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Scattered Data Interpolation and Approximation on the Sphere and Similar Surfaces

Peter Alfeld, Univ. of Utah

This is a report on ongoing joint work with Larry Schumaker and Marian Neamtu. We developed a natural way to define Bernstein-Bézier polynomials on the sphere and on similar surfaces. This leads to a theory that largely parallels the familiar bivariate case. The talk will describe the theory and focus on the differences between the planar and the spherical case.

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A multistage methodology for scattered data approximation

Erlend Arge and Morten Dæhlen

Important aspects to consider when constructing methods for approximating scattered data are such things as the distribution and behaviour of the data, size of the given data set, the choice of the underlying function space and various requirement on quality of the final result, e.g shape preservation. In order to handle as many of these problems as possible we have constructed a multistage methodology which consists of three major steps:

1. A set of functions which describe the distribution and behaviour of the data are constructed. Typical examples are density functions based on (inverse) distance measures, local estimates on how well the given points span the plane and local estimates on the variation in the given data.
2. Based on the function values found in step 1. we perform local approximations to the data in certain regions, e.g. where the point density is higher than some prescribed threshold.
3. Finally, the approximation is constructed over the whole region of interest by applying some global smoothing technique.

An overview of the methodology will be presented with emphasis on important aspects and possibilities within the three steps. Moreover, we will consider global problems related to constraints and the choice of function spaces. Several examples from CAGD and cartography will be presented.

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Scattered Data Approximation with Control of Global Shape

Erlend Arge and Øyvind Hammer, SINTEF

When approximating scattered data, global shape is often controlled via minimization of roughness criteria. Most often such criteria reduces to require that some higher order derivatives of the surface should vanish, and as such, they do not incorporate more knowledge of global shape than what is inherent in the underlying scattered data set. In this talk we discuss how shape features can be incorporated in the roughness critera by means of certain weight functions. Several examples will be given where we demonstrate how these techniques can be used to model user defined features such as ridges and contour trends.

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Approximations Maintaining the Likely Behaviour of the Data

Mira Bozzini

Given a set of functional values on the interval (a, b) corrupted by noise, the problem is that of reconstructing a function such that its behaviour maintains the characteristics that the unknown function likely possesses.

To this purpose we solve a variational problem that gives, as solution, a cubic spline $S(x) \in C^2(a, b)$ satisfying the above-mentioned requirements.

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A C^1 scattered data interpolant with local support

Jeff Brown

Given a set of points (data sites) $\{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n\}$ scattered in the plane, real numbers $\{f_i\}$ and vectors (normal directions) $\{\mathbf{g_i}\}$, the problem is to construct a C^1 function $F(\mathbf{x})$ such that $F(\mathbf{p_i}) = \mathbf{f_i}$ and the normal diection to the graph of F at $\mathbf{p_i}$ is $\mathbf{g_i}$ for all i.

For each i, let $s_i(\mathbf{x})$ be the natural neighbor coordinate of $\mathbf{x}$ with respect to $\mathbf{p_i}$. These functions provide a crude interpolant

$$f_1 s_1(\mathbf{x}) + \ldots + \mathbf{f_n s_n}(\mathbf{x})$$

which ignores the gradient data $\mathbf{g_i}$ and is only C^0 at the data sites.

In this talk I will show how to alter the functions s_i to produce functions t_i which are C^1, still have local support, and such that

$$f_1 t_1(\mathbf{x}) + \ldots + \mathbf{f_n t_n}(\mathbf{x})$$

interpolates and has normal direction $\mathbf{g_i}$ at the data site $\mathbf{p_i}$.

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Radial Function Approximants to Scattered Data

M.D. Buhmann

We study radial basis function approximants to scattered data, in particular quasi-interpolation. The best known examples of radial basis functions in applications are multiquadrics and thin plate splines, and both of them are included in the general class of radial functions we consider. Our main concern is the existence of suitable generators for our approximants in *all* dimensions and their properties, especially the L^p approximation orders obtainable with them for $p \in [1, \infty]$.

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On a Class of Polynomial Triangular Macro-elements

Paolo Costantini and Carla Manni

One of the most popular tools for the interpolation of scattered data is given by the *Clough-Tocher cubic macro-elements*, which can be used to produce a continuously differentiable surface. It is well- known that the resulting method is local and has the advantage of a simple polynomial representation, at the price of a high dependence on the adopted triangulation.

