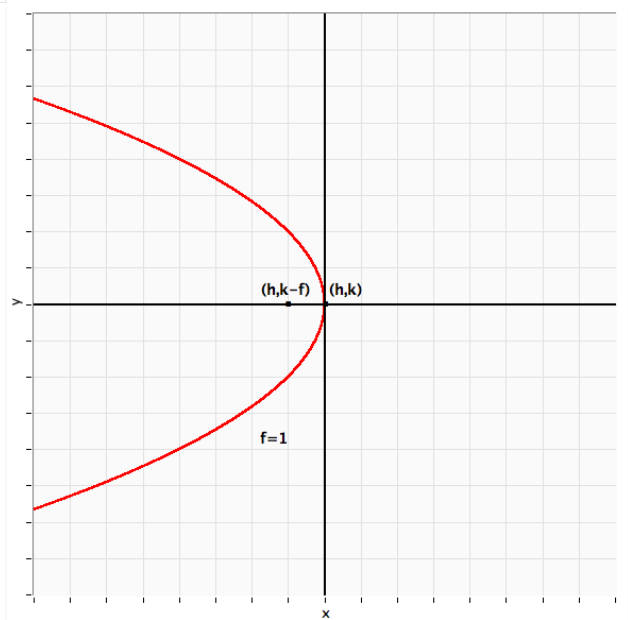
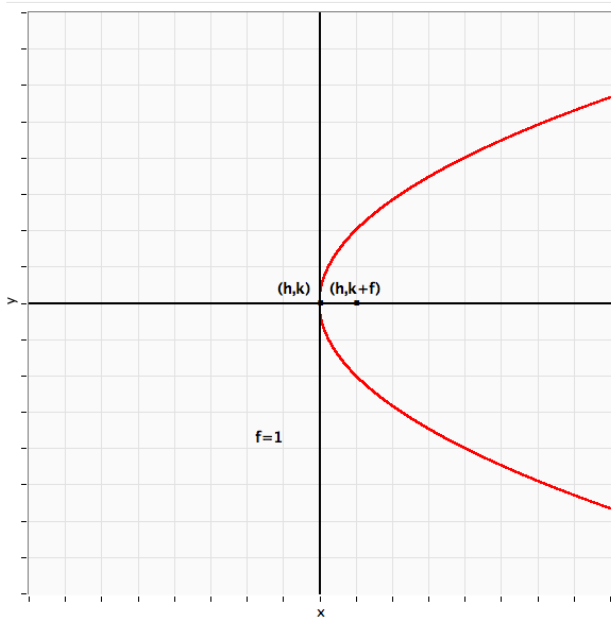
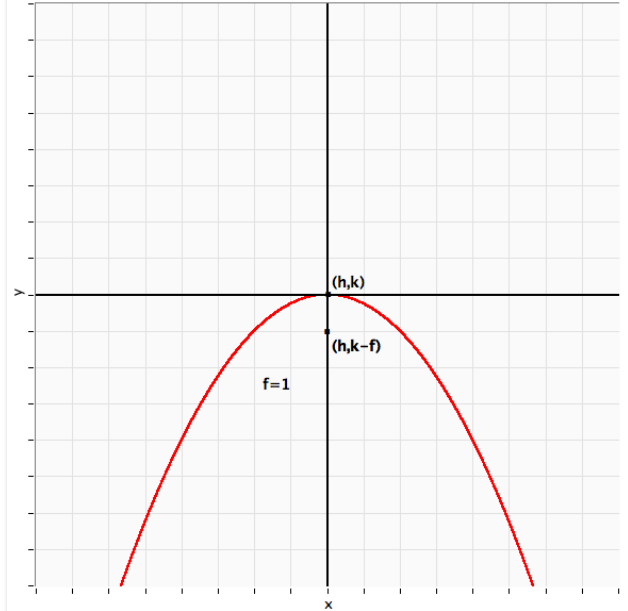
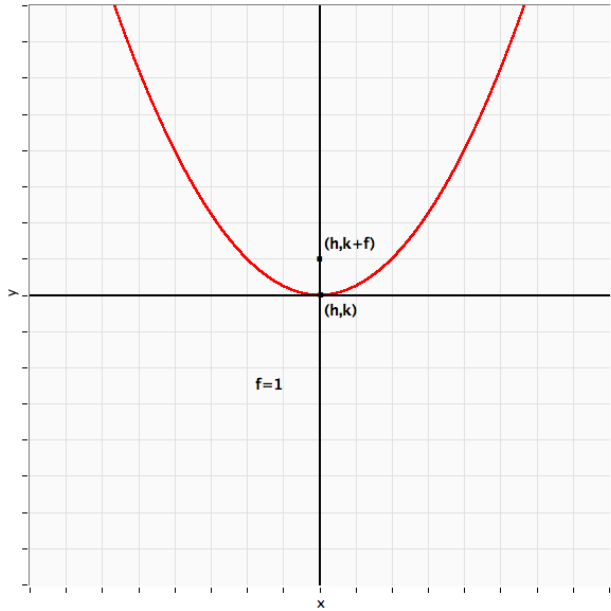
Vertices: $(h + a, k)$, $(h - a, k)$, Foci: $(h + f, k)$, $(h - f, k)$

