# Spline Spaces on $k$-dimensional Triangulations

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- Definitions
- What can be expected?
- Lower Bounds
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Emphasis on bounds, and the basic ideas of their derivation. Details in the paper.

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**Definition 1.** *Let*

$$\mathcal{T} = \{\sigma_i : i = 1, 2, \cdots, T\} \tag{1}$$

*be a finite set of non-degenerate simplices in* $\mathbb{R}^k$ *and let*

$$\Omega = \bigcup_{i=1}^{T} \sigma_i. \tag{2}$$

*Let*

$$\mathcal{X} = \{x_i \in \Omega, i = 1, 2, \cdots, N\} \tag{3}$$

*be the set of all vertices of simplices in* $\mathcal{T}$*. Then* $\mathcal{T}$ *is a triangulation of* $\Omega$ *provided that*

1. *The interiors of the simplices in* $\mathcal{T}$ *are pairwise disjoint,*
2. *Each* $(k-1)$*-dimensional face of a simplex in* $\mathcal{T}$ *is either on the boundary of* $\Omega$ *or else is a common face of exactly two simplices in* $\mathcal{T}$*,*
3. *Each simplex in* $\mathcal{T}$ *contains no points of* $\mathcal{X}$ *other than its vertices.*

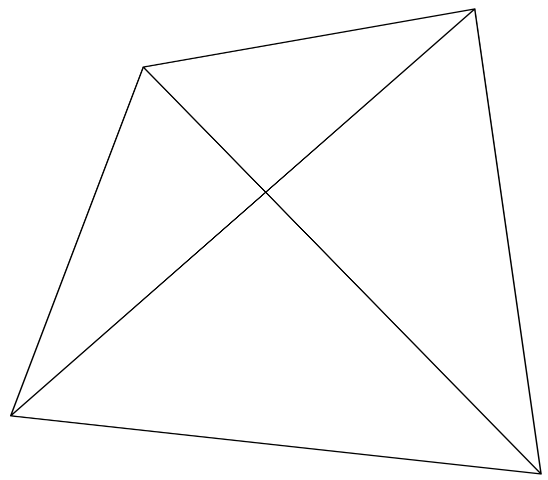
**Definition 2.** *A triangulation* $\mathcal{T}$ *is regular if* $\Omega$ *is homeomorphic to the unit cube.*

**Definition 3.** *A regular triangulation* $\mathcal{T}$ *is said to be shellable if it consists of a single simplex or a smaller shellable triangulation* $\tilde{\mathcal{T}}$ *can be obtained by removing one simplex from* $\mathcal{T}$*.*

**Definition 4.** *Let* $\mathbf{P}_d^k$ *denote the* $\binom{k+d}{d}$*-dimensional space of polynomials in* $k$ *variables of degree* $d$*. Then the space* $S_d^r(\mathcal{T})$ *is the set of all functions that can be represented as a polynomial in* $\mathbf{P}_d^k$ *on each simplex in* $\mathcal{T}$*, and that are* $r$*-times differentiable across each interior facet of* $\mathcal{T}$*.*

## Generic Dimension

- The dimension of $S_d^r(\mathcal{T})$ depends on the *geometry* as well as the *topology* of a triangulation $\mathcal{T}$.
- The *generic* dimension is such that if $\dim S_d^r(\mathcal{T})$ does not equal it then there is an arbitrarily small perturbation in the location of the vertices that will cause $\dim S_d^r$ to equal the generic value.
- The generic dimension is always the smallest possible value for a given topology of $\mathcal{T}$.



