Multivariate Splines

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Outline of Talk

- Splines
- Multivariate Splines
- Generic Dimension
- History of Multivariate Splines
- Unsolved Problems $\boxed{k = 2}$
- Unsolved Problems $\boxed{k > 2}$
- The Generic Dimension of S_d^1, $k = 3$, $d \geq 9$
- Algebraic Techniques
- Geometric Techniques
- Fundamental Issue
- Combinatorial Techniques
- Putting it Together
- Additional Remarks

Splines

Splines = smooth piecewise polynomial functions.

smooth usually = C^1 or C^2.

Example:

$$a = x_0 < x_1 < \cdots < x_N = b$$

$$S_d^r\big([a,b]\big) = \Big\{ s \in C^r\big([a,b]\big) : s\big|_{[x_{i-1},x_i]} \in \mathbb{P}_d^1, \quad i = 1, 2, \cdots, N \Big\}$$

$\mathbb{P}_d^k$ is $\binom{k+d}{d}$ dimensional space of polynomials of degree d in k variables.

Used ubiquitously, e.g., for:

— Interpolation of data and functions.
— Approximation of data and functions.
— Data Representation.
— Curve Design.
— Numerical Differentiation.
— Numerical Integration.
— Numerical ODEs.
— Numerical Integral Equations.

Multivariate Splines

How to generalize to $k > 1$ variables?

Most Natural:
$\Omega \subset \mathbb{R}^k$ is a triangulated domain, $\mathcal{T}$ is the (k-dimensional) triangulation.

$$S_d^r(\Omega) = \left\{ s \in C^r(\Omega) : s|_\triangle \in \mathbb{P}_d^k \ \forall \triangle \in \mathcal{T} \right\}.$$

Questions:
— Dimension?
— Basis?
— Local Basis?
— Interpolation?
— Algorithms?

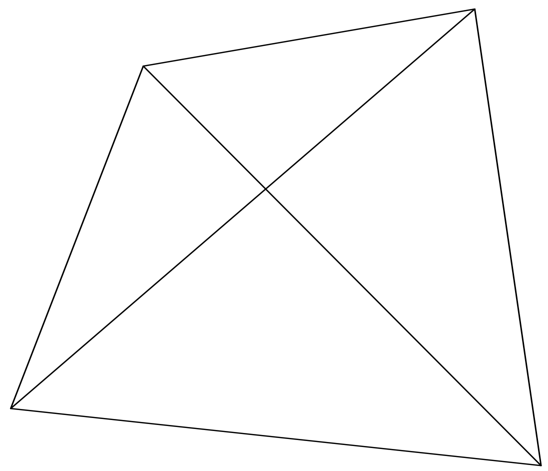
Notes:

- The structure of S_d^r for $k > 1$ is extremely complicated and poorly understood in general.
- Only sub- and superspaces of S_d^r (with simpler structure) are used for applications.
- The dimension of $S_d^r(\mathcal{T})$ is affinely and projectively invariant!
- For $k > 1$, dimension, basis, and interpolation depend on the geometry as well as the combinatorics of the triangulation. For $k = 1$, only the combinatorics matter. But there is a *generic* dimension.

Generic Dimension

The **generic** dimension is such that if $\dim S_d^r(\mathcal{T})$ does not equal it then there is an arbitrarily small perturbation in the location of the vertices that will cause $\dim S_d^r$ to equal the generic value.

The generic dimension is always the smallest possible value for a given topology of $\mathcal{T}$.



A Singular Vertex.

Jiggle the interior vertex and the dimension of S_d^1 will drop by 1. (For S_d^2 it will drop by 1 or 3—but not 2.)

(very) Selected History of Multivariate Splines

Disclaimer: $r = 0$ is trivial. Special triangulations are studied widely. Subdivision of triangles done in finite elements. Approximation order is widely studied. Supersplines form subspaces. All of these topics will be omitted.

year	k	r	d	**Result**	**Author**
1973	2	1	3	dim., conjecture	Strang
1973	2	r	d	dim., conjecture	Strang
1975	2	1	≥ 5	basis	Morgan & Scott
1979	2	r	d	dim., lower bound	Schumaker
1984	2	r	d	dim., upper bound	Schumaker
1986	3	$\{1, 2\}$	$\geq 3, 5$	dim., lower bound	A
1987	2	r	$> 4r$	dimension	A&S
1987	2	$\{1, 2, 3\}$	$> 4r$	basis	A&S&Piper
1987	2	1	4	basis	A&S&Piper
1988	2	1	d	generic dimension	Billera, Whiteley
1990	2	r	$3r + 1$	generic dimension	A & S
1991	2	r	$> 3r + 1$	basis	Ibrahim & S
1991	2	r	$> 3r + 1$	basis	D. Hong
1992	3, k	r	$d > 8r$	ex. loc. basis	A & S & Sirvent
1993	3	1	≥ 8	generic dimension	A & S & Whiteley
1995	k	r	d	upper, lower bounds	A
1995	2	spline spaces on the sphere			A & S & Neamtu

Unsolved Problems $\boxed{k=2}$

Estimated difficulty rated on a scale of 1–10, 10 being the hardest, the Four Color Map Problem rating about 7 and a good Ph.D. thesis rating about 4.