The aim of this talk is to describe a generalization to macro- elements of degree n , which possess the same simple structure of the Clough-Tocher ones and, therefore, can be defined using the function's values and the gradients at the interpolation points. The main feature of this new class is that these macro-elements tend, as the degree n tends to infinity, to the plane interpolating the data at the vertices of the triangle. In other words, the degree plays the role of a tension parameter, and can be used to reproduce the shape induced by the triangulated data. We will discuss how to locally compute an *adaptive degrees interpolating surface* with local tension parameters.

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Radial Interpolation

Fabrice Derrien

We present some results about the spectrum of the collocation matrix arising in multivariate radial interpolation in reliance on the geometry and the cardinality of the node set, some properties of the basis function and the dimension of the Euclidean space wich contains the nodes. Applications of these results give necessary or sufficient conditions for a function to define a basis function, characterize the boundedness, the compactness and the condition number of the bi-infinite matrix operator in case of infinite nodes, permit to control the interpolant under data peturbation and contribute to the knot removal problem which is investigated by Pr. A. Le Méhauté.

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Smooth surface reconstruction from cross-sections

Michael Floater and Geir Westgaard

Given a sequence of parallel cross-sections, each containing contours, we discuss various ways of constructing a smooth triangulation approximating the data. The basic philosophy is to choose a suitable bounding region $\Omega \subset \mathbb{R}^3$, and build a smooth function $f : \Omega \to \mathbb{R}$, having the contours as zeros. The whole zero set of f is then a smooth surface (or manifold) approximation to the data. The zero set can be approximated by a triangulation by evaluating f at every node of a cubical lattice and using any of a variety of known techniques in computer graphics, such as the 'marching cubes' algorithm, for triangulating the isosurface.

In particular, if each slice is normal to the z axis, we can, making some reasonable assumptions, solve the biharmonic equation in (x, y) in each slice with appropriate boundary conditions and applying a cubic spline approximation in z to compute f. This yields a C^1 surface approximation without needing a great deal of computing time. In principle f and its zero set can be made to be C^k.

This implicit method has many advantages over traditional (interpolatory) triangulations for cross-sections [3]. The main one is that the notorious 'branching problem' is automatically solved in a natural way. Another is that we can smoothly interpolate between slices.

The idea of using an implicit method for cross-sectional data was presented recently by Jones and Chen [1]. However implicit methods for tackling scattered data problems can, at least, be dated back to Hoppe et al. [2]. The advantage of the f we present here is that it is smooth. The implicit function theorem guarantees that the zero set is similarly smooth away from critical points.

There are many potential applications of this method in, for example, tomography and reverse engineering.

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Interpolation by Stripping Boundaries of Convex Hulls

T.A. Foley, T. Kim and G. Weiss

Given scattered data points in a plane and associated function values, we construct an interpolating surface which is based entirely on shape preserving curve interpolation. The domain data is partioned into a sequence of nested convex polygons which are computed by repeatedly stripping off the boundary of the convex hull of a subset of the scattered data. Convex curves interpolate these polygons, monotone curves-on-curves are computed over the convex domain curves, and radial monotonicity preserving curves are used to define the interpolating surface.

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Least Squares Multiquadric Methods: Practical Approximations

Richard Franke

Taking measurements at scattered locations is a very common occurrence in most fields of science and engineering. Usually the data is subject to errors which may or may not have reasonably well-known statistics. Methods such as Kriging or smoothing splines can be used in such cases but may be be rather expensive for large data sets. We propose some methods using multiquadric functions in a least squares setting. In the simplest version, the location of the basis functions (referred to here as knots, in analogy with splines) are chosen adaptively. Basis functions are added sequentially until either (1) a given number of basis functions are used, or (2) the rate of increase of a smoothness measure of the surface relative to the rate of decrease of the residuals indicates that complexity is being added to the surface without a commensurate increase in accuracy. Another method treats the location of the knots as parameters in an optimization process. Some results of tests with known and unknown underlying functions, and some practical observations are given.

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Scattered Quasi-interpolation in $\mathbf{R}^2$

P. Gonzalez-Casanova

Based on a Hilbertian approach to quasi-interpolation methods, we recently constructed two new scalar and vectorial algorithms in $\mathbf{R}^2$. Here we study their numerical performance by using some test functions in order to evaluate their behavior for different distribution of scattered data.