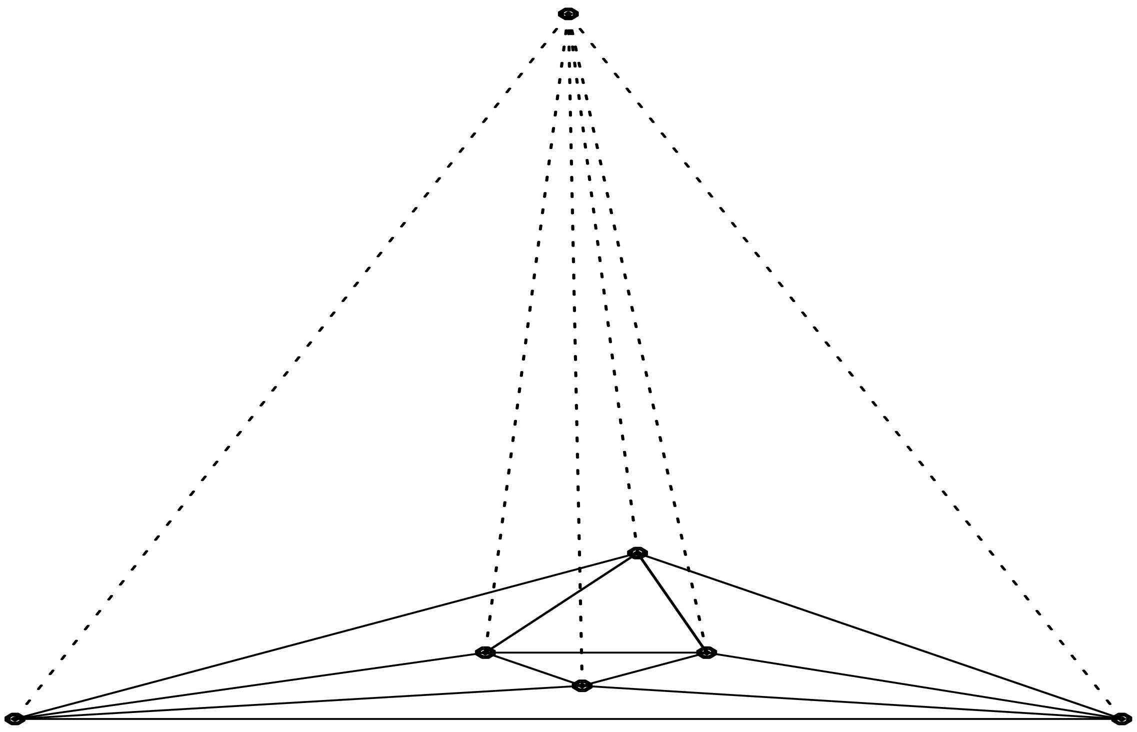
A Singular Vertex.

- Jiggle the interior vertex and the dimension of $S_d^1$ will drop by 1.

# What can be Expected?

## Very Little!

Solving a problem for $k$ variables and large $d$ requires the solution of the same problem for $k-1$ and **All** values of $d$.



Coning.

# A Commercial

Known: generic dimension of $S_d^1$, $k = 3$, $d \geq 8$.

Details see: Peter Alfeld, Larry L. Schumaker, Walter Whiteley: "The Generic Dimension of the Space of $C^1$ Splines of degree $d \geq 8$ on Tetrahedral Decompositions", SIAM JNA 30 (June 1993), pp. 889-920.

Required techniques from several areas: geometry, combinatorics, analysis, rigidity theory.

# B-form Basic Facts

**0.** A $k$-variate polynomial $p$ of degree $d$ can be written in its Bernstein Bézier form associated with a simplex $\sigma$.

**1.** The Bézier ordinates of $p$ are associated with $\binom{d+k}{k}$ *domain points* that are distributed over $\sigma$.

**2.** Let $p$ and $\tilde{p}$ be two polynomials defined on two simplices $\sigma$ and $\tilde{\sigma}$, respectively, that share a facet $\phi$. Assume the Bézier ordinates of $p$ are given. Then, in order for $p$ and $\tilde{p}$ to join smoothly of order $r$ the the smoothness conditions determine uniquely the Bézier ordinates in the first $r+1$ layers in $\tilde{\sigma}$ parallel to $\phi$.

**3.** The number of domain points in $\sigma$ remaining after removing a total of $s$ layers parallel to faces of $\sigma$ is $\binom{d+k-s}{k}$.

**4.** When joining a simplex to a regular triangulation at $\kappa$ facets, *continuity* can always be obtained by choosing the Bézier ordinates at the faces to be identical to the corresponding Bézier ordinates on the adjoining simplices. The remaining Bézier ordinates can be chosen arbitrarily.

## How Close can Bounds Be?

By coning repeatedly a degenerate configuration a non-generic dimension can be obtained that differs from the generic value by a polynomial of degree $k = 2$.

