Effects of Non-Generic Triangulations on Bivariate Spline Interpolants

Vanderbilt University, May 1, 1997

Peter Alfeld

Department of Mathematics

University of Utah

Salt Lake City, UT 84112

801-581-6842 or 801-581-6851

alfeld@math.utah.edu

URL: http://www.math.utah.edu/~alfeld/

Outline of Talk

- **Bivariate Splines**
- **Triangulations**
- **Spline Spaces on Triangulations**
- **The Bernstein-Bézier Form**
- **A lower bound on the dimension of S_1^3**
- **Singular Vertices**
- **Interpolating with S_3^1**
- **The Commutation Problem**
- **Examples**
- **Conclusions**

Bivariate Splines

"bivariate" = two independent variables.

- Two meanings of "spline":
  - Minimize a variational principle
  - smooth piecewise polynomial function

Used for many purposes:

- Interpolation of data and functions.
- Approximation of data and functions.
- Data Representation.
- Curve Design.
- Surface Design
- Numerical Differentiation.
- Numerical Integration.
- Numerical ODEs, PDEs.

Triangulations

Let

$$\mathcal{T} = \{\tau_i : i = 1, 2, \cdots, N\} \tag{1}$$

be a finite set of non-degenerate triangles in $\mathbb{R}^2$ **and let**

$$\Omega = \bigcup_{i=1}^{T} \tau_i. \tag{2}$$

Let

$$\mathcal{X} = \{x_i \in \Omega, i = 1, 2, \cdots, V\} \tag{3}$$

be the set of all vertices of triangles in $\mathcal{T}$**. Then** $\mathcal{T}$ **is a** *triangulation* **of** Ω **provided that**

1. **The interiors of the triangles in** $\mathcal{T}$ **are pairwise disjoint.**
2. **Each edge of a triangle in** $\mathcal{T}$ **is either on the boundary of** Ω **or else is a common edge of exactly two triangles in** $\mathcal{T}$**.**
3. **Each triangle in** $\mathcal{T}$ **contains no points of** $\mathcal{X}$ **other than its vertices.**
4. Ω **is homeomorphic to the unit square.**

Combinatorics of Triangulations

Let

$$\begin{aligned} V_B &= \textbf{number of boundary vertices in } \mathcal{T} \\ V_I &= \textbf{number of interior vertices in } \mathcal{T} \\ E_B &= \textbf{number of boundary edges in } \mathcal{T} \\ E_I &= \textbf{number of interior edges in } \mathcal{T} \\ N &= \textbf{number of triangles in } \mathcal{T} \end{aligned}$$

Then

$$\begin{aligned} E_B &= V_B \\ E_I &= V_B + 3V_I - 3 \\ N &= V_B + 2V_I - 2 \end{aligned}$$

Spline Spaces on Triangulations

$$S_d^r(\mathcal{T}) = \{s \in C^r(\Omega) : s|_\tau \in \mathbf{P}_d, \quad \forall \tau \in \mathcal{T}\}$$

where $\mathbf{P}_d$ is the $\binom{d+2}{2}$-dimensional space of bivariate polynomials of degree d.

Notes:

- **Triangle base finite elements are a special case**
- **Local Bases are known if $d \geq 3r + 2$**
- **Dimension depends on the geometry of the triangulation**
- **Focus on $d = 3, r = 1$.**

The Bernstein Bézier Form

v_1, v_2, v_3 **vertices of a general triangle.**

Define *barycentric coordinates***:**

$$x = \sum_{i=1}^{3} b_i v_i \quad \textbf{where} \quad \sum_{i=1}^{3} b_i = 1.$$

Any polynomial $p \in \mathbf{P}_d$ **can be written uniquely in its** *Bernstein-Bézier form* **as:**