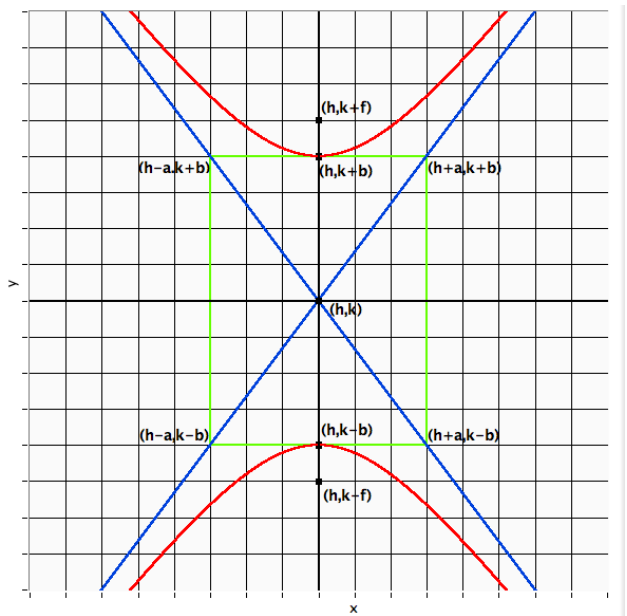
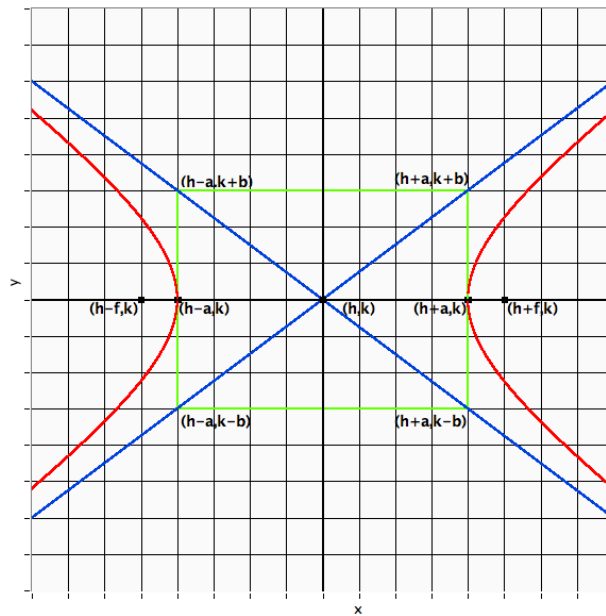
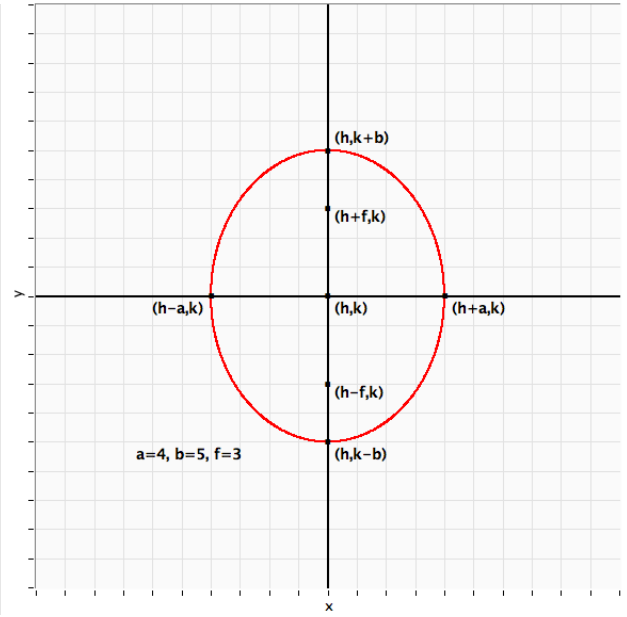
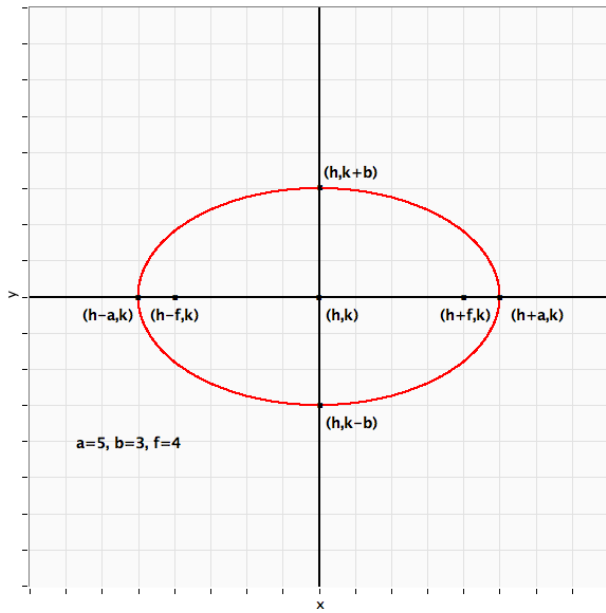
$$+\frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1, \quad (\text{Vertical Major Axis})$$

Vertices: $(h, k + b)$, $(h, k - b)$, Foci: $(h, k + f)$, $(h, k - f)$

Algebraic Equations in Standard Form and Geometric Information

Standard Form of Parabolas with vertex (h, k) and focal length f :





$$y - k = +\frac{(x-h)^2}{4f} \quad (\text{Opening up})$$