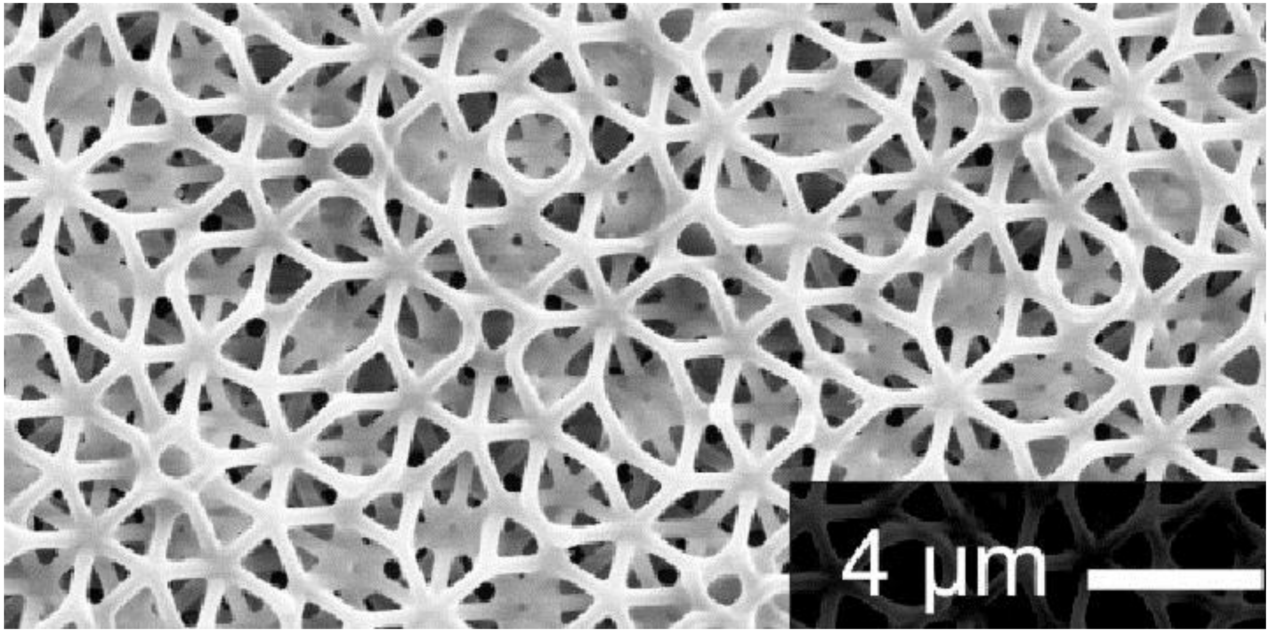
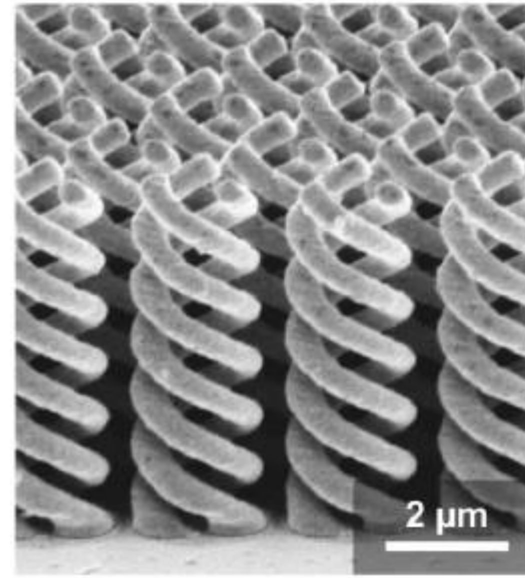
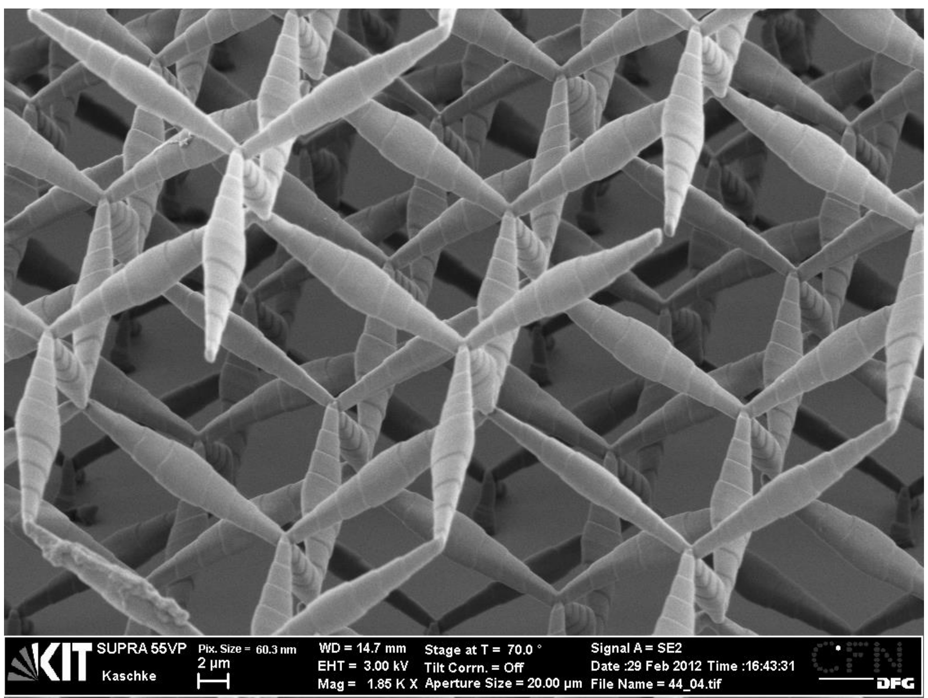
On the elasticity tensors of 2d and 3d printed materials

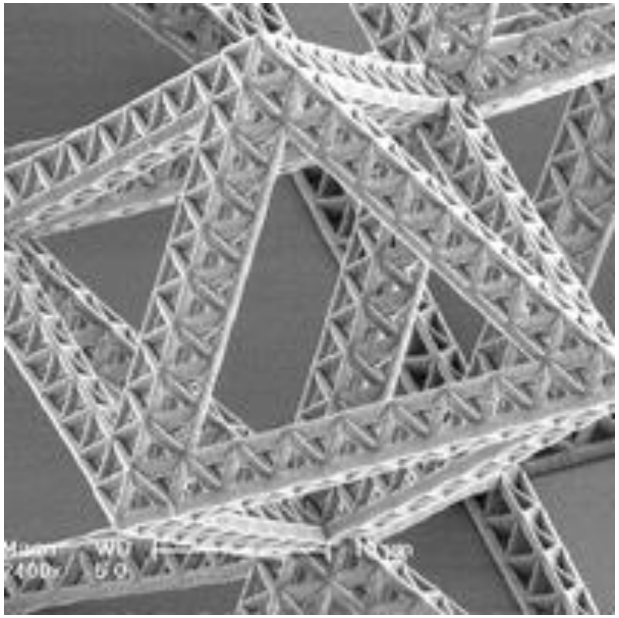
Graeme W. Milton,
Marc Briane, and
Davit Harutyunyan