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k.$$

where

c_{ijk} : *Bézier ordinates*

$P_{ijk} = \dfrac{iv_1 + jv_2 + kv_3}{d} \in \mathbb{R}^2$: *Domain Points*

$(P_{ijk}, c_{ijk}) \in \mathbb{R}^3$: *Bézier control points*

Set of Bézier control points = *Bézier control net, or B-net***.**

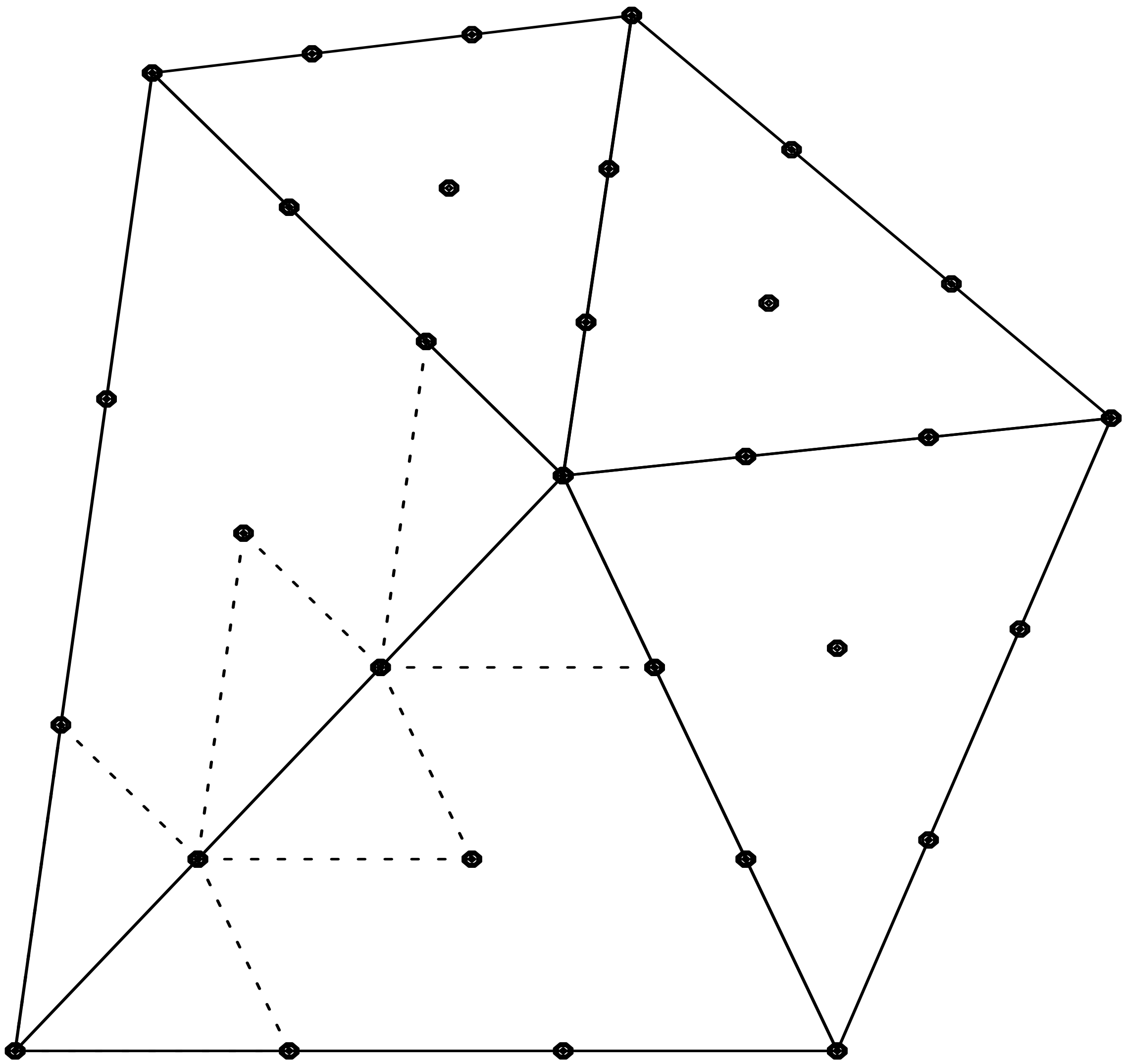

Figure 1. The B-net.

A lower bound on the dimension of S_3^1

Define the space S to be the subspace of S_d^0 that has continuous gradients at the vertices.

$$\mathbf{dim} S = 3\,(V_B + V_I) + N$$

To obtain differentiability across interior edges it is sufficient to enforce planarity of the central quadrilateral of each interior edge. Thus:

$$\begin{aligned}\mathbf{dim} S_3^1 &\geq \mathbf{dim} S - E_I \\ &= 3\,(V_B + V_I) + (V_B + 2V_I - 2) - (V_B + 3V_I - 3) \\ &= 3V_B + 2V_I + 1\end{aligned}$$

But: Are the central quadrilateral conditions linearly independent?

It turns out that sometimes they are not!

A singular vertex

Consequence: The dimension of a spline space depends on the geometry as well as the combinatorics of a triangulation.

Modify lower bound:

$$\mathbf{dim} S_3^1 \geq 3V_B + 2V_I + 1 + \sigma$$

where σ is the number of singular vertices.

Note: we have enough parameters to interpolate to function values at the vertices of the triangulation.

$$s(v) = f(v) \forall v \in \mathcal{T},$$

for some function f (or other data).
Very attractive! (S_2^1 is too small.)