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Industrial Surface Reconstruction

Thomas Grandine

Many exciting new methods for fitting and approximating scattered data have been developed over the last several years, but many of these methods produce functions incompatible with commercial CAD systems. Practical surface reconstruction in an industrial setting typically requires fitting scattered data with tensor product splines. This process not only requires fitting data to possibly noisy measurements, but discovering a viable underlying parametrization as well. This talk will describe efforts undertaken during the last several years to produce viable surface reconstruction tools in this setting.

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Bivariate Hermite Interpolation
over Triangular Grids

A. W. Habib

Hermite interpolation of bivariate data has received little attention as a separate research problem. The Hermite problem is usually addressed as a special case of Lagrange interpolation obtained by coalescing data [de Boor, Gasca]. In this paper we address the bivariate Hermite interpolation problem for a special arrangement of the interpolation points (nodes).

The nodes are assumed to be the intersection points of three sets of lines; each point lies on the intersection of three lines, one from each set. This configuration is sometimes referred to as a *geometric mesh* configuration [Gasca]. The data we want to interpolate at the nodes are values and partial derivatives up to some fixed order r.

We propose an interpolation algorithm based on dynamic programming that builds a polynomial interpolant for the data at the nodes. Our algorithm has a pyramidal structure. The final interpolant is built up recursively by constructing partial interpolants for the data over smaller and smaller triangular arrays, down to individual triangles with three data points, then blending these partial interpolants. We derive a set of necessary and sufficient conditions that must be satisfied by any blending functions for this structure and then give a formula for these functions. Interpolants over individual triangles are also computed recursively, using dynamic programming, from interpolants with fewer interpolation conditions until we bottom out at Taylor's theorem or the univariate version of Neville's algorithm.

The degree of the interpolant that solves a problem of order r, with n lines per direction is $(2r + 1) + (n - 2) * (3r + 1)$ and the algorithmic complexity of computing points on the surface using dynamic programming is $O(r^3 n^2 + n^3)$. The algorithm specializes to Neville's bivariate algorithm over geometric meshes when no derivative data is specified. The interpolant is also coordinate–system independent. Basis functions can be precomputed for fixed settings. Local changes to the data do not require recomputing the entire interpolant.

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A Trust Region Algorithm for Parametric Curve and Surface Fitting

H.-P. Helfrich, and D. Zwick

Let a family of curves or surfaces be given in parametric form via the model equation $x = f(s, \beta)$, where $x \in I\!R^n$, $\beta \in I\!R^m$, and $s \in I\!R^d$ for $d < n$. We present an algorithm for solving the problem:

> Find a shape vector β^* such that the contour $C^* = \{f(s, \beta^*)\}$ is a best fit to scattered data $\{x_i\}_{i=1}^N \subset I\!R^n$ in the sense that, for some $\{s_i^*\}_{i=1}^N$, the sum of the squared orthogonal distances $\sum_{i=1}^N \|x_i - f(s_i^*, \beta)\|^2$ from the data points to the contour C^* is minimal.

For robustness, our algorithm uses a globally convergent trust region approach in which, at each iteration, an approximation to the objective function is minimized in a given neighborhood of the current iterate.

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Basic Aspects of Stable Multilevel Decompositions for Scattered Data Fits

Kurt Jetter, Universität Duisburg, Germany

This talk deals with central aspects of scattered data interpolation with translates of a single basis function. One of the most popular basis function used in the past few years was the famous multiquadric. Its growth at infinity and its smoothness creates typical difficulties concerning the condition of the interpolation matrix. The intent of the lecture is to show some ways how to overcome these difficulties.

One of the central ideas is that the interpolation problem is closely related to an L_2-problem. In the latter case, we have access to notions such as Riesz bases of $L_2(\mathbb{R})$ etc., which are central to the stable numerical computation of wavelet expansions (as one important particular case of multilevel expansions). We are going to elaborate on these ideas and, in particular, will show that we are lead to multilevel expansions based on regularized spline functions. These "spline wavelet" decompositions for scattered data problems were very much advanced in recent cooperation with J. Stöckler from the University of Duisburg and with C.K. Chui and J.D. Ward from the Texas A&M University.

While the lecture will address the theoretical foundations as well as the practical computational aspects, the main viewpoint will focus on the presentation of stability theorems as they appear to be adequate for scattered data fits.

