Scattered Data Interpolation and Approximation on the Sphere and Similar Surfaces

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joint work with Marian Neamtu and Larry Schumaker.

(Mistakes and disagreeable statements are my own.)

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- **Bernstein-Bézier Theory on the Sphere**
- **much like planar case**
- **Talk focuses on Differences**

Papers to appear, submitted, in the works. Ask us!

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The Idea

v_1, v_2, v_3, v: **points on the (unit) sphere.**

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3$$

no Normalization!

b_i: spherical barycentric coordinates (SBCs).

SBCs are unique (Their span is the only rotation invariant three dimensional space of functions on the sphere that for every great circle contains a non-zero function which vanishes on that circle).

$$p(v) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k$$

p: spherical Bernstein-Bézier polynomial.

What's a graph?

Graph: $\left\{x \in \mathbb{R}^3 : x = p(v)v\right\}$

graphs do indeed look different!

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Earth

Spheres are important because we live on one.

But we don't!

Earth more closely resembles an ellipsoid with a semimajor axis of

$$6,378,137\mathbf{m}$$

and a semiminor axis of

$$6,356,752\mathbf{m}.$$

$$\textbf{Difference} = 21,385m.$$

The "geoid" = an equipotential surface of gravity.

Definition of "sea level".

It matters e.g., when modeling altitude.

What's higher: Mount Everest or K2?

Homogeneous Extension

Major observation: the Equations

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3$$

and

$$p(v) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k$$

make sense for all $v \in \mathbb{R}^3$!

They are ordinary trivariate polynomials!

They are homogeneous:

$$b_i(\alpha v) = \alpha b_i(v) \quad \textbf{and} \quad p(\alpha v) = \alpha^d p(v)$$

Moreover: The b_i are ordinary coordinates of v with respect to the basis

$$\{v_1, v_2, v_3\}\,.$$

Sphere-Like Surfaces

Functions on the Sphere are obtained by the restriction of trivariate functions to the Sphere.

Correspondence is one-to-one by virtue of homogeneity!

Don't need to restrict to sphere, can restrict to similar surface.

$$S = \{u \in \mathbb{R}^3 : u = \rho(x)x, \|x\|_2 = 1\}$$

($\rho(x) > 0$, ρ smooth). Somewhat more special than star like surface.

A piecewise defined function on a sphere-like surface will be smooth on the surface if and only if its homogeneous extension is smooth in $\mathbb{R}^3$.

Consequence: Apply standard trivariate Bézier Technology on trihedral decompositions (which reduce to spherical triangulations).

Analogs

To be sure, many things do carry over:

- **de Casteljau Algorithm**
- **Subdivision**
- $b_i(v_j) = \delta_{ij}$
- **Signs of SBCs define (eight) spherical triangles (corresponding to seven planar subdomains)**
- **SBCs vanish on edges of spherical triangles**
- **Span of SBCs is independent of spherical triangle**
- **SBCs are ratios of volumes of tetrahedra**
- **SBCs are invariant under rotation (corresponding to affine transform in the plane).**
- **Smoothness Conditions**
- **Clough-Tocher Scheme**
- **Powell-Sabin Scheme**
- **Quintic Finite Elements**
- **Least Squares**
- **Simplex Splines (M. Neamtu)**

Differentiation

Differentiation with respect to SBCs makes sense.

$$\nabla_v p = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} \nabla_b p$$

where

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \textbf{and} \quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Differentiation with respect to planar barycentric coordinates does not make sense:

$$\beta_1 + \beta_2 + \beta_3 = 1$$

but

$$\frac{\partial}{\partial \beta_i} 1 \neq \frac{\partial}{\partial \beta_i} (\beta_1 + \beta_2 + \beta_3).$$

Nesting of Spaces

Let

$$H_d$$

denote the space of all trivariate homogeneous polynomials of degree d**.**

Clearly,

$$d \neq \bar{d} \quad \Longrightarrow \quad H_d \cap H_{\bar{d}} = \{0\}\,.$$

Thus the trivariate spaces surely are not nested.

The spaces of restrictions to sphere-like surfaces may be nested.

Examples:

$$S = \textbf{sphere} \quad \Longrightarrow \quad H_d|_S \subset H_{d+2}|_S.$$

$$S = \left\{ v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^4 + y^4 + z^4 = 1 \right\} \quad \Longrightarrow \quad H_d|_S \subset H_{d+4}|_S.$$

In such cases, degree raising is possible.

But usually (Earth!) the restrictions of H_d **to sphere-like surface will not be nested.**

Polynomial Precision

For a general sphere-like surface the restriction of H_d will not contain constants.

Higher degree polynomial precision?

Study things in $\mathbb{R}^3$ and extend functions being approximated to $\mathbb{R}^3$ by homogeneity.

Spline Spaces

Defining an interpolation or approximation scheme by having a polynomial on each triangle that joins smoothly with the polynomials on neighboring triangles gives rise to spline spaces.