## Bounds: Basic Ideas

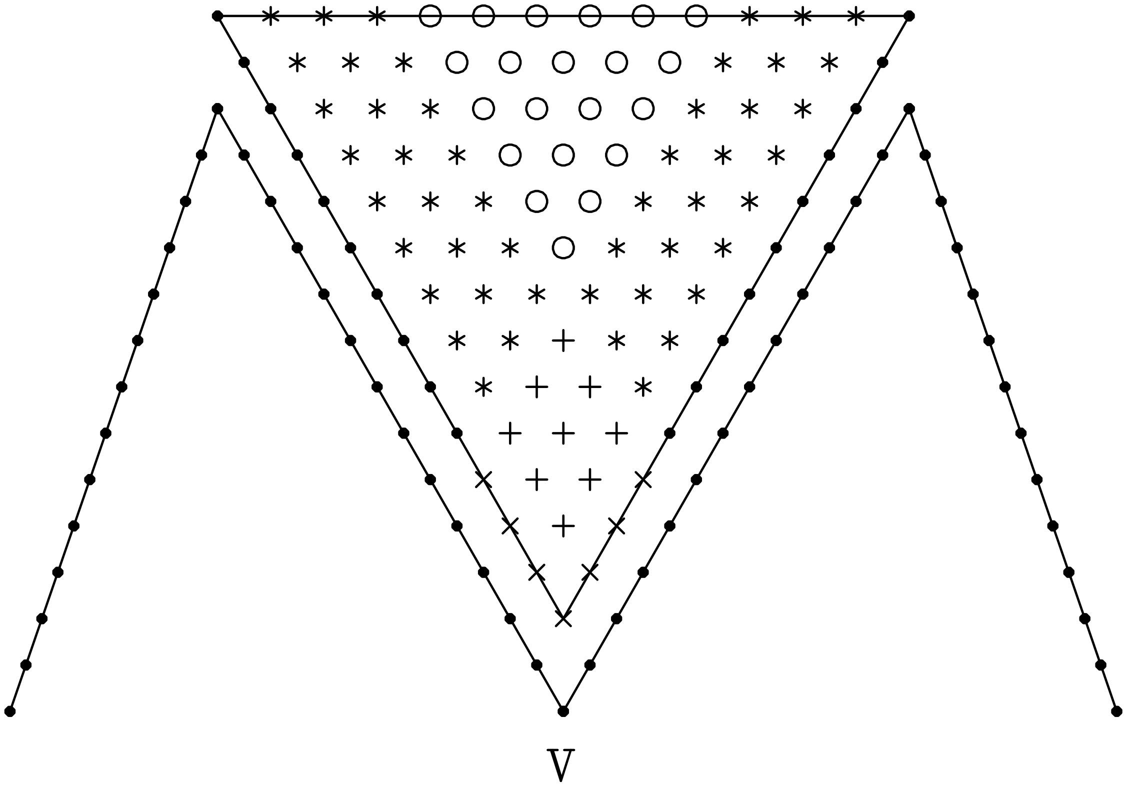
Build the triangulation one simplex at a time. Let $a_\kappa$ denote the number of times a new simplex was joined at $\kappa$ faces. The set $\{a_\kappa\}$ is a set of *dismantling parameters* (or *shelling parameters* if at every step the growing triangulation is a regular triangulation).

Think of $S_d^r$ as a subspace of $S_d^0$ and count parameters and smoothness conditions.

To get a lower bound subtract at each step a sufficient number of smoothness conditions from the number of new parameters.

To get an upper bound subtract those parameters determined by the smoothness conditions and assume the smoothness conditions are always consistent and do not reduce the dimension.

Illustrate ideas in a figure:




Basic Ideas.

$k = 2 \quad \kappa = 2 \quad r = 3 \quad d = 13 \quad \psi = 5 \quad \gamma = 12 \quad \Psi = 21$

# Lower Bounds

**Theorem 5.** *Let $\mathcal{T}$ be a triangulation, $\{a_\kappa : \kappa = 0, \cdots, k+1\}$ a set of dismantling parameters, and $d \geq r \geq 1$ integers. Then*

$$dimS_d^r(\mathcal{T}) \geq \max\left\{\binom{d+k}{k}, \quad \sum_{i=0}^{\kappa+1} \psi_\kappa a_\kappa\right\} \tag{4}$$

*where*

$$\psi_\kappa = \binom{d+k}{k} - \kappa \sum_{\mu=0}^{r} \binom{d+k-\mu}{k-1}. \tag{5}$$

**Theorem 6.** *Let $\mathcal{T}$ be a shellable triangulation, $\{a_\kappa : \kappa = 0, \cdots, k\}$ a set of shelling parameters, and $d \geq r \geq 1$ integers. Then*

$$dimS_d^r(\mathcal{T}) \geq \max\left\{\binom{d+k}{k}, \quad \sum_{i=0}^{\kappa} \gamma_\kappa a_\kappa\right\} \tag{6}$$

*where*

$$\gamma_\kappa = \binom{d+k-\kappa}{k} - \kappa \sum_{\mu=1}^{r} \binom{d+k-\kappa-\mu+1}{k-1}. \tag{7}$$

# An Upper Bound

**Theorem 7.** *Let $\mathcal{T}$ be a triangulation, $\{a_\kappa : \kappa = 0, \cdots, k+1\}$ a set of dismantling parameters, and $d \geq r \geq 1$ integers. Then*

$$dim S_d^r(\mathcal{T}) \leq \sum_{\kappa=0}^{\kappa+1} \Psi_\kappa a_\kappa \tag{8}$$