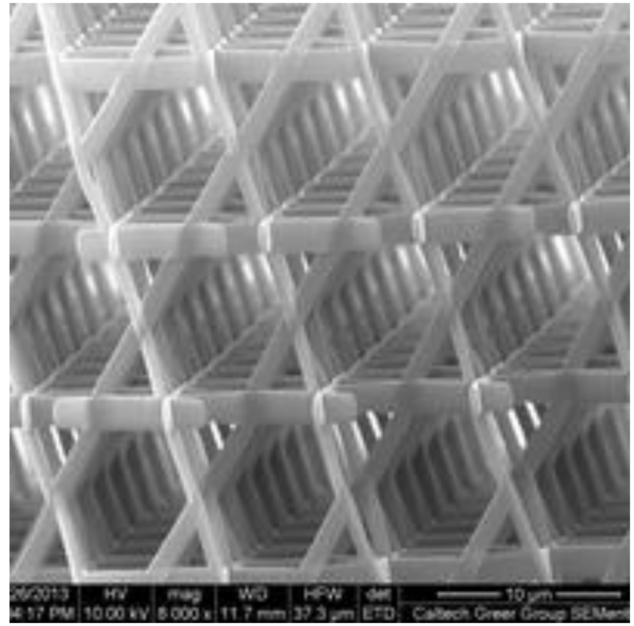


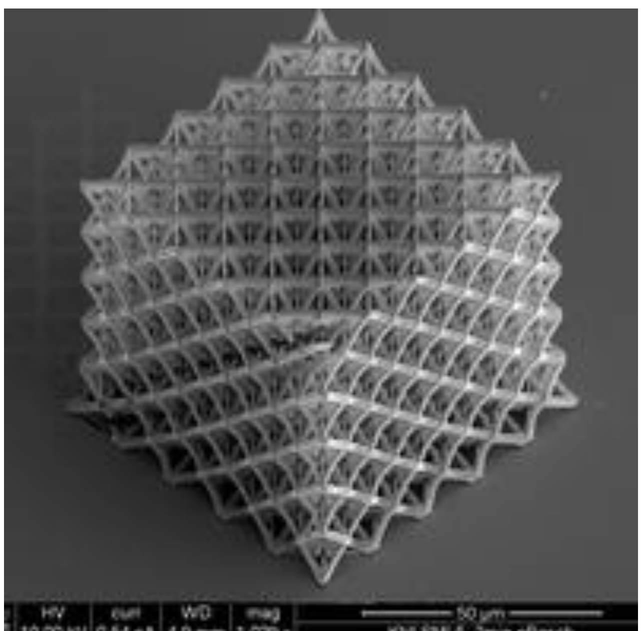




Group of
Martin Wegener







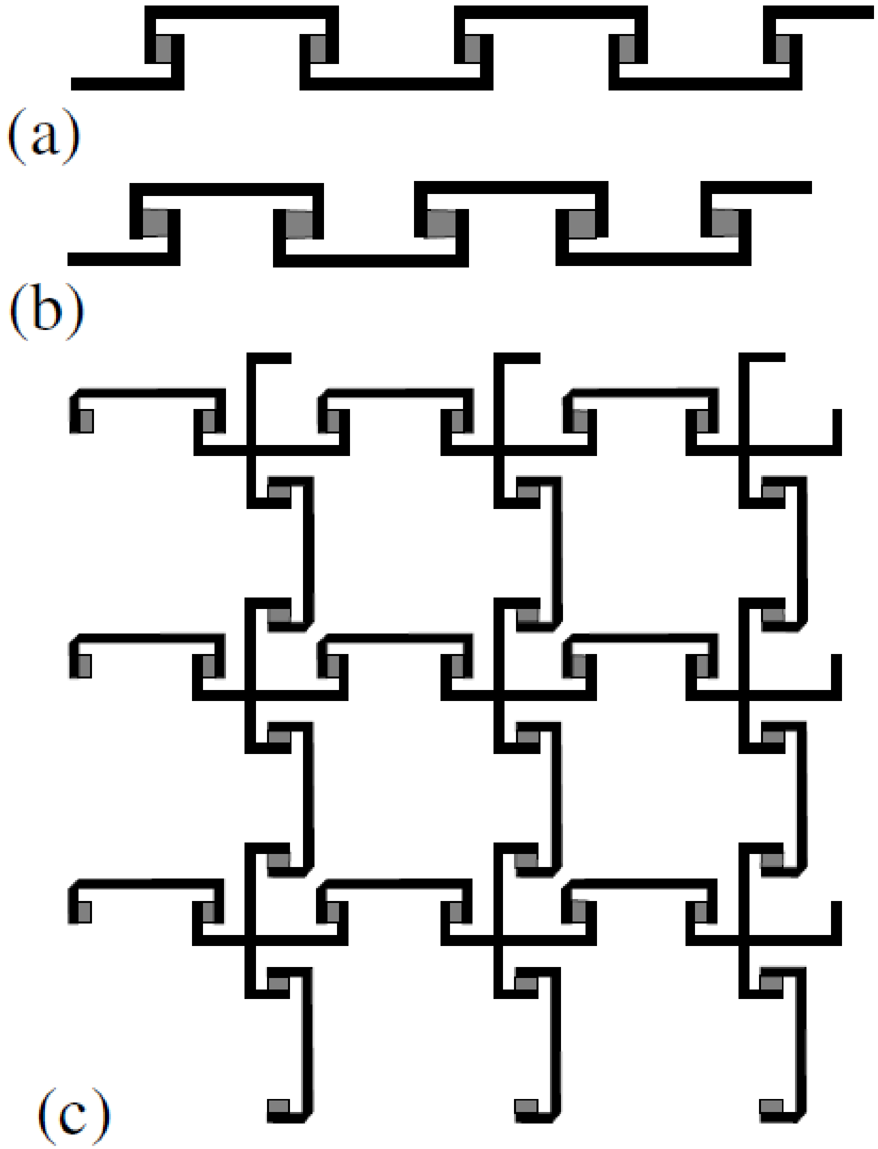
Group of Julia Greer

A fundamental question:

What elasticity tensors can be realized in 3-d printed materials, given the volume fraction and given the elastic constants of the constituent material?

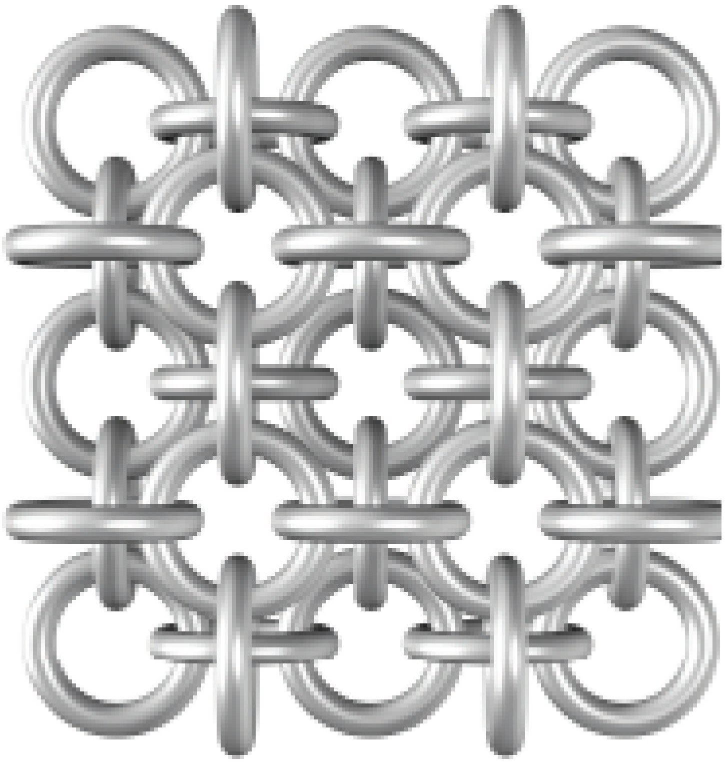
The set of possible elasticity tensors is known as the G-Closure GU_f

Another example: negative expansion from positive expansion



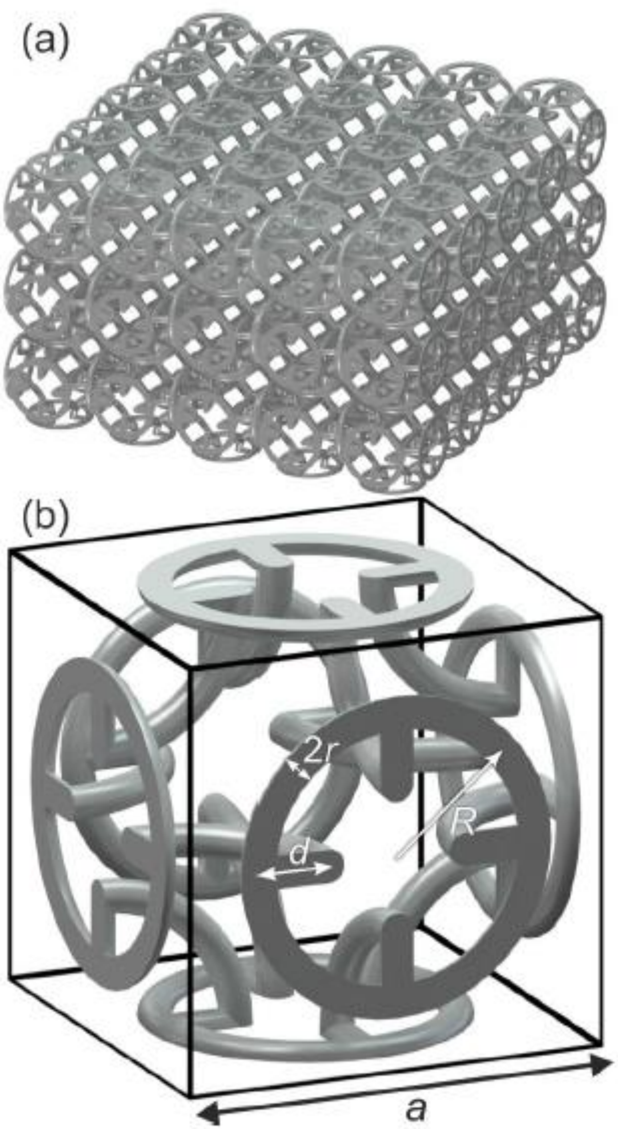
Original designs: Lakes (1996); Sigmund & Torquato (1996, 1997)