1: What is the precise dimension of S_3^1? (The generic dimension is $3V_B + 2V_I + 1$, as shown by Billera and Whiteley. Conjecture:

$$\dim S_3^1 = 3V_B + 2V_I + 1 + \sigma$$

with σ being the number of singular vertices.) .... **—8—**

2: What is the precise dimension of S_2^1? (The generic dimension is $V_B + 3$, as shown by Billera and Whiteley. The equivalent of the above S_3^1 conjecture is false!) .... **—9—**

3: Can one interpolate to function values at the vertices of $\mathcal{T}$ with S_3^1? Conjecture: Yes. .... **—8—**

4: Can one interpolate in general to function and gradient values at the vertices of $\mathcal{T}$ with S_4^1? (Generically it's possible. That problem would rate 1.) Conjecture: Yes. .... **—5—**

5: What is the generic dimension of S_d^r, for $r > 1$ and $\frac{3r+1}{2} < d \leq 3r$? (Whiteley call this question "an essential challenge to all known techniques".) .... **—6—**

Unsolved Problems $\boxed{k > 2}$

6: What is the precise dimension of S_d^1, $k = 3$ for $d \geq 9$? . . —**9**—

7: What is the *generic* dimension of S_d^r, $k = 3$, in the case that $r > 1$ and $d \geq 8r + 1$? . . . . —**6**—

8: What are tight upper and lower bounds on the dimension of S_d^r, $k > 2$? (Rating depends on degree of *tightness*.) . . . . —**n**—

9: What is the generic dimension of S_d^r for $k > 3$? . . . . —**8**—

10: Does the Morgan-Scott idea generalize to $k > 3$? Conjecture: yes, it does. . . . . —**8**—

11: What are minimal values of d as a function of k, for S_d^r

— to have a local basis
— to have full approximation order
— to be able to interpolate to function values at vertices
— to be large enough to interpolate to function values at vertices?

Conjecture: They all grow exponentially. . . . . —**10**—

The Generic Dimension of S_d^1, $k = 3$, $d \geq 9$

To illustrate techniques discuss:

Theorem. *Let $\Omega \subset \mathbb{R}^3$ be a triangulated polyhedral domain, $\mathcal{T}$ its triangulation, and suppose*

$$\boxed{d \geq 9.}$$

Then, generically,

$$dim S_d^1(\Omega) = \frac{d(d-1)(d-5)}{6} T + 3(d-1)V_I + d(d-1)V_B + 1 + 5d - 2d^2$$

where

$$V_B = \text{the number of boundary vertices of } \mathcal{T},$$
$$V_I = \text{the number of interior vertices of } \mathcal{T},$$
$$T = \text{the number of tetrahedra of } \mathcal{T}.$$

Remainder of Talk: Sketch ingredients of proof. Details in paper:
P. Alfeld, L.L. Schumaker, W. Whiteley: The generic dimension of the space of C^1 splines of degree $d \geq 8$ on tetrahedral decompositions. SIAM J.N.A., **30** (1993), 889-920.

Algebraic Techniques

Will illustrate for $k = 2$.

Given $\Omega \subset \mathbb{R}^2$ and its triangulation $\mathcal{T}$.
v_1, v_2, v_3 vertices of a general triangle.

Define *barycentric coordinates*:

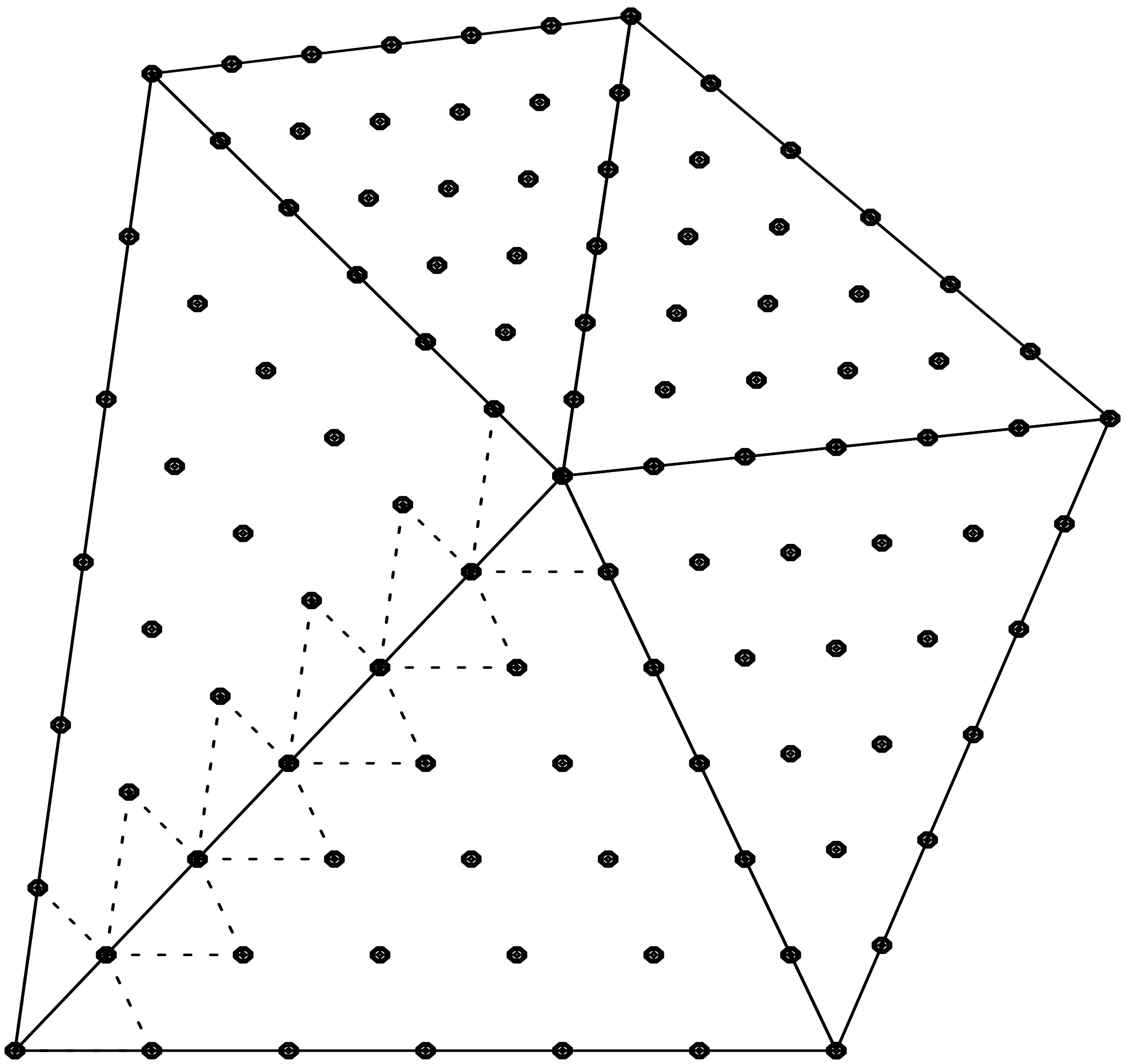
$$x = \sum_{i=1}^{3} b_i v_i \quad \text{where} \quad \sum_{i=1}^{3} b_i = 1.$$

Any polynomial $p \in \mathbb{P}_d^2$ can be written uniquely in its *Bernstein-Bézier form* as:

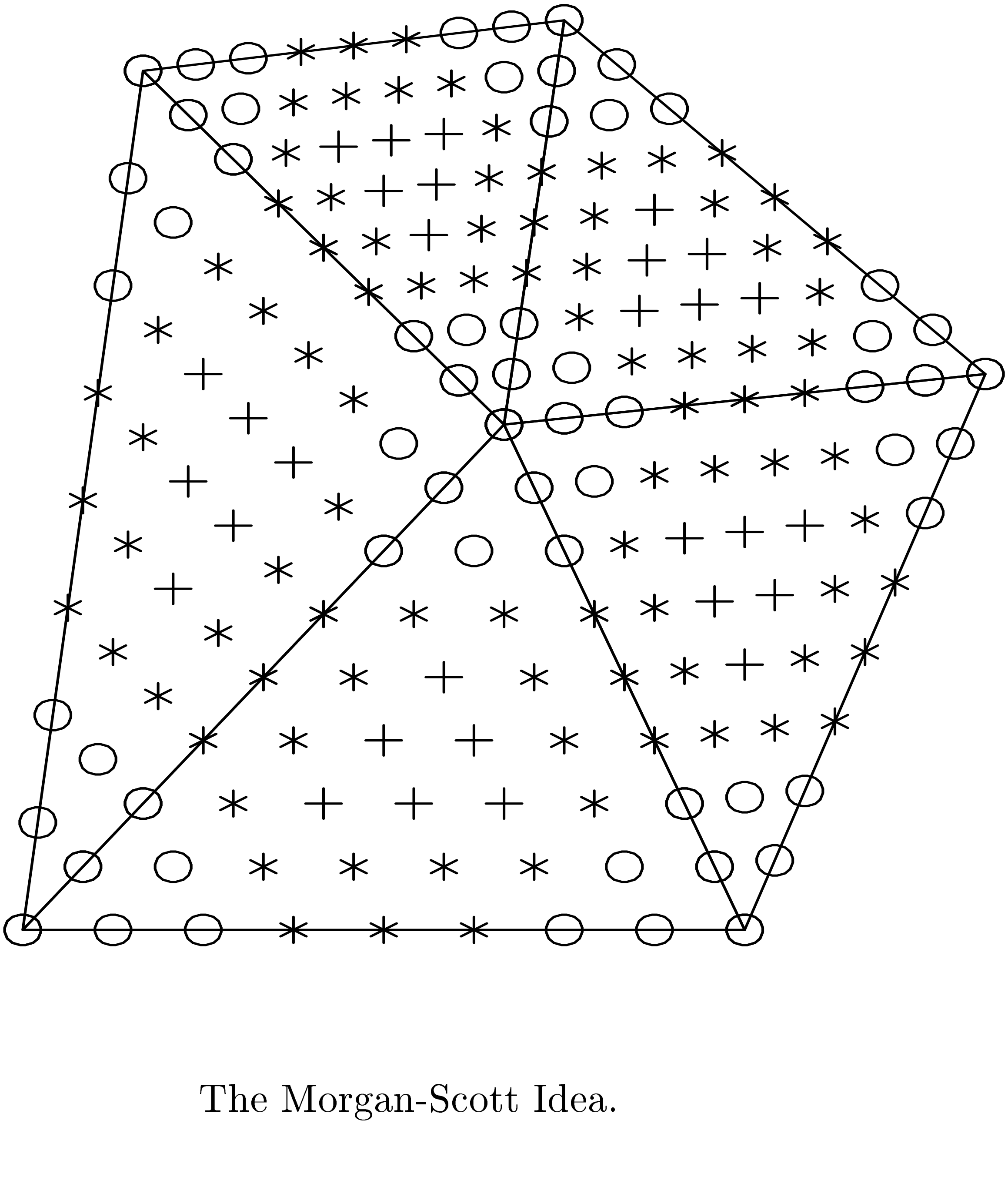
$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k.$$

where

c_{ijk} : *Bézier ordinates*
$P_{ijk} = \dfrac{iv_1 + jv_2 + kv_3}{d} \in \mathbb{R}^2$: *Domain Points*
$(P_{ijk}, c_{ijk}) \in \mathbb{R}^3$: *Bézier control points*
Set of Bézier control points = *Bézier control net, or B-net.*



The B-net.



The Morgan-Scott Idea.

Generalization to Three-Dimensional Triangulations

Note: for $k = 2$, in order to avoid overlaps of edge sets the vertex sets must have radius 2. To avoid overlap of vertex sets, d must be at least 5. For $k = 3$, in order to avoid overlaps of face sets, edge sets must have radius 2. Then, in order to avoid overlaps of edge sets, vertex sets must have radius 4. In order to avoid overlap of vertex sets d must be at least nine.

Divide domain points into 4 types:

0. Those of distance up to 4 from a vertex.

1. Those of distance up to 2 from an edge, but not of type 0.

2. Those of distance up to 1 from a triangular face, but not of type either 0 or 1.

3. Those that are left. (These are precisely the points that are not involved in any smoothness conditions at all.)

Stumbling Block: Type 0 points around individual vertices. Analysis of these is equivalent to analyzing S_4^1 on the star of the vertex.

Elaboration on Problem 10.

Remember problem 10: Does the Morgan-Scott idea generalize to

$$k > 3?$$

Previous categorization generalizes in an obvious way to $k > 3$, yielding points of type 0 through k (and requiring $d \geq r2^k + 1$).

Basic Issue boils down to this:

Do induction in the dimension of simplices, say $m \leq k$. This leads to a block-triangular system

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} c_m \\ c_{m+1} \end{pmatrix} = 0.$$

First block row are smoothness conditions for points of types up to m, second block row involves points of type $m + 1$

Question:

$$\text{Does} \quad \text{rank} \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \text{rank}A + \text{rank}C?$$

For $k \in \{2, 3\}$ the answer is yes. For $k > 3$ the answer is not known.