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Interpolatory Approximation Schemes in $S^r_{2r+1}(\mathbf{PS}(\triangle))$ for small r

Ming-Jun Lai

We are interested in constructing interpolatory approximation schemes in bivariate spline spaces S^r_{2r+1} over a Powell-Sabin's refinement $\mathrm{PS}(\triangle)$ of an arbitrary triangulation for small integer $r > 0$. The study of those constructions in $S^1_3(\mathrm{PS}(\triangle))$ is contained in [Lai'93] since $\mathrm{PS}(\triangle)$ is a FVS triangulation and the study of $S^2_5(\mathrm{PS}(\triangle))$ were carried out in [Lai'94]. In this talk, I will show how to construct an interpolatory approximation scheme in $S^3_7(\mathrm{PS}(\triangle))$ and $S^4_9(\mathrm{PS}(\triangle))$. The study of each of these bivariate spline spaces requires different techniques to disentangle the rings of the smoothness conditions around the vertices of the triangulation $\mathrm{PS}(\triangle)$. These techniques may not work for $S^r_{2r+1}(\mathrm{PS}(\triangle))$ for larger integer r.

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A finite element approach to C^k polynomial defined on a triangulation

A. Le Méhauté

Let $S(k, N, \mathcal{T})$ be the set of C^k-polynomial splines of degree N defined on a triangulation $\mathcal{T}$ of a bounded domain in $\mathbb{R}^2$ and involving as data derivatives up to order k at most. It is well known that if $N \geq 4k+1$ this set is non-empty and its dimension was given by Schumaker and Alfeld. Following Ženíšek, finite element's people are using another space of C^k-polynomial splines, say $Z(k, N, \mathcal{T})$, with a lower dimension, but involving derivatives up to order $2k$ at the vertices and what Chui called "super-splines".

In this presentation, we show that a carefull study of Morgan & Scott's result can provide a basis of $S(k, N, \mathcal{T})$ which is convenient to finite elements and does not involve all the derivatives of higher orders.

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Recovering a Function with Faults

L. Lenarduzzi

We deal with the problem of recovering a function f, defined on D squared, from exact data on a regular grid; the function is assumed to be smooth except along known edges, not necessarily straight, where the surface has a fault.

Functions like these often occur in geology and in seismology.

An interpolating patch is constructed for each subdomain D_k bounded with edges as follows. A set of pseudodata is obtained on D by reflecting the data from D_k on suitable pivots. The pseudodata are interpolated globally with an algorithm by Jetter and Stöckler [1]; finally the interpolating function is restricted to D_k.

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Variational Approach to Radial Basis Functions

Will Light

We will discuss the variational approach to radial basis functions. The theory of Duchon, which covers "polyharmonic splines" in even dimensions and powers of the norm in odd dimensions is now of some antiquity (20 years old!). The more recent theory mainly due to Madych and Nelson covers examples such as the multiquadric. There are some interesting 'conflicts' between these two theories. Our chief concern throughout is to expond results on errors in interpolation to scattered data.

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Hybrid Triangle Patches on the Sphere

X.-L. Liu and Larry L. Schumaker*

Using the natural set of barycentric coordinates associated with spherical triangles and the associated spherical Bernstein-Bézier polynomials introduced recently by Alfeld, Neamtu, and Schumaker, we discuss a new method for fitting scattered data on the sphere which is the analog of a planar method due to Foley and Opitz. It uses certain cubic elements, and involves splitting the center control point to create a rational element which provides a C^1 interpolant without having to split the initial triangles.

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Scattered Data Interpolation for Quadrilaterals and Distorted Tensor Product Grids

Suresh Lodha

We compare at least six simple easy-to-implement popular techniques for data interpolation that have been used in several applications including image processing, texture mapping, volumetric rendering of data and meteorological data reconstruction. The six interpolants are bilinear interpolant, rational interpolant, line interpolant, de Boor's interpolant, multiquadrics and triangular interpolants. Although all these interpolants degenerate to the well-known and well-accepted tensor product interpolation on a rectangular grid, we establish that each of these six techniques yield very different solutions even in the simple case of a quadrilateral. The behavior and properties of these interpolants are also quite different. We discuss the possibility of extending these simple techniques for data interpolation on quadrilaterals to scattered data interpolation on distorted tensor product grids and analyze their strengths and weaknesses. Applications to visualizing uncertain data will also be discussed.

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Bivariate Polynomial Interpolation for Scattered Data

José Javier Martinez

Although it is well known that polynomials of even moderate degree exhibit a considerable oscillatory character and the resulting surface is often too undulating, the approach of global polynomial interpolation for fitting surfaces to scattered data has been considered in the literature (see [11] for early references). On the other hand, some of the global schemes can be made local (with the necessary modifications) by subdividing the domain.

For one or another reason, the "classical" approach has not ceased to attract considerable attention in recent years (see, for example, [1,3,5,7,8,9,10].

However, it should be noticed that the approach used in [1,7,8,9,10], which is perhaps the most popular, is mainly based on the concept of *almost regularity*, and so is far from being constructive in nature.

Our approach is that of *Hack interpolation* (see [4,5,6]): for the Lagrange and Hermite cases, and for arbitrary (not necessarily gridded) data on the plane, it is shown how to construct an interpolating polynomial. For the Lagrange case, a very efficient algorithm is given, based on previous work for the univariate case (see [2]).

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Domain Decomposition and Spectral Decomposition for Parallel Data Fitting

L. Bacchelli Montefusco

Two different strategies for parallel data fitting have been considered: domain decomposition and spectral decomposition. The first one is classical: the physical domain is decomposed into slightly overlapping subdomains, each handled by a differnt processor; then a smooth, global approximation is obtained by suitably weighting the local approximations [1].

The second strategy is a new one: spectral decomposition has been achieved using a wavelet packet basis for the approximation space; then the parallelism is obtained by imposing this basis to be semiorthogonal. More precisely, we consider an extension to the wavelet packet case , of the general C.K. Chui's construction of semiorthogonal polynomial spline wavelets[2]. Using this basis the approximation matrix presents a block diagonal form which is a good starting point for a parallel algorithm. The solutions corresponding to the different blocks are still global in the physical domain, but they are local in the frequency domain as they belong to different frequency bands. The final approximation, which is global in both frequency and physical spaces, is easily obtained by means of the wavelet packet reconstruction algorithm.

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Spherical Simplex Splines

M. Neamtu

Recently, together with P. Alfeld and L. Schumaker, we introduced a natural analog of spaces of piecewise polynomials on the sphere. In this talk we will discuss the definition and properties of spherical simplex splines. Spaces of these splines exhibit most of the important properties of the classical simplex splines in a Euclidean space. The main tools used in the construction will be the cone splines and spherical divided differences.

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Vertical Fault Detection from Scattered Data

M. C. Parra, M. C. López de Silanes, J. J. Torrens

In Geology or Geophysics, the reconstruction of a faulted surface from a set of scattered data points is a common problem. Usually, such a surface admits an explicit representation of the form $z = f(x, y)$, where f is a function defined on a bounded open set Ω that corresponds to the terrestrial region of interest and that is singular on a subset F of Ω (i.e. f or some of its derivatives are discontinuous on F).

There exist different methods for solving the above problem. To apply them, however, it is necessary to previously determine the position of the fault lines, that is, the set F must be known. Hence the interest in disposing of a *fault detection algorithm* that provides an approximation of F using the same set of data points that later will be employed for the surface fitting process.

In this work, we focus on a method for detecting vertical faults, modelized as jump discontinuities of the function f on the set F.

Essentially, the proposed method is as follows:

[1] For all $n \in \mathbb{N}$, construct a triangulation T_n of $\overline{\Omega}$ by means of equal rectangles of area $\text{meas}(\Omega)/4^n$.

[2] Consider all sequences of rectangles $(K_n)_{n\in\mathbb{N}}$ such that $K_n \in T_n$ and $K_{n+1} \subset K_n$.

[3] For any sequence $(K_n)_{n\in\mathbb{N}}$, consider the sequence $\big(J_{K_n}(f)\big)_{n\in\mathbb{N}}$, where

$$J_{K_n}(f) = \frac{1}{\text{meas}(K_n)} \int_{K_n} \|\nabla(\Pi_{K_n} f)(x)\|^2 \, dx \,,$$

$\Pi_{K_n} f$ being a polynomial least squares approximant of f.

[4] Analyze the convergence of the sequences $\big(J_{K_n}(f)\big)_{n\in\mathbb{N}}$.