*where*

$$\Psi_\kappa = \binom{d + k - \kappa(r+1)}{k}. \tag{9}$$

## Comparison of Bounds

**Theorem 8.** *For $\kappa = 0, 1, \cdots, k$, let $\psi_\kappa$, $\gamma_\kappa$, $\Psi_\kappa$ be defined as in Theorems 5, 6, and 7, respectively. Then*

$$\psi_\kappa \leq \gamma_\kappa \leq \Psi_\kappa. \tag{10}$$

*Moreover,*

$$\Psi_\kappa - \psi_\kappa = \mathcal{O}\left(d^{k-2}\right) \tag{11}$$

*as $d \longrightarrow \infty$.*

## Examples.

| $d$ | $\omega$ | $\psi$ | $\gamma$ | **gdim** | $H$ | dim | $\Psi$ |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 6 | 6 | 6 [4] | 6 | 6 | 6 |
| 3 | 4 † | 10 | 10 | 10 | 10 [5] | 10 [6] | 13 |
| 4 | 8 †[1] | 15 | 15 | 15 [4] | 18 [5] | 18 [6] | 24 [7] |
| 5 | 15 [1] | 21 | 27 [3] | 30 [4] | 33 [5] | 33 [6] | 39 [7] |
| 6 | 22 [1] | 34 [2] | 49 [3] | 52 [4] | 55 [5] | 55 [6] | 61 [7] |
| 7 | 29 [1] | 63 [2] | 78 [3] | 81 [4] | 84 [5] | 84 [6] | 90 [7] |
| 8 | 36 [1] | 99 [2] | 114 [3] | 117 [4] | 120 [5] | 120 [6] | 126 [7] |
| 9 | 43 [1] | 142 [2] | 157 [3] | 160 [4] | 163 [5] | 163 [6] | 169 [7] |

**Table 9**
**The 2-dimensional Morgan-Scott Split, $r = 2$**

| | | | | | |
|---|---|---|---|---|---|
| [1] : | $\omega(d)$ | $=$ | | $7d$ | $-20$ |
| [2] : | $\psi(d)$ | $=$ | $7d^2/2$ | $-33d/2$ | $+7$ |
| [3] : | $\gamma(d)$ | $=$ | $7d^2/2$ | $-33d/2$ | $+22$ |
| [4] : | **gdim** | $=$ | $7d^2/2$ | $-33d/2$ | $+25$ |
| [5] : | $H(d)$ | $=$ | $7d^2/2$ | $-33d/2$ | $+28$ |
| [6] : | $\dim(d)$ | $=$ | $7d^2/2$ | $-33d/2$ | $+28$ |
| [7] : | $\Psi(d)$ | $=$ | $7d^2/2$ | $-33d/2$ | $+34$ |

| $d$ | $\omega$ | $\psi$ | $\gamma$ | **gdim** | $H$ | dim | $\Psi$ |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 10 | 10 | 10 | 10 | 10 | 10 |
| 3 | 4 † | 20 | 20 | 20 | 20 [5] | 20 | 24 |
| 4 | 5 † | 35 | 35 | 35 | 35 [5] | 35 | 51 |
| 5 | 6 †[1] | 56 | 56 | 56 | 56 [5] | 56 | 96 |
| 6 | 21 †[1] | 84 | 84 | 84 | 98 [5] | 87 | 170 [7] |
| 7 | 36 †[1] | 120 | 120 | 132 [4] | 176 [5] | 154 [6] | 284 [7] |
| 8 | 51 [1] | 165 | 165 | 243 [4] | 305 [5] | 274 [6] | 449 [7] |
| 9 | 66 [1] | 220 | 256 [3] | 420 [4] | 500 [5] | 460 [6] | 680 [7] |
| 10 | 81 [1] | 286 | 496 [3] | 678 [4] | 776 [5] | 727 [6] | 992 [7] |
| 11 | 96 [1] | 364 | 832 [3] | 1032 [4] | 1148 [5] | 1090 [6] | 1400 [7] |
| 12 | 111 [1] | 455 | 1279 [3] | 1497 [4] | 1631 [5] | 1564 [6] | 1919 [7] |
| 13 | 126 [1] | 728 [2] | 1852 [3] | 2088 [4] | 2240 [5] | 2164 [6] | 2564 [7] |