Geometric Techniques

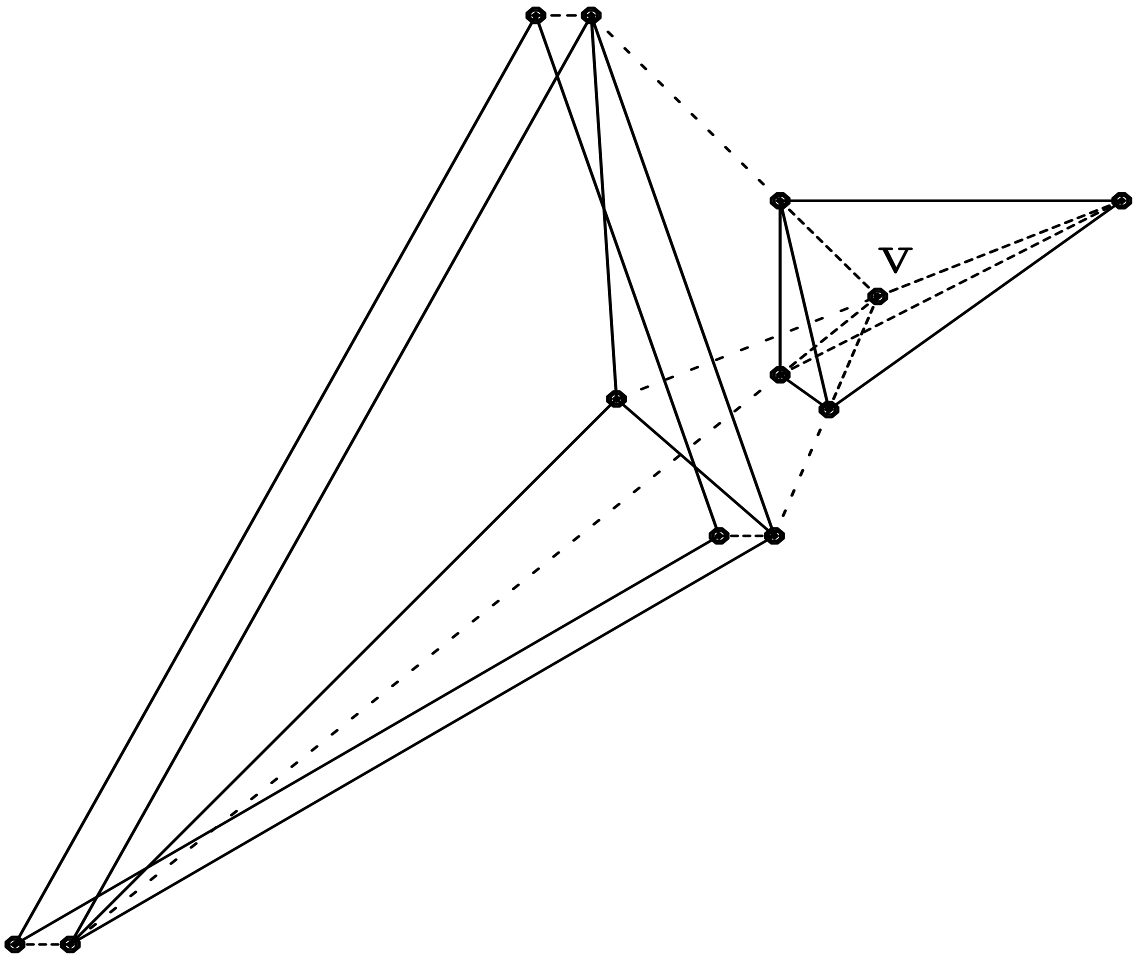
Need to analyze S^1_4 on a vertex star.

Let $\mathcal{T}$ be the triangulation of a the star Ω of a vertex v.

Project vertices in $\mathcal{T}$ from v into a plane.