Let $\mathcal{T} := \{T^{[i]}\}_1^N$ **be a planar triangulation of a set** $\Omega \in \mathbb{R}^2$**.**
Usual bivariate spline space:

$$\mathcal{S}_d^r(\Delta) := \{s \in C^r(\Omega) \ : \ s|_{T^{[i]}} \in \mathcal{P}_d, \quad i = 1, \ldots, N\}$$

$\mathcal{P}_d$ **= space of bivariate polynomials of degree** d**.**

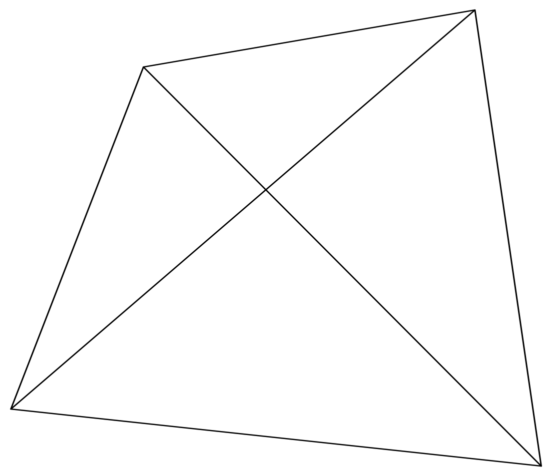
Questions:

- **Dimension (formula)**
- **Basis**
- **Solvability of interpolation problems.**
- **Algorithms**

Well known: Answers to some questions depend on geometry as well as the topology of the triangulation.

Generic Dimension

- The **generic** dimension is such that if dim$S_d^r(\mathcal{T})$ does not equal it then there is an arbitrarily small perturbation in the location of the vertices that will cause dimS_d^r to equal the generic value.
- Result: The generic dimension is always the smallest possible value for a given topology of $\mathcal{T}$.



A Singular Vertex.

- Jiggle the interior vertex and the dimension of S_d^1 will drop by 1. (For S_d^2 it will drop by 1 or 3—but not 2.)

Degenerate Edges

An edge is degenerate at one of its end points if the two neighboring edges emanating from that point are parallel.

Obvious: An edge of a planar triangulation can be degenerate at at most one of its end points.

On a sphere-like surface an edge may be degenerate at both its endpoints, and neighboring vertices (connected by an edge) may both be singular.

Triangulations without Boundary

Obvious property of spherical triangulation: it may not have a boundary.

Consequences for spline spaces?

Trihedral Decompositions

Let $\{v_1, v_2, v_3\}$ be a set of linearly independent unit vectors in $\mathbb{R}^3$. We call

$$T = \{v \in \mathbb{R}^3 : v = b_1 v_1 + b_2 v_2 + b_3 v_3 \quad \textbf{with} \quad b_i \geq 0\}$$

the trihedron generated by $\{v_1, v_2, v_3\}$.

Let $\mathcal{T} = \{T^{[i]}\}_{i=1}^N$ be a non-empty set of trihedra, and let $\Omega := \cup T^{[i]}$. Then we call $\mathcal{T}$ a trihedral decomposition of Ω provided

1) **the interiors of the trihedra in $\mathcal{T}$ are pairwise disjoint,**
2) **the set $\Omega \cap S$ is homeomorphic to a two-dimensional disk or equals S, where S is the unit sphere,**
3) **each face of a trihedron in $\mathcal{T}$ is either on the boundary of Ω or it is a common face of precisely two trihedra in $\mathcal{T}$.**

$\mathcal{H}_d^r$ = space of functions in $C^r(\Omega)$ that on each trihedron reduce to a homogeneous polynomial of degree d.