**Table 10**
**The 3-dimensional Morgan-Scott Split, $r = 2$**

| | | | | | | |
|---|---|---|---|---|---|---|
| [1] : | $\omega(d)$ | $=$ | | | $15d$ | $-69$ |
| [2] : | $\psi(d)$ | $=$ | $5d^3/2$ | $-27d^2$ | $-29d/2$ | $-13$ |
| [3] : | $\gamma(d)$ | $=$ | $5d^3/2$ | $-27d^2$ | $+151d/2$ | $-59$ |
| [4] : | **gdim** | $=$ | $5d^3/2$ | $-27d^2$ | $+187d/2$ | $-57$ |
| [5] : | $H(d)$ | $=$ | $5d^3/2$ | $-27d^2$ | $+223d/2$ | $-139$ |
| [6] : | $\dim(d)$ | $=$ | $5d^3/2$ | $-27d^2$ | $+205d/2$ | $-98$ |
| [7] : | $\Psi(d)$ | $=$ | $5d^3/2$ | $-27d^2$ | $+295d/2$ | $-283$ |

## An Estimate of the Dimension.

**Idea:** Consider a simple (non-generic) geometry that is easy to analyze.

**Which one?** Assume all ($k-1$-dimensional) interfaces are parallel (and infinite).

That's not a triangulation! No, but an arrangement of interfaces, for which we can consider smoothness conditions.

Resulting dimension will be larger than generic dimension.

Of spline space on arrangement of interfaces, not triangulation.

Unfortunately, triangulations can be even more degenerate, e.g., in the three dimensional Clough-Tocher split 6 planes meet at the centroid, whereas the generic value would be 3 planes.

$S_d^r(H_1)$: Have a $k+1$ tuple of points in $\mathbb{R}$, being the intersections of the interfaces with the line. Associate a univariate polynomial with each such $k+1$ tuple. Require $r$-th order smoothness across each such point. Consider the corresponding spline space.

$$\dim S_d^r(\mathcal{T}) \approx \dim S_d^r(\Pi) = \sum_{i_k=0}^{d} \sum_{i_{k-1}=0}^{i_k} \cdots \sum_{i_2=0}^{i_3} \dim S_{i_2}^r(H_1) \qquad (12)$$

## Another Example.

| $d$ | $\omega$ | $\psi$ | $\gamma$ | **gdim** | $H$ | dim | $\Psi$ |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 10 | 10 | 10 | 10 | | 10 |
| 3 | 4 † | 20 | 20 | 20 | 20 [5] | | 21 |
| 4 | 5 † | 35 | 35 | 35 [4] | 35 [5] | | 39 |
| 5 | 6 †[1] | 56 | 56 | 57 [4] | 56 [5] | | 66 |
| 6 | 10 †[1] | 84 | 84 | 89 [4] | 87 [5] | | 105 [7] |
| 7 | 14 †[1] | 120 | 120 | 135 [4] | 132 [5] | | 159 [7] |
| 8 | 18 [1] | 165 | 165 | 199 [4] | 195 [5] | | 231 [7] |
| 9 | 22 [1] | 220 | 233 [3] | 285 [4] | 280 [5] | | 325 [7] |
| 10 | 26 [1] | 286 | 337 [3] | 397 [4] | 391 [5] | | 445 [7] |
| 11 | 30 [1] | 364 | 471 [3] | 539 [4] | 532 [5] | | 595 [7] |
| 12 | 34 [1] | 455 | 639 [3] | 715 [4] | 707 [5] | | 779 [7] |
| 13 | 38 [1] | 596 [2] | 845 [3] | 929 [4] | 920 [5] | | 1001 [7] |

**Table 11**
**The 3-dimensional Clough-Tocher Split, $r = 2$**

| | | | | | | |
|---|---|---|---|---|---|---|
| [1] : | $\omega(d)$ | $=$ | | | $4d$ | $-14$ |
| [2] : | $\psi(d)$ | $=$ | $2d^3/2$ | $-5d^2$ | $-5d/3$ | $-2$ |
| [3] : | $\gamma(d)$ | $=$ | $2d^3/3$ | $-5d^2$ | $+55d/3$ | $-13$ |
| [4] : | **gdim** | $=$ | $2d^3/3$ | $-5d^2$ | $+79d/3$ | $-33$ |
| [5] : | $H(d)$ | $=$ | $2d^3/3$ | $-5d^2$ | $+76d/3$ | $-29$ |
| [7] : | $\Psi(d)$ | $=$ | $2d^3/3$ | $-5d^2$ | $+103d/3$ | $-65$ |

Why ask about bounds? Eventually one would like to solve the following

## Open Problems

What are minimal values of $d$ as a function of $k$, for $S_d^r$

— to have a local basis
— to have full approximation order
— to be able to interpolate to function values at vertices
— to be large enough to interpolate to function values at vertices?