[5] The fault lines result from all the intersections of sequences of nested rectangles $(K_n)_{n\in\mathbb{N}}$ whose associated sequences $\big(J_{K_n}(f)\big)_{n\in\mathbb{N}}$ are divergent.

In practice, the fault lines are approached by unions of rectangles, since only the first terms of every sequence $\big(J_{K_n}(f)\big)_{n\in\mathbb{N}}$ can be actually computed.

We give a theoretical justification of this method and we show several examples of its performance.

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Radially symmetric Thin Plate Splines

Christophe Rabut

This work intends to give a good approximation tool for scattered data problems with radial symmetry.

The origin of the work is for fitting points from the pools used for water treatment (most of them are circular). Measurements of various quantities are scattered on the surface of the basin, and all the quantities are supposed to be radially symmetrical. So, we want to get a bivariate, radially symmetric function which interpolates (or approximates) scattered data.

To do so, we of course could determine a univariate polynomial spline, and extend it by using the radial symmetry. But this is not good, as we can easily see by noticing that the so-obtained function usually is not $\mathcal{C}^1$ in 0.

We define the interpolating function as the one minimising

$$\int_\Omega \left(\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2 \right) \, \mathrm{d}x \; \mathrm{d}y$$

over all radially symmetric functions interpolating the data (Ω is the domain such that $R_1^2 \leq x^2 + y^2 \leq R_2^2$, for some given R_1 and R_2). Let $f(x,y) = \varphi(\sqrt{x^2+y^2})$, then

$$\int_\Omega \left(\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2 \right) \mathrm{d}x\, \mathrm{d}y = \int_{R_1}^{R_2} \left(\frac{(\varphi'(r))^2}{r} + r(\varphi''(r))^2 \right) dr.$$

We so have to determine the univariate function $\varphi \in D^{-2}L^2(\Omega)$ that minimises $|\varphi| = \int_{R_1}^{R_2} \left(\frac{(\varphi'(r))^2}{r} + r(\varphi''(r))^2 \right) \mathrm{d}r$ over all functions that interpolate the data. We then go back to the two variables problem by using radial symmetry.

An analog problem can be stated for fitting data; the general form of the resulting function is the same one.

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Localization of Surface Spline Interpolants

K. Salkauskas

By combining the moving least squares interpolation method using a polynomial basis with the smoothing spline technique using a penalized energy, it is possible to generate a local interpolation method which is locally, though not globally, a thin-plate spline. While such interpolants can be C^1, excellent results can be obtained from even discontinuous interpolants when they are subject to rendering processes in visualization. These discontinuous interpolants are a limiting form of the smooth ones, and are splines on the tiles of a data-dependent tiling of the plane. The tiling depends on the support of the weight function in the moving least squares part of the energy.

In the limiting case, the value of the interpolant at any point is a piecewise constant function of the diameter of the support of the weight, and averaging of interpolants corresponding to different diameters leads to smoother interpolants.

As the diameter of the support increases, instability of the calculation is expected because of the poor conditioning of the associated linear system with matrix V, say. This is studied by examining the behaviour of $||(V - \lambda I)^{-1}||$ as a real bivariate function of the real and imaginary parts of λ.

Some of these results are joint work with A. Le Méhauté.

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Tools for Constructing Positive Definite

Radial Basis Functions

Robert Schaback, Göttingen

Any univariate real-valued function ϕ on $\mathbb{R}_{\geq 0}$ can be interpreted as a multivariate **radial** function

$$\phi_d(\cdot) = \phi(\|\cdot\|_{2,\mathbb{R}^d})$$

on R^d. Such functions have proven to be highly useful for scattered data analysis, if they are positive definite. This contribution focuses on tools for the construction of new positive definite radial functions. These tools will consist of purely univariate operators that can be effectively applied to generate explicit representations for both the radial functions and their Fourier transforms. A new characterization for positive definiteness of radial functions follows as a byproduct, and this characterization involves only univariate ingredients.

Several new cases of positive definite radial basis functions will also be presented.

This is joint work with Wu Zong-min, Fudan University, Shanghai.

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Range Restricted Interpolation of Scattered Data Using Powell-Sabin Splines

Jochen W. Schmidt, Technical University of Dresden, Germany

Let be given a set $P_0, \ldots, P_n$ of scattered data sites, and corresponding data values $z_0, \ldots, z_n$. For range restricted interpolation let Δ be an admissible triangulation of *conv* $\{ P_0, \ldots, P_n \}$. On the triangles $\Delta_{ijk} \in \Delta$ with the vertices P_i, P_j, P_k lower and upper bounds $L_{ijk} \leq U_{ijk}$ are prescribed which are compatible with the data values z_i, z_j, z_k, i.e., $L_{ijk} \leq z_i, z_j, z_k \leq U_{ijk}$. The problem is to construct an interpolating range restricted C^1 function s, i.e., we require

$$(i\ )\quad s\,(P_i) = z_i \quad \text{for} \quad i = 0, \ldots, n,$$

$$(ii)\quad L_{ijk} \leq s\,(P) \leq U_{ijk} \quad \text{for} \quad P \in \Delta_{ijk},\ \ \Delta_{ijk} \in \Delta.$$

In this way, for instance unwanted oscillations can be avoided. It can be shown that such interpolation problems are always solvable when using quadratic C^1 splines on Powell-Sabin refinements of the original triangulation Δ. This smooth result is no longer valid if the staircase-like bounds (ii) are substituted by continuous piecewise linear ones. In addition, it seems to be an open problem how to proceed if an interpolating range restricted function of C^2 continuity is required.

In the C^1 case the number of splines solutions from above is infinite. In our numerical experiments we have selected suitable interpolants by a so-called fit and modify approach.

The lecture is mainly based on a paper joint with B. Mulansky. This paper will appear in the journal Computing.

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Generalization of Natural Neighbor Interpolant on Scattered Data Fitting

Leonardo Traversoni

Universidad Autónoma Metropolitana (Iztapalapa)

On Natural neighbor interpolation weights of neighbors are taken one at a time and the interpolant is an expression of the type: $\sum_{i=1}^{n} W_i V_i$. In this paper is explored the use of expressions of the type $\sum_{i,j}^{n} W_{i,j} V_i V_j$ and its possible advantages and disadvantages in some applications. It is also discussed the computational implementation of an algorithm performing the new expression.

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Generalized Hermite Interpolation of Scattered Data in R^s and Other Manifolds

Joe Ward, Texas A&M University, College Station, TX

In this talk, we will discuss some recent progress in multivariate interpolation using radial basis or related functions. Our particular emphasis will center on a broad class of interpolation problems, for both scalar and vector-valued multivariate functions subject to linear side-conditions, such as being divergence free, where the data are generated via integration against compactly supported distributions. In particular, we will focus on the invertibility and stability of the associated generalized interpolation matrices. In addition to R^s, our results extend to certain other manifolds including the m-dimensional torus and the spheres S^n.

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Fitting a Nonnegative or a Monotone Surface to a Set of Scattered Data

Karin Willemans and Paul Dierckx

We consider the problem of fitting a surface to a set of noisy measurement data, corresponding to points arbitrarily scattered in a bounded area. In many applications it is important that the fitted surface preserves certain geometric properties of the surface underlying the data. For, in some cases the shape preservation is essential for the fitted surface to be physically meaningful. In others, the use of a constrained surface fitting method can avoid improper bumps and wholes which might be present in a surface fitted by an unconstrained method.

In earlier work [1] we considered the fitting of a convex surface or a surface with boundary conditions. Here we will present algorithms for smoothing with a nonnegativity or a monotonicity preserving surface.

As for the approximation functions, Powell-Sabin splines on triangulations are used. These piecewise quadratics with C^1-continuity are expressed as a linear combination of locally supported basis functions (B-splines), which are fairly easy to construct. The necessary and sufficient nonnegativity and monotonicity conditions are then formulated in terms of the B-spline coefficients. Due to the local support of the B-splines these constraints are sparse, which can be exploited to efficiently solve the problem. A refinement procedure for the triangulation that automatically takes account of the difficulties in the function underlying the data can be built in.

The presented nonnegativity preserving algorithm can easily be adapted so that one can specify for each triangle of the triangulation whether the approximation should be either nonnegative, nonpositive or even unconstrained on the triangle at issue; the monotonicity preserving algorithm can be adapted in a similar way.

The methods will be illustrated with several practical examples.

References

1. Willemans, K., and P. Dierckx, Constrained Surface Fitting Using Powell-Sabin Splines, in *Wavelets, Images and Surface Fitting*, P. J. Laurent, A. Le Méhauté, L. L. Schumaker (eds.), A. K. Peters, Wellesley, Massachusetts, 1994, 511–520.