Obtain a *generalized triangulation* Δ where:

1. Triangles may overlap.
2. All edges may be interior. (They are if v is an interior vertex.)



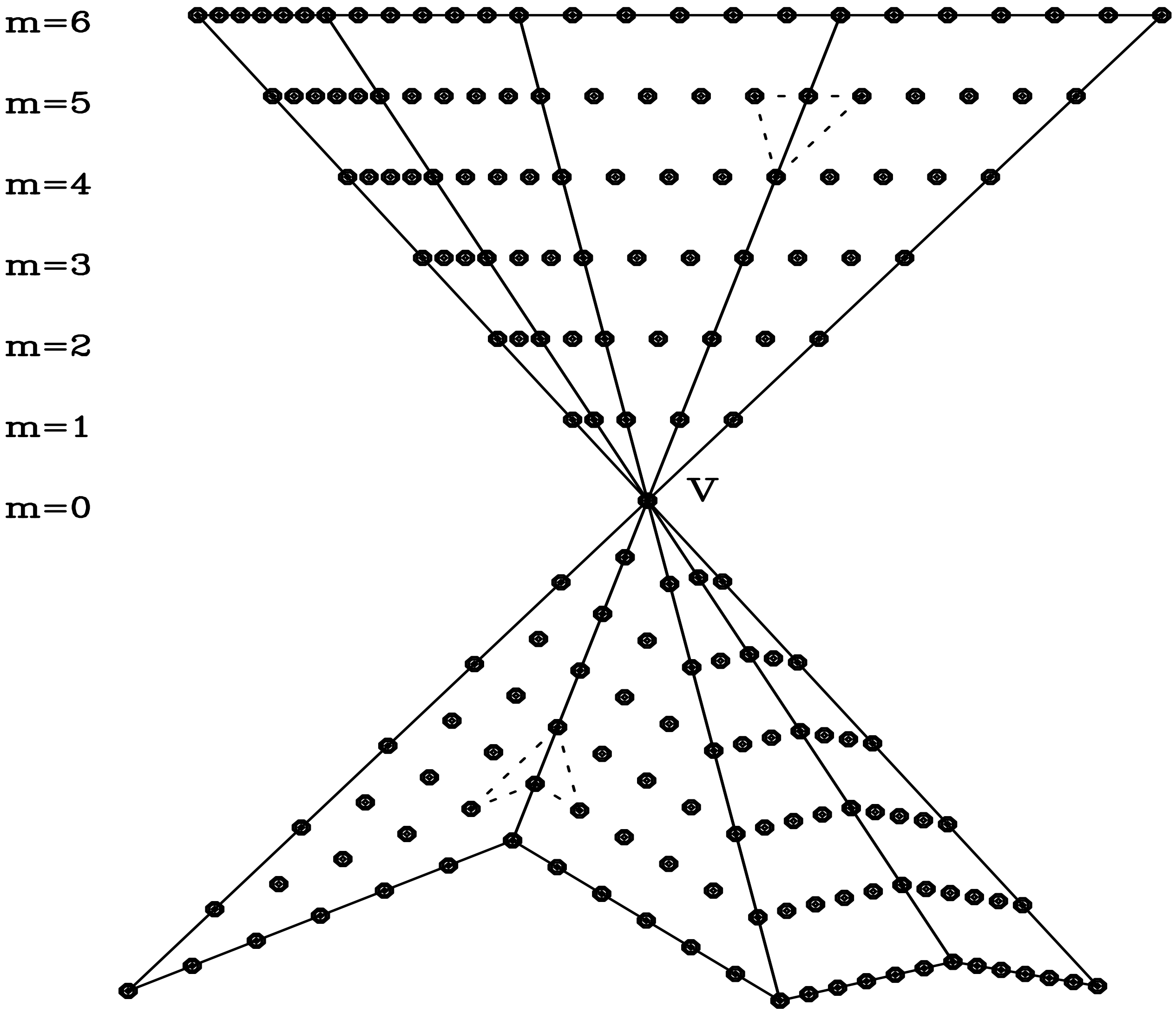
Projecting vertex stars.

Major tool:

Theorem.