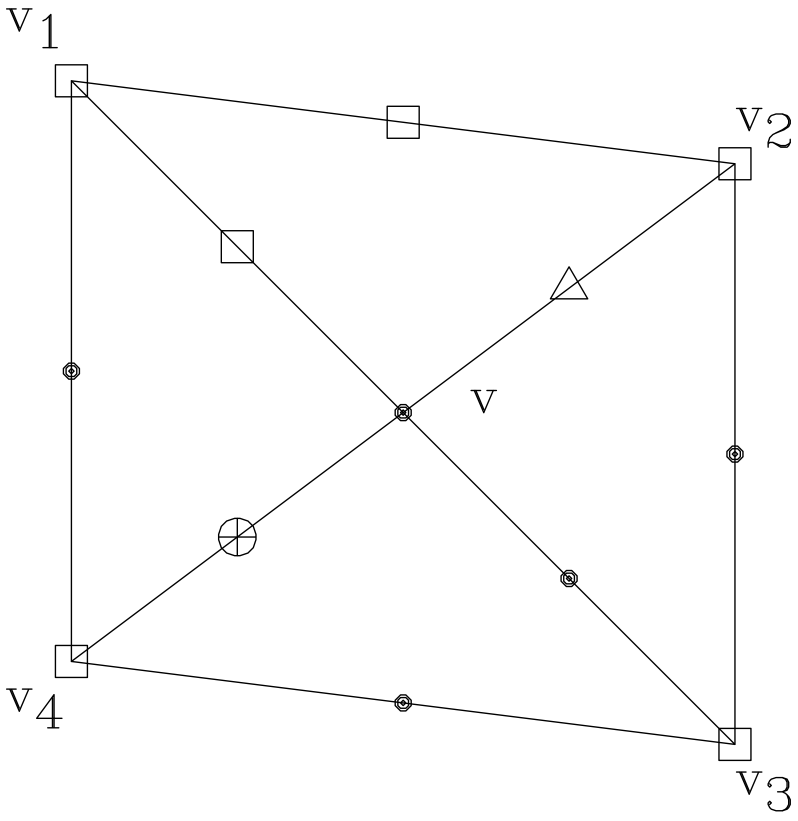
Degenerate Faces

Let F be an interior face of a trihedral decomposition $\mathcal{T}$, and let e be one of its two edges. We say that F is degenerate at e if the two faces sharing e and neighboring F are coplanar.

In other words: if we cut across e with a plane the intersection of f with the plane is degenerate at the intersection of e with the plane.

An interior edge e of a trihedral decomposition is *singular* if each of the faces sharing e is degenerate at e.

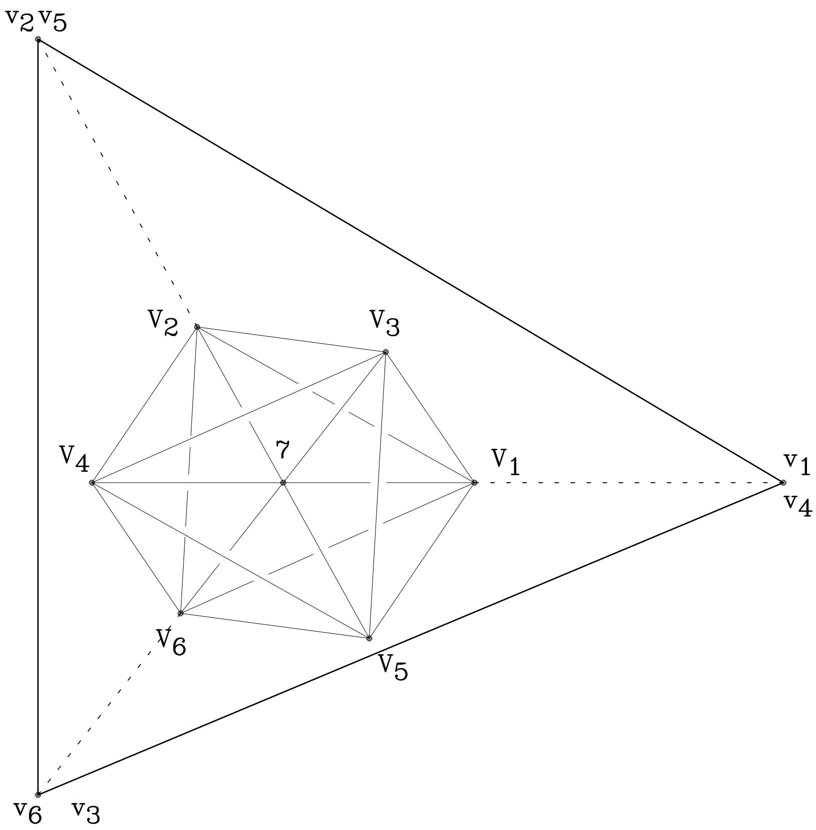
Minimal Determining Sets



Minimal Determining Set.

$v_3 = -v_1$ **and** $v_4 = -v_2$

The Regular Octahedron



Regular Octahedron.

More on the Octahedron

All interior edges are singular!

Generic dimension of dim $H^1_3 = 14$.

Dimension on regular octahedron: dim $H^1_3 = 19$.

This does not equal generic dimension plus number of singular edges.

(In the planar case we conjecture that the dimension of S^1_3 equals the generic dimension plus the number of singular vertices, i.e.,

$$\mathbf{dim} S^1_3 = 3V_B + 2V_I + 1 + \sigma$$

(famous open problem).

Geometric Interpretation.

Lacking so far:

- **control points**
- **Design Handles**
- **Geometric Interpretation of Smoothness Conditions.**

Summary

- **Drop Normalization .... spherical barycentric coordinates**
- **Homogeneous Trivariate Functions**
- **Restriction to Sphere-Like Surfaces**
- **Uniqueness of SBCs**
- **employ trivariate technology**
- **Most things are just like in the planar cases, except:**
- **Graphs look bizarre**
- **Nesting of Spaces**
- **Degree Raising**
- **Edges can be degenerate at both end points**
- **Neighboring vertices may both be singular**
- **Triangulations may have no boundaries**
- **Geometric Interpretation of Smoothness Conditions is lacking**