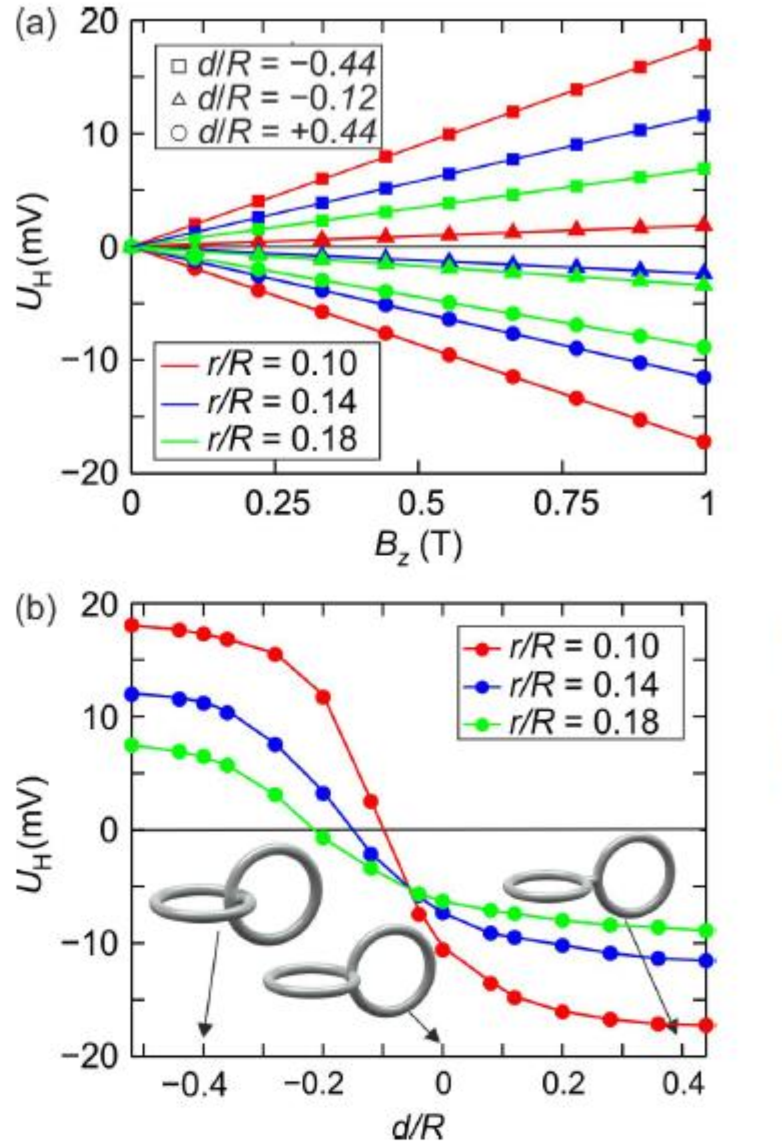
Geometry suggested by artist Dylon Whyte

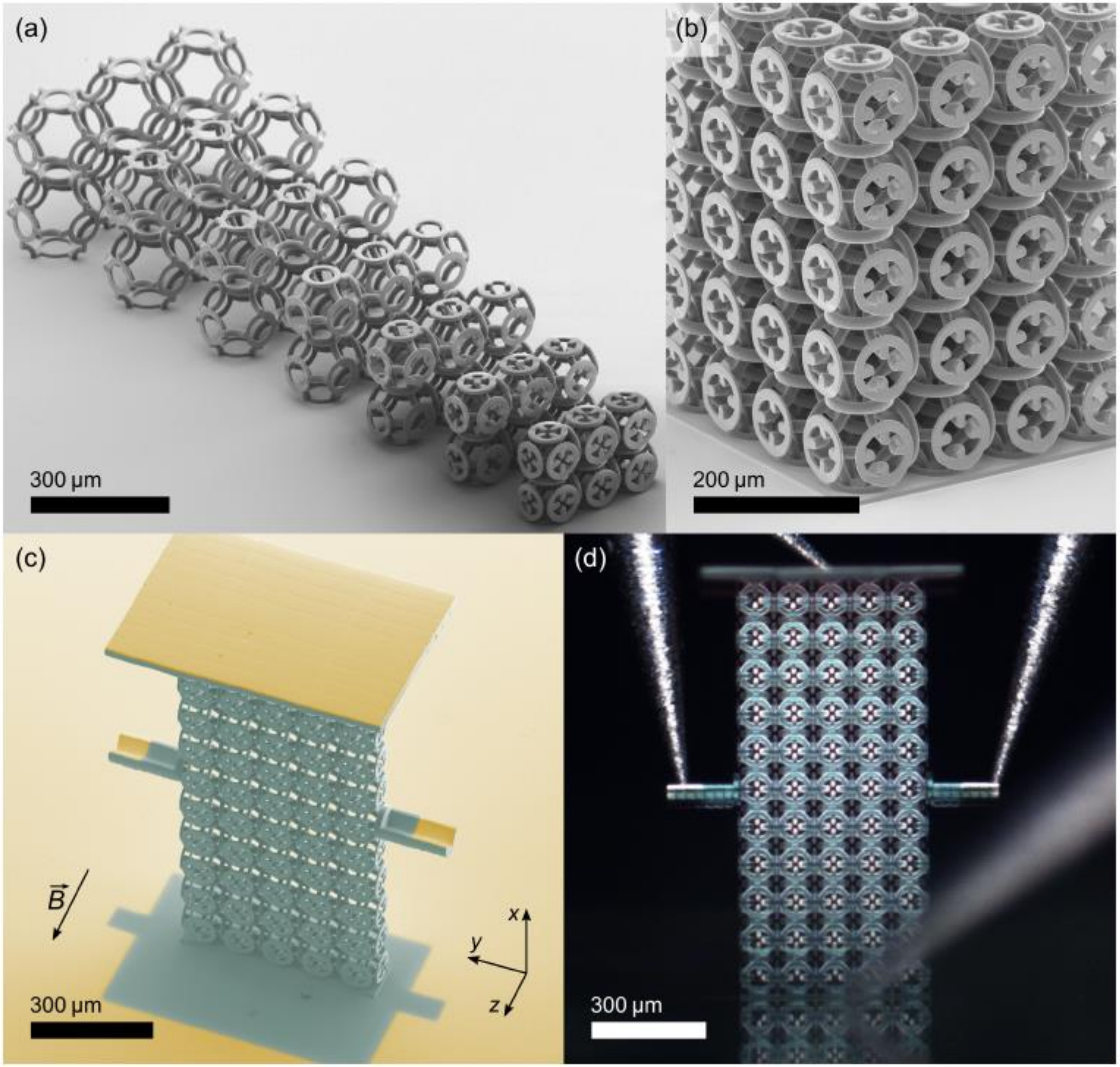


A material with cubic symmetry having a Hall Coefficient opposite to that of the constituents (with Marc Briane)