$$dim S_d^r(\Omega) = \sum_{m=0}^{d} dim S_m^r(\Delta).$$

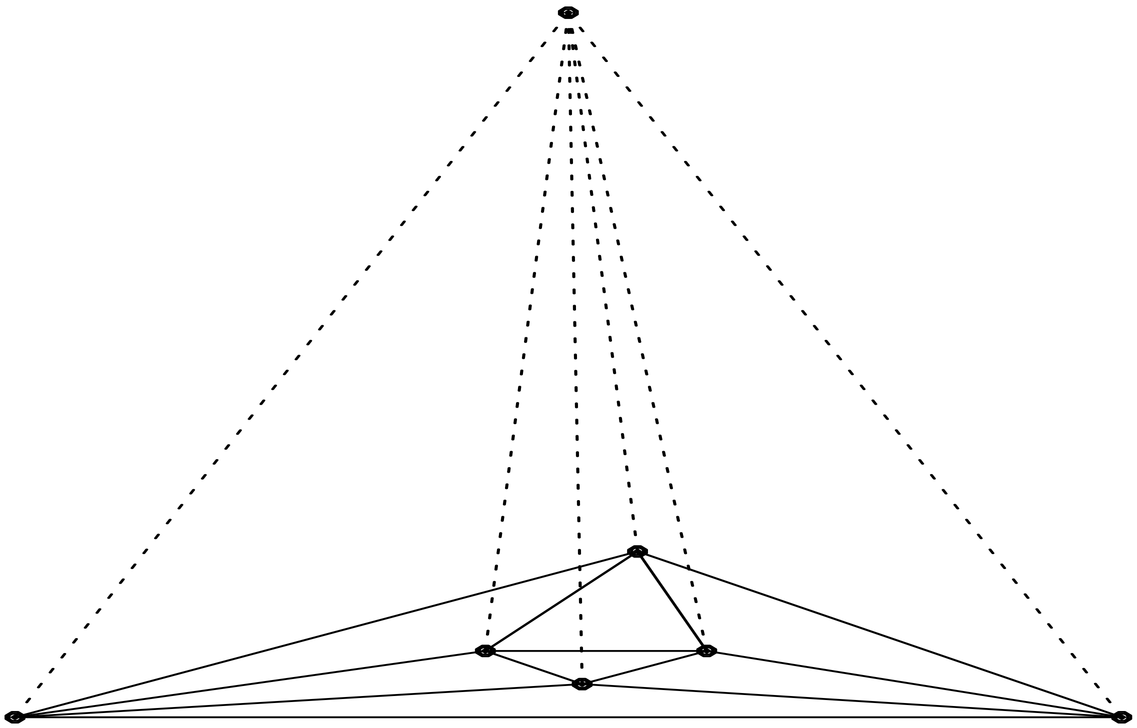
Illustrate proof in terms of B-form for $k = 2$.



Proof of Projection Result.

Fundamental Issue

Any dimension statement for $S_d^r(\Omega)$, for a general triangulated domain $\Omega \in \mathbb{R}^k$, implies similar statements for $S_m^r(\omega)$, for domains $\omega \in \mathbb{R}^{k-1}$, and *for all* $m \leq d$.



Coning.

Combinatorial Techniques

Need to analyze the generic dimension of spline spaces on generalized triangulations.

Write down smoothness conditions in terms of *smoothing cofactors.*

Obtain a homogeneous block system with
— one block row for each interior vertex, and
— one block column for each interior edge.

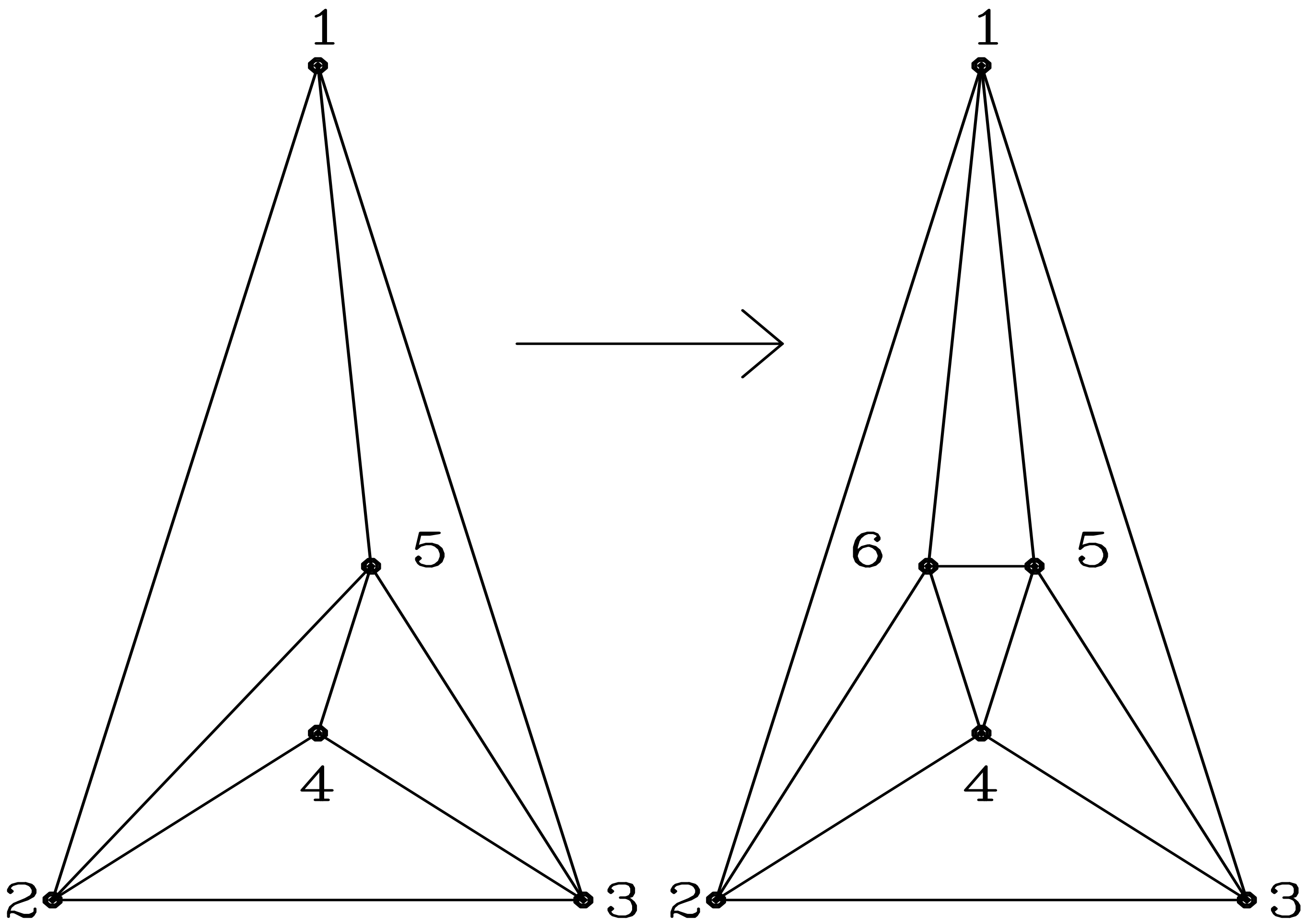
Need to analyze the rank of that system.