Open Problems:

- **Does**
$$\mathbf{dim} S_3^1 = 3V_B + 2V_I + 1 + \sigma?$$
- **Are the interpolation and smoothness conditions always consistent.**

Interpolating with S_3^1

Suppose we can always interpolate. (Could choose a larger polynomial degree).

Which of the many interpolants do we pick?

Most natural: minimize a suitable variational principle subject to interpolation.

Example (clamped elastic plate functional)
Find $s \in S_3^1(\mathcal{T})$ **such that**

$$\iint_\Omega \left(s_{xx}^2 + 2s_{xy}^2 + s_{yy}^2\right) \mathbf{d}x\mathbf{d}y = \min$$

subject to

$$s(v) = f(v) \qquad \forall v \in \mathcal{T}.$$

The Commutation Problem

Using Lagrange Multipliers this gives rise to the linear system:

$$\begin{bmatrix} A & C^T & D^T \\ C & 0 & 0 \\ D & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

where

$$\int\!\!\int_\Omega \left(s_{xx}^2 + 2s_{xy}^2 + s_{yy}^2\right) \mathbf{d}x\mathbf{d}y = \frac{1}{2}c^T A c$$

- A **is positive semi-definite and has rank-deficiency 3 (since the clamped elastic plate functional is zero for linear functions).**
- c **is the vector of coefficients of** $s \in S$
- λ **and** μ **are Lagrange multipliers (for interpolation and smoothness conditions, respectively).**
- $Cc = b$ **are the interpolation conditions**
- $Dc = 0$ **are the (central quadrilateral) smoothness conditions.**
- D **is rank deficient when the dimension of** S_1^3 **does not equal the generic value** $(3V_B + 2V_I + 1)$**.**

The coefficient matrix depends on the triangulation.

Consider a situation where the triangulation approaches degeneracy (i.e., rank deficiency of D).

Questions

- **Does the interpolant change discontinuously at degeneracy. (Sometimes)**
- **In other words, does the diagram**

$$\begin{array}{ccc} S_3^1\big(\Delta(\mathcal{T})\big) & \to & S_3^1\big(\Delta(\mathcal{T}_0)\big) \\ \downarrow & & \downarrow \\ s_{\mathcal{T}} & \to & s_{\mathcal{T}_0} \end{array} \tag{4}$$

  commute? (Not always)
- **If it does not commute for the clamped elastic plate functional, are there other (reasonable) functionals for which the diagram always commutes? (Don't know)**
- **How do we handle degeneracy numerically. (Only know of ad hoc procedures.)**

Note special structure (most of the system is homogeneous).

Homogeneity due to nature of variational principle.

Buckling

Example 1.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & \varepsilon \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\varepsilon \\ 1 & 1 & 1 & 0 & 0 \\ \varepsilon & 0 & -\varepsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution (for $\varepsilon \neq 0$):

$$c_1 = \frac{2}{5}, \quad c_2 = \frac{1}{5}, \quad c_3 = \frac{2}{5}, \quad \lambda = -\frac{1}{5}, \quad \mu = \frac{1}{5\varepsilon}$$

Solution (for $\varepsilon = 0$):

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 1, \quad \lambda = 0, \quad \mu = \textbf{arbitrary}$$

The diagram does not commute. Buckling occurs.

Example 2.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & \varepsilon \\ 0 & 1 & 0 & 1 & -\varepsilon \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ \varepsilon & -\varepsilon & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution (for all ε):

$$c_1 = c_2 = \lambda = 0, \quad c_3 = 1,$$

and $\mu = 0$ is $\varepsilon \neq 0$ and arbitrary otherwise.

So in this case the diagram does commute!

- **Remaining Examples: $\mathcal{T}$ consists of four triangles with one interior vertex that approaches singularity. Results are computed numerically.**
- **Let the vertex approach singularity in a certain direction and monitor the maximum difference between the interpolant at singularity and the generic interpolant.**
- $\xi = 2^{-n}, \quad \eta = a\xi, \quad n = 1, 2, \ldots$
- **Commutation if the minimum discrepancy is (roughly) independent of the a (the direction), and it keeps changing until numerical singularity is reached.**
- **Colored pictures on a web site:**

URL: http://www.math.utah.edu/˜alfeld/commutation.html

- **Notation like 00011 means**

$$f(v_1) = f(v_2) = f(v_3) = 0 \quad \textbf{and} \quad f(v_4) = f(v_5) = 0,$$

where v_5 is the interior vertex.

Example 3.