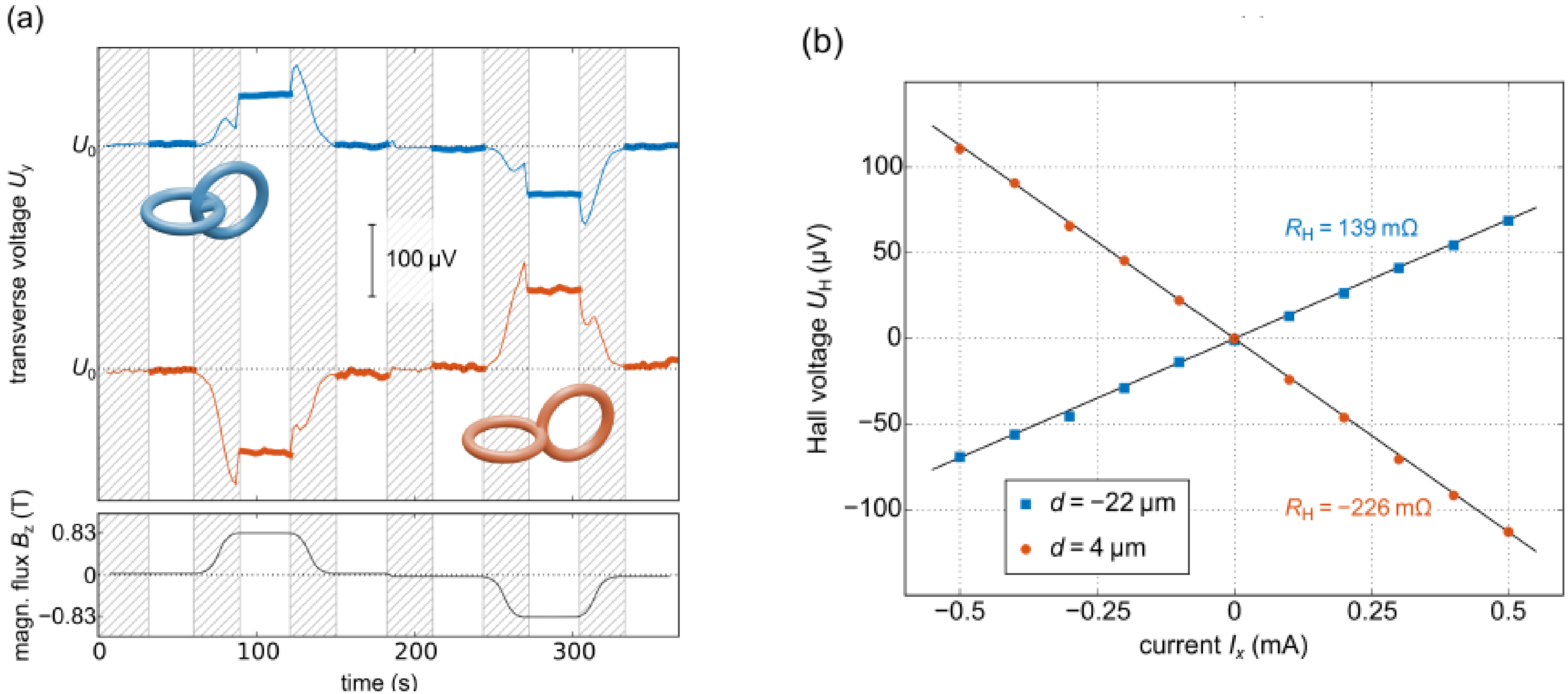
Simplification of Kadic et.al. (2015)

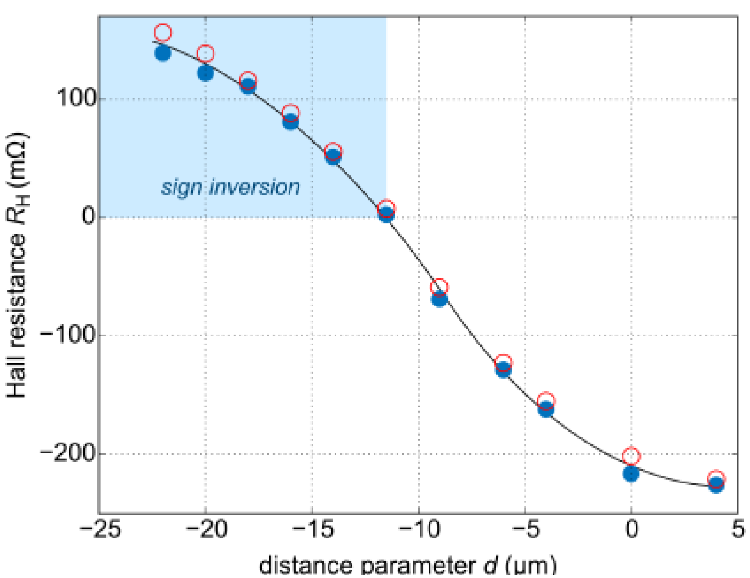






Experimental Realization of Kern, Kadic, Wegener

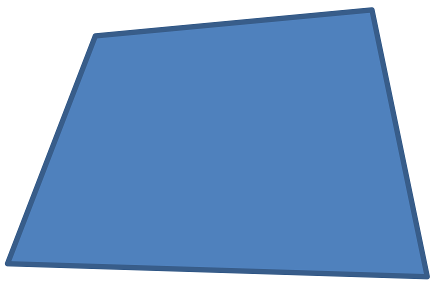




Their experimental results confirming Hall-effect reversal

Back to the question of finding the set of possible elasticity tensors: these have 18 invariants

Not an easy question:



A distorted square in 2-dimensions is specified by eight parameters:

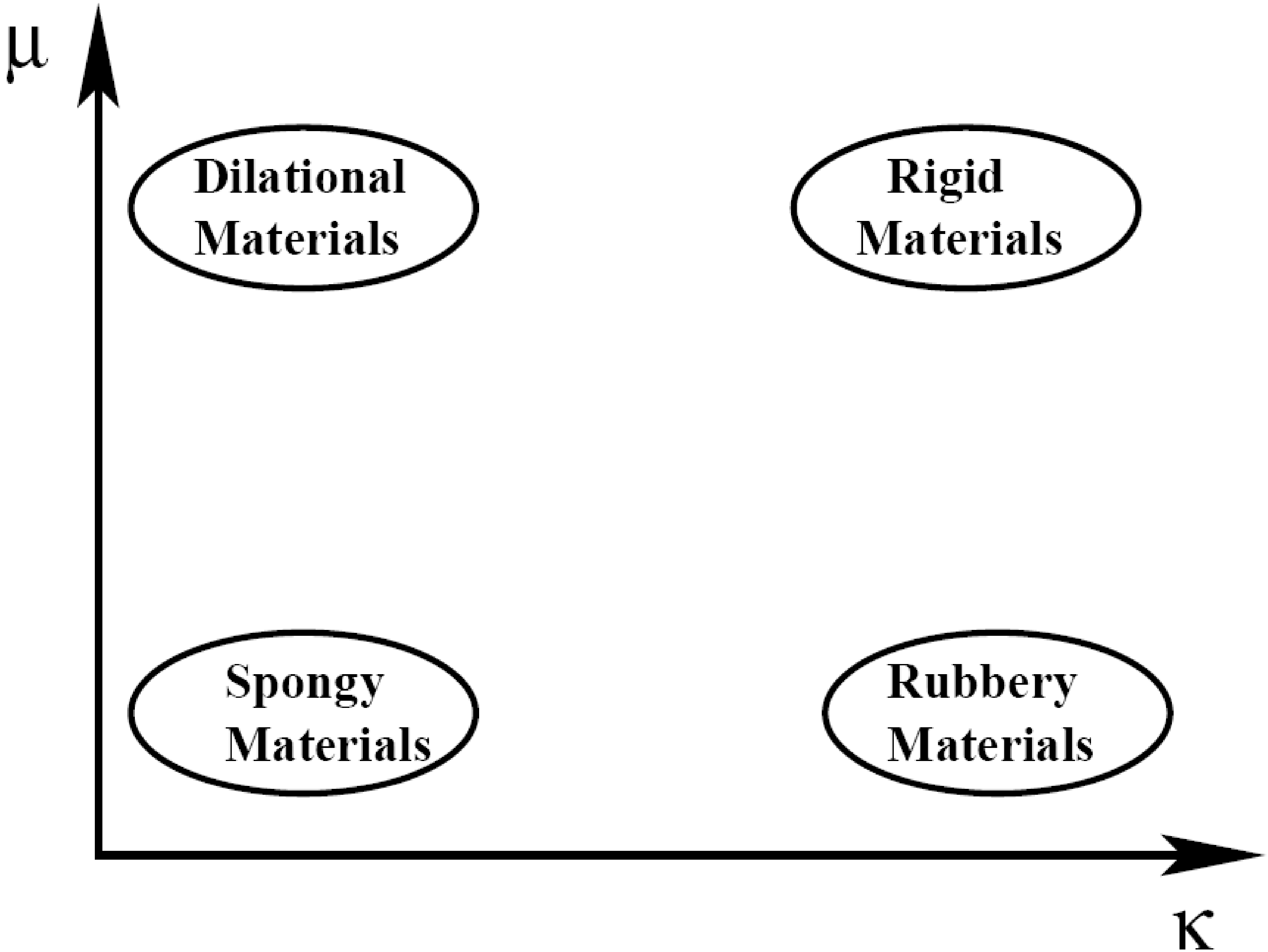
$$(x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3), \quad (x_4, y_4)$$

In 18 dimensions, need 4.7 million numbers to specify a distorted hypercube.

What linearly elastic materials can be realized?

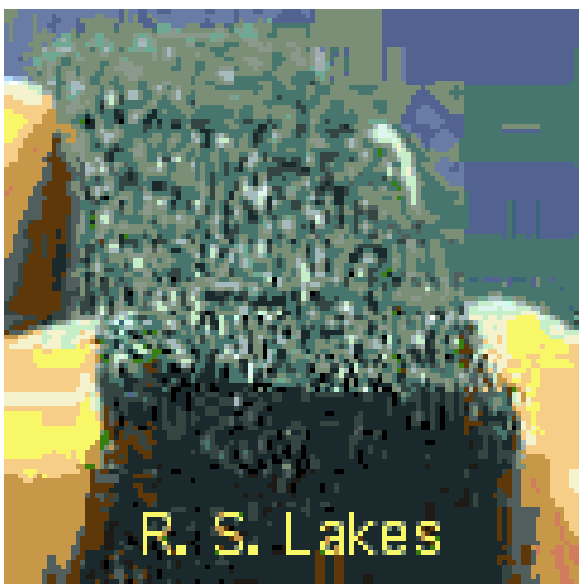
(joint with Andrej Cherkaev, 1995)

Landscape of isotropic materials

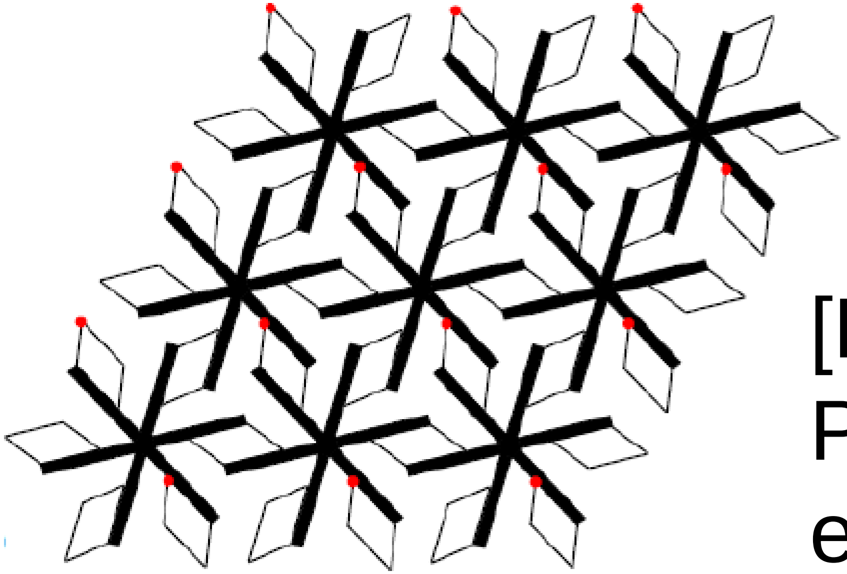


Experiment of R. Lakes (1987)

Normal Foam

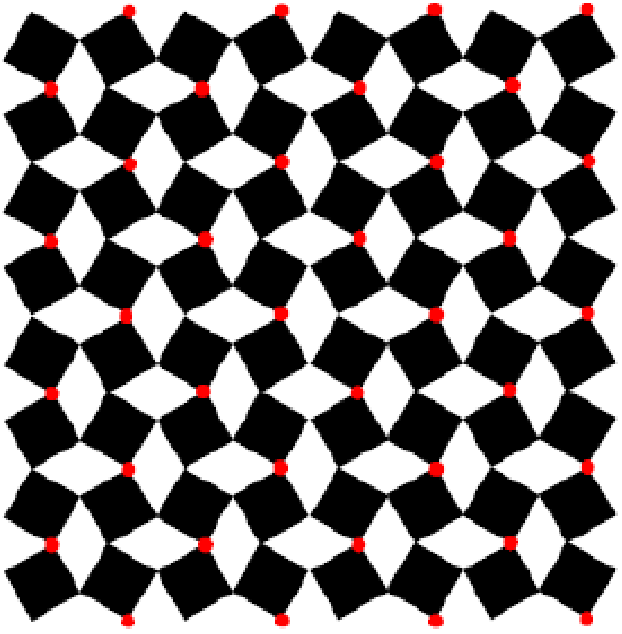


Two dimensional Dilational materials



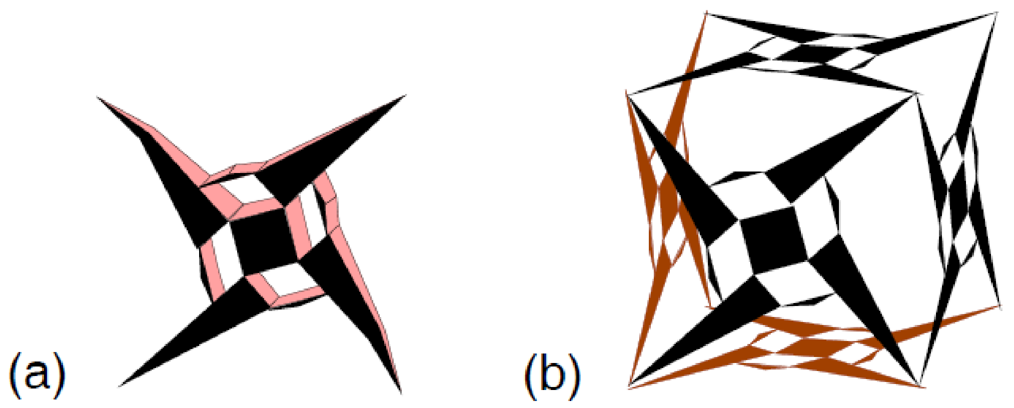
Milton 1992

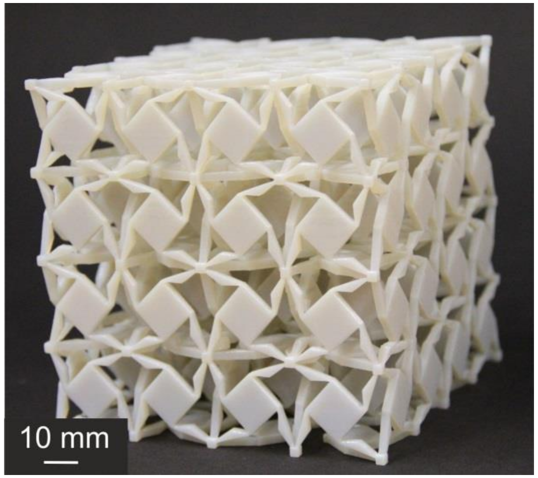
[First proof of negative Poission ratio in continuum elasticity]

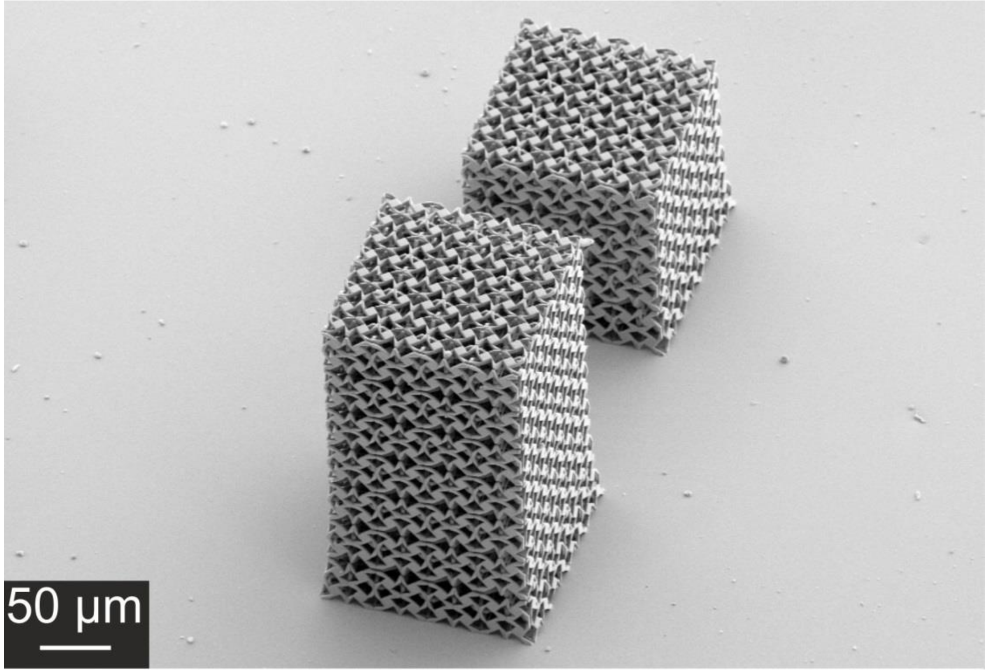


Grima and Evans 2000

Three Dimensional Dilational materials







Buckmann,, Schittny, Thiel, Kadic, Milton Wegener (2014)

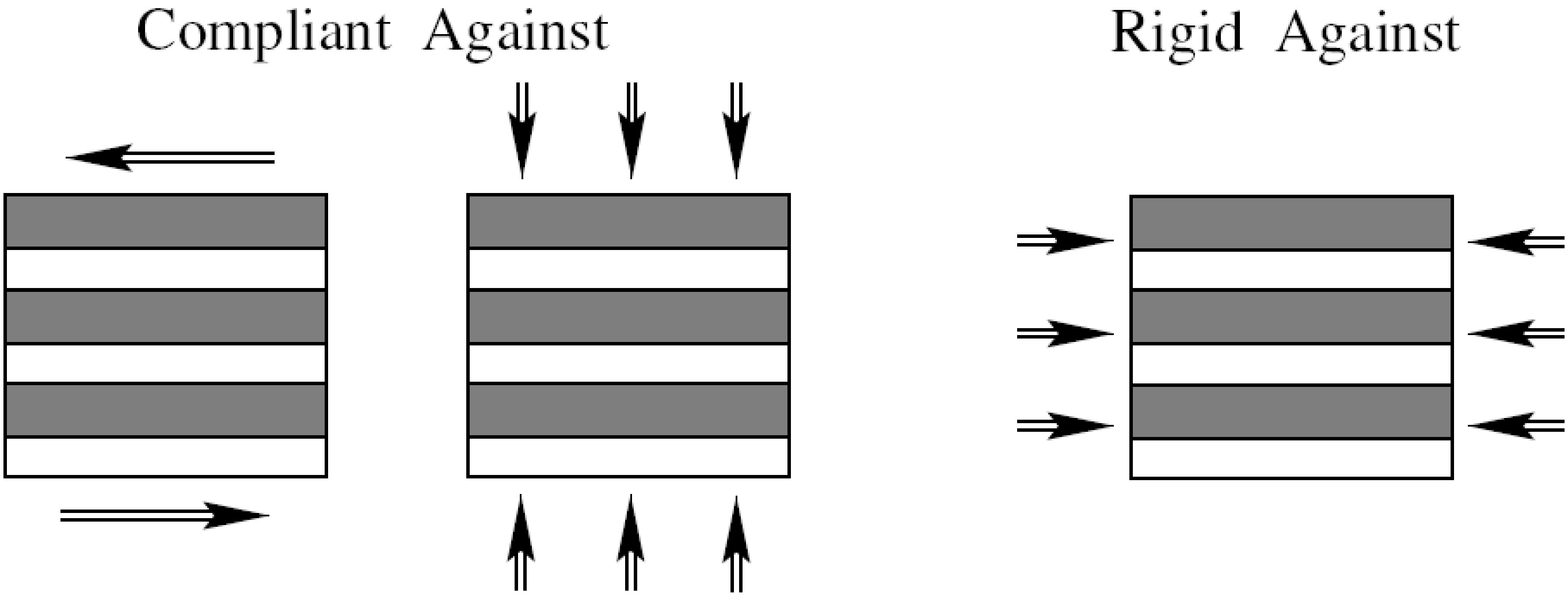
A material with Poisson's ratio close to -1 (a dilational material) is an example of a unimode extremal material.

It is compliant with respect to one strain (dilation) yet stiff with respect to all orthogonal loadings (pure shears)

The elasticity tensor has one eigenvalue which is small, and five eigenvalues which are large.

Can one obtain all other types of extremal materials?

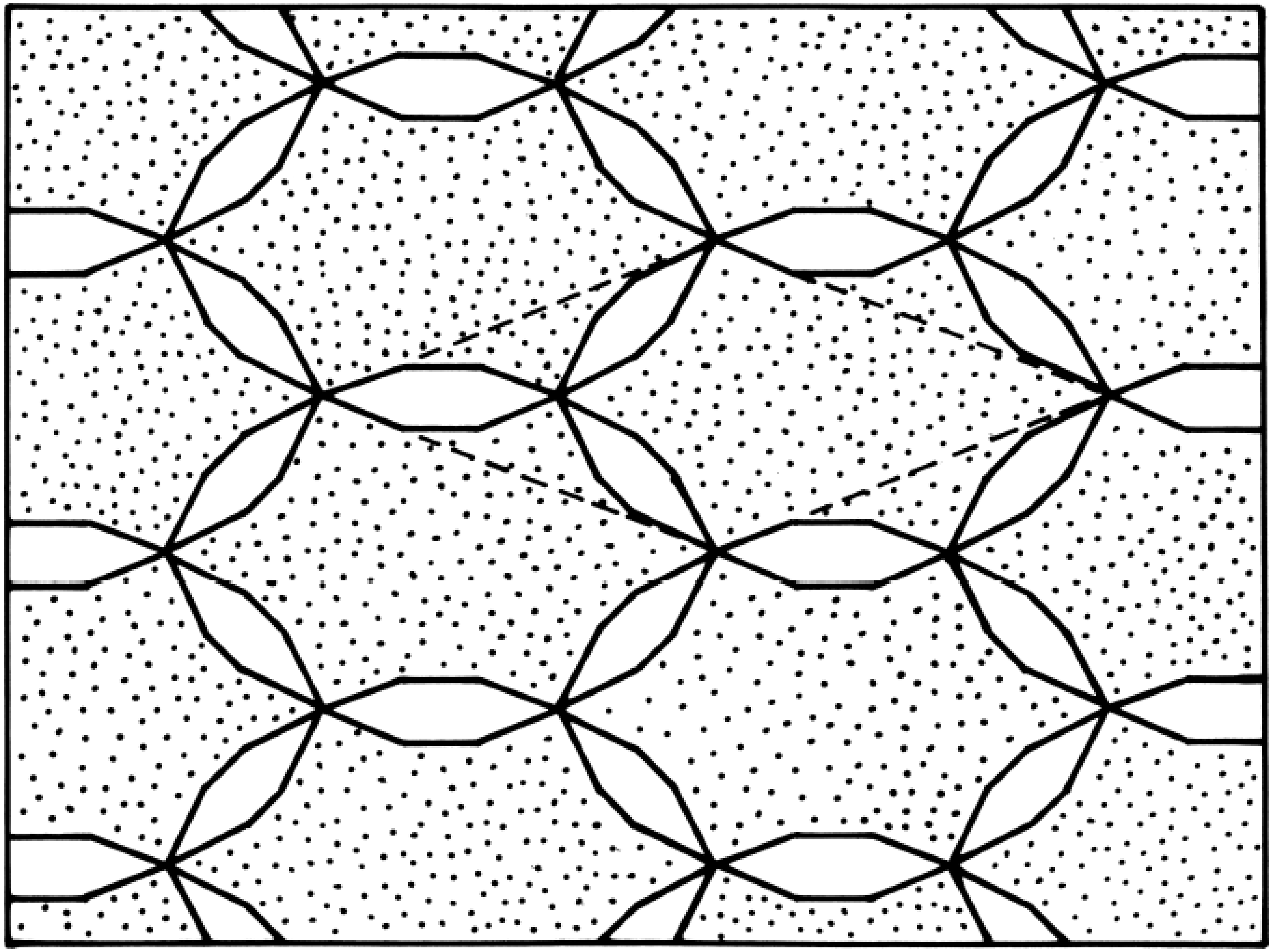
A two-dimensional laminate is a bimodal material



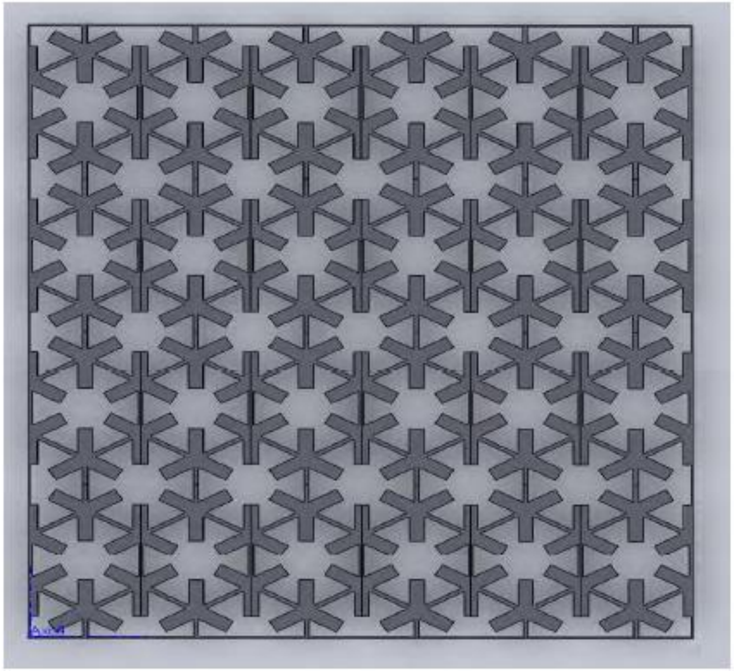
Two eigenvalues of the elasticity tensor are small

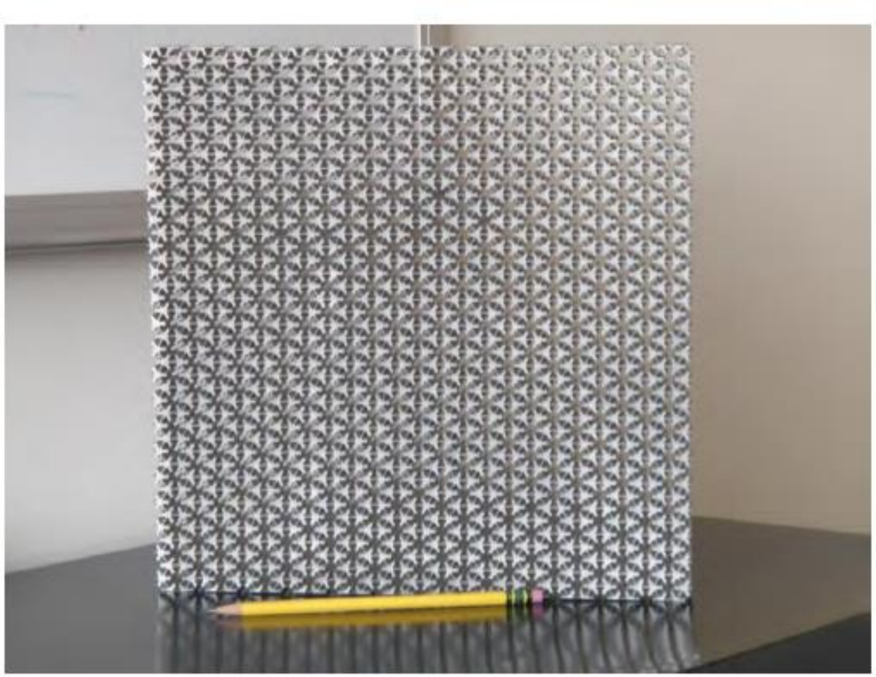
In three-dimensions such a laminate is a trimode extremal material

A bimode material which supports any biaxial loading with positive determinant



Two-dimensional Metal-water constructed by the group of Norris (2012)



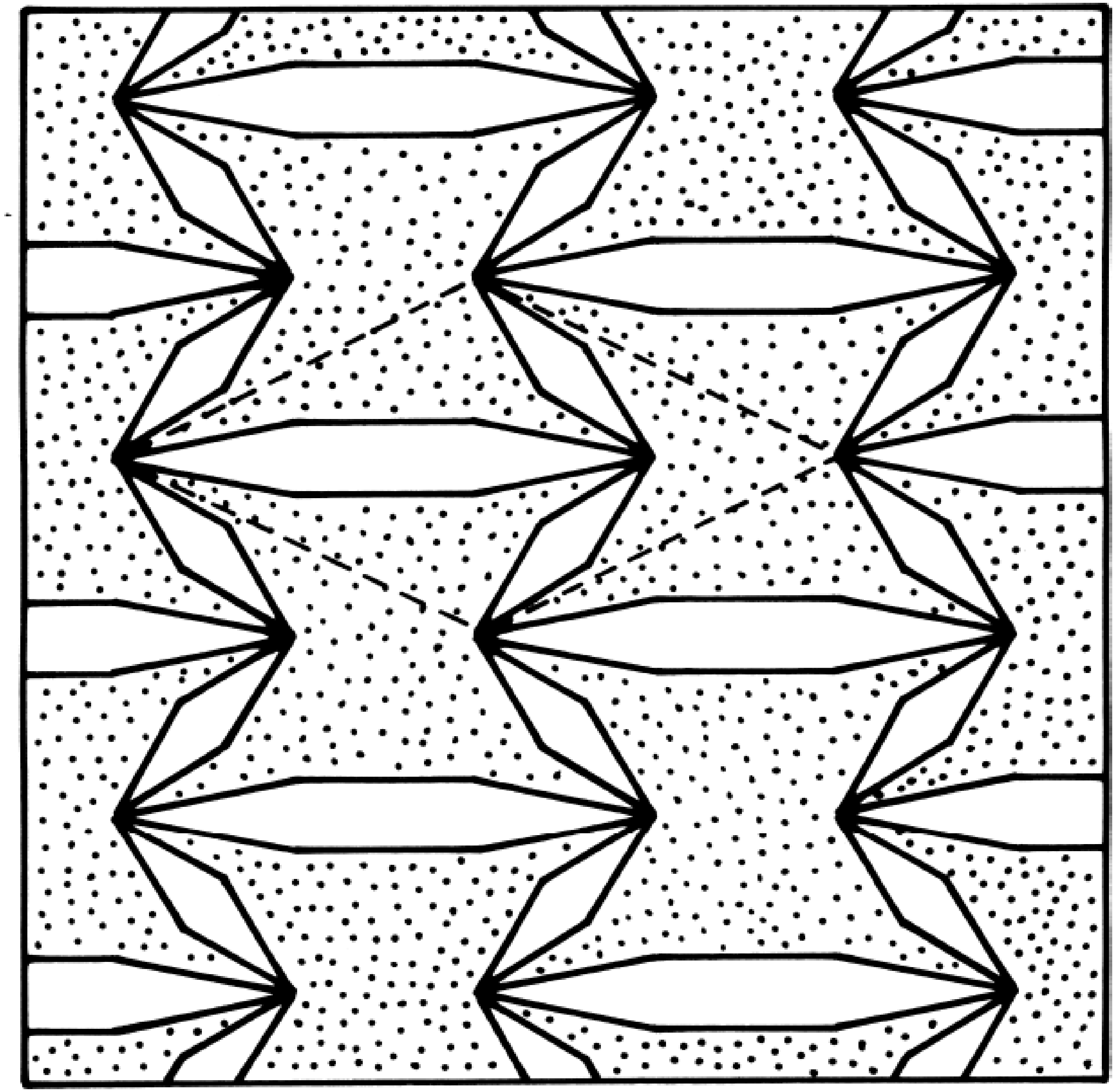


Bulk modulus = 2.25 Gpa

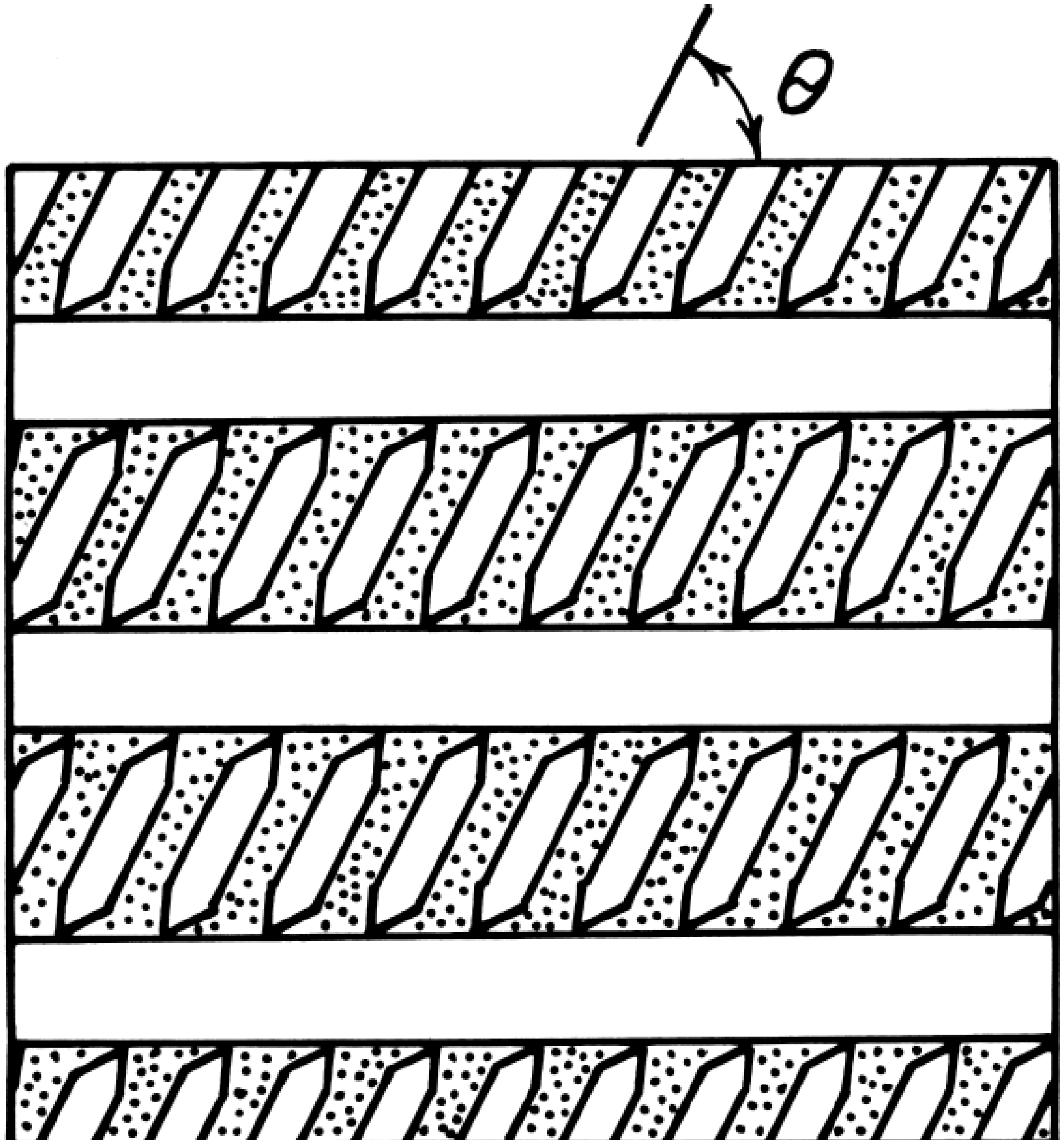
Density = 1000 kg/m^3

Shear modulus = 0.065 GPa

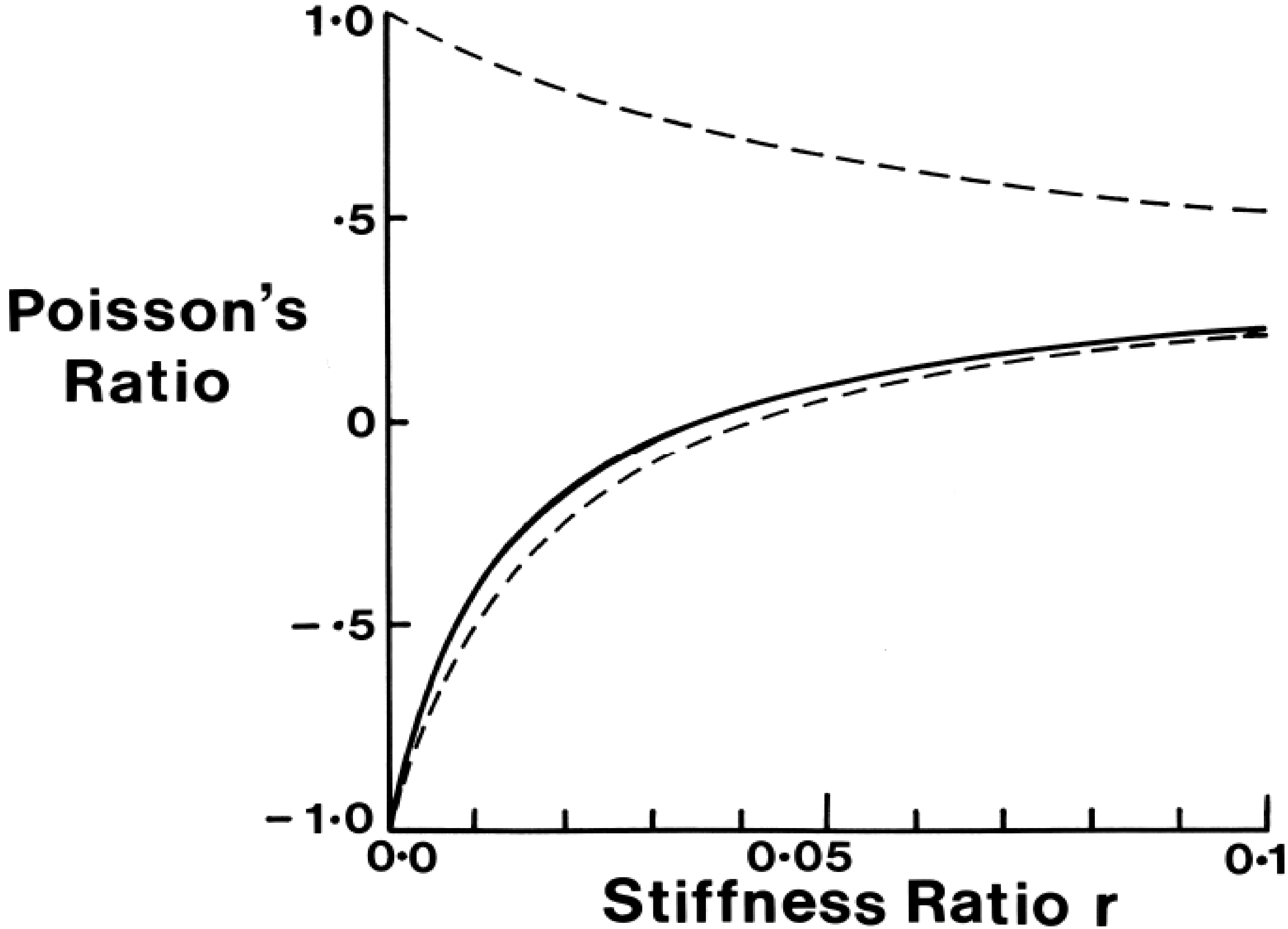
A bimode material which supports any biaxial loading with negative determinant



A unimode material which is compliant to any loading with negative determinant