Use induction in the number of interior vertices V_I^Δ to show that

For g-triangulations with boundary the rank is full = the number of rows.

(For closed g-triangulations the rank deficiency is constant, i.e., independent of V_I^Δ.)

Use **Vertex Splitting** (Barnette, Whiteley).



Example: The Morgan-Scott Split.

$$\bar{A} = \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 \\ 0 & 0 & A_{54} & A_{53} & A_{51} & A_{52} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{54} & A_{53} & A_{51} & A_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{51} & A_{65} & A_{54} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{54} & A_{53} & A_{51} & A_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{52} & A_{51} & A_{65} & A_{54} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{54} & A_{53} & A_{51} & 0 & -A_{51} & -A_{65} & -A_{54} \\ 0 & 0 & 0 & 0 & 0 & A_{52} & A_{51} & A_{65} & A_{54} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & A_{45} \\ 0 & 0 & A_{54} & A_{53} & A_{51} & 0 & 0 & -A_{65} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{52} & A_{51} & A_{65} & A_{54} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & A_{46} \\ 0 & 0 & A_{54} & A_{53} & A_{51} & 0 & 0 & A_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{62} & A_{61} & A_{65} & A_{64} \end{pmatrix}$$

Putting it Together

$$
\begin{aligned}
\dim S_d^1(\mathcal{T}) = \quad & 20E_B + 16E_I + 6V_I + 5V_B \\
& \qquad \text{(from the vertices)} \\
+ & \sum_{e \text{ an interior edge}} \big[(3d-25) + (d-7)n(e)\big] \\
& \qquad \text{(from the interior edges)} \\
+ & \sum_{e \text{ a boundary edge}} \big[(3d-25) + (d-7)(n(e)+1)\big] \\
& \qquad \text{(from the boundary edges)} \\
+ & \left[\binom{d-7}{2} + \binom{d-5}{2}\right] F \\
& \qquad \text{(from the triangular faces)} \\
+ & \binom{d-5}{3} T \\
& \qquad \text{(from the interiors of} \\
& \qquad \quad \text{the individual tetrahedra)} \\
= & \text{ desired value}
\end{aligned}
$$

where $n(e)$ is the number of triangular faces attached to the edge e.

Additional Remarks

1. Our proof does not yield an explicit basis. The difficulty again lies with the vertex stars. If the analysis can be made more incisive the resulting basis would be minimally supported (i.e., the support of a basis function would be contained in the star of a vertex).

2. Extension to $r > 1$ may be possible, but is not straightforward.

3. Analysis is intrinsically generic.

4. Extension to $k > 3$ requires knowledge for all d in the case $k = 3$ and appears difficult.

5. Extension to $d < 9$, e.g., $d = 8$ or $d = 7$ may be possible, but is not straightforward.

6. Techniques may be applicable towards analysis of interpolation problems.

7. Curious offshot: $s \in S^1_d$, $d \geq 0$, $k = 3$ is *twice* differentiable at interior vertices of $\mathcal{T}$.

8. Main point of Talk: interdisciplinary nature of proof.