Find $s \in S_3^1(\mathcal{T})$ such that

$$\iint_\Omega \left(s_{xx}^2 + 2s_{xy}^2 + s_{yy}^2\right) \mathbf{d}x\mathbf{d}y = \min$$

a	n	10000	11000	00001	00011
1000	24	$-1.7e-5$	$-3.4e-5$	$\pm 2.9e-5$	$3.0e-5$
2	16	$-1.3e-5$	$-1.8e-5$	$\pm 1.5e-5$	$1.3e-5$
1	15	$-1.7e-5$	$-2.0e-5$	$\pm 1.6e-5$	$1.2e-5$
$1/2$	15	$-1.3e-5$	$-1.2e-5$	$\pm 1.5e-5$	$-1.3e-5$
0	14	$-1.8e-5$	$\pm 1.5e-5$	$\pm 3.1e-5$	$-3.1e-5$
$-1/2$	15	$-6.0e-6$	$1.2e-5$	$\pm 1.5e-5$	$-1.8e-5$
-1	15	$\pm 3.9e-6$	$2.0e-5$	$\pm 1.6e-5$	$-2.3e-5$
-2	16	$6.0e-6$	$1.8e-5$	$\pm 1.5e-5$	$-1.8e-5$
-1000	24	$1.7e-5$	$3.5e-5$	$\pm 3.0e-5$	$-3.0e-5$

n**: largest value for which the matrix is numerically non-singular.**

The solution at singularity is a pure quadratic!

Example 4.

Singular vertices do not always occur by themselves!

Find $s \in S_3^1(\mathcal{T})$ such that

$$\int\int_\Omega \left(s_{xx}^2 + 2s_{xy}^2 + s_{yy}^2\right) \mathbf{d}x\mathbf{d}y = \min$$

subject to the additional condition

$$\frac{\partial s}{\partial y}(P) = 0.$$

where P is the midpoint of the edge joining v_1 and v_2.

a	n	10000	11000	00001	00011
1000	24	$-2.4e-5$	$-4.8e-5$	$1.8e-5$	$4.1e-5$
2	16	$3.8e-3$	$-7.6e-3$	$1.3e-5$	$7.6e-3$
1	15	$-6.4e-3$	$-1.2e-2$	$1.7e-5$	$1.2e-2$
1/2	15	$-7.9e-3$	$-1.6e-2$	$1.2e-5$	$1.6e-2$
0	14	$9.3e-3$	$1.8e-2$	$1.8e-5$	$-1.8e-2$
$-1/2$	15	$-7.8e-3$	$-1.5e-2$	$5.9e-6$	$1.6e-2$
-1	15	$-63e-3$	$-1.2e-2$	$4.1e-6$	$1.2e-2$
-2	16	$-3.8e-3$	$5.5e-3$	$-6.2e-6$	$-5.5e-3$
-1000	24	$2.4e-5$	$4.8e-5$	$-1.8e-5$	$3.6e-5$

Example 5.

Find $s \in S_3^1(\mathcal{T})$ such that

$$\iint_\Omega \left(s_x^2 + s_y^2\right) \mathbf{d}x\mathbf{d}y = \min$$

a	n	10000	11000	00001	00011
1000	27	$\pm 2.5e-2$	$\pm 5.1e-5$	$\pm 4.3e-5$	$\pm 2.4e-2$
2	18	$\pm 3.6e-2$	$\pm 2.4e-2$	$\pm 6.6e-6$	$\pm 1.2e-2$
1	17	$\pm 3.6e-2$	$\pm 3.6e-2$	$\pm 8.8e-6$	$\pm 6.8e-6$
$1/2$	17	$\pm 3.6e-2$	$\pm 4.8e-2$	$\pm 6.6e-6$	$\pm 1.2e-2$
0	17	$\pm 2.5e-2$	$\pm 4.9e-2$	$\pm 5.5e-6$	$\pm 2.5e-2$
$-1/2$	17	$\pm 1.2e-2$	$\pm 4.8e-2$	$\pm 6.6e-6$	$\pm 3.6e-2$
-1	17	$\pm 2e-6$	$\pm 3.6e-2$	$\pm 8.8e-6$	$\pm 3.6e-2$
-2	18	$\pm 1.2e-2$	$\pm 2.4e-2$	$\pm 6.6e-6$	$\pm 3.6e-2$
-1000	24	$\pm 2.5e-2$	$7.4e-5$	$\pm 4.3e-5$	$\pm 2.4e-2$

Conclusions

- **Non-commutation means that the limit depends on the precise manner in which singularity is obtained, and the span of all possible limits has dimension greater than 1.**
- **Conceivably we could come up with a scheme for which the diagram always commutes.**
- **We don't know what degeneracies there are for S_3^1.**
- **It is not clear how to handle the numerical complications.**
- **Perhaps the most prudent way out is to avoid degeneracies by using appropriate subspaces of S_d^r (finite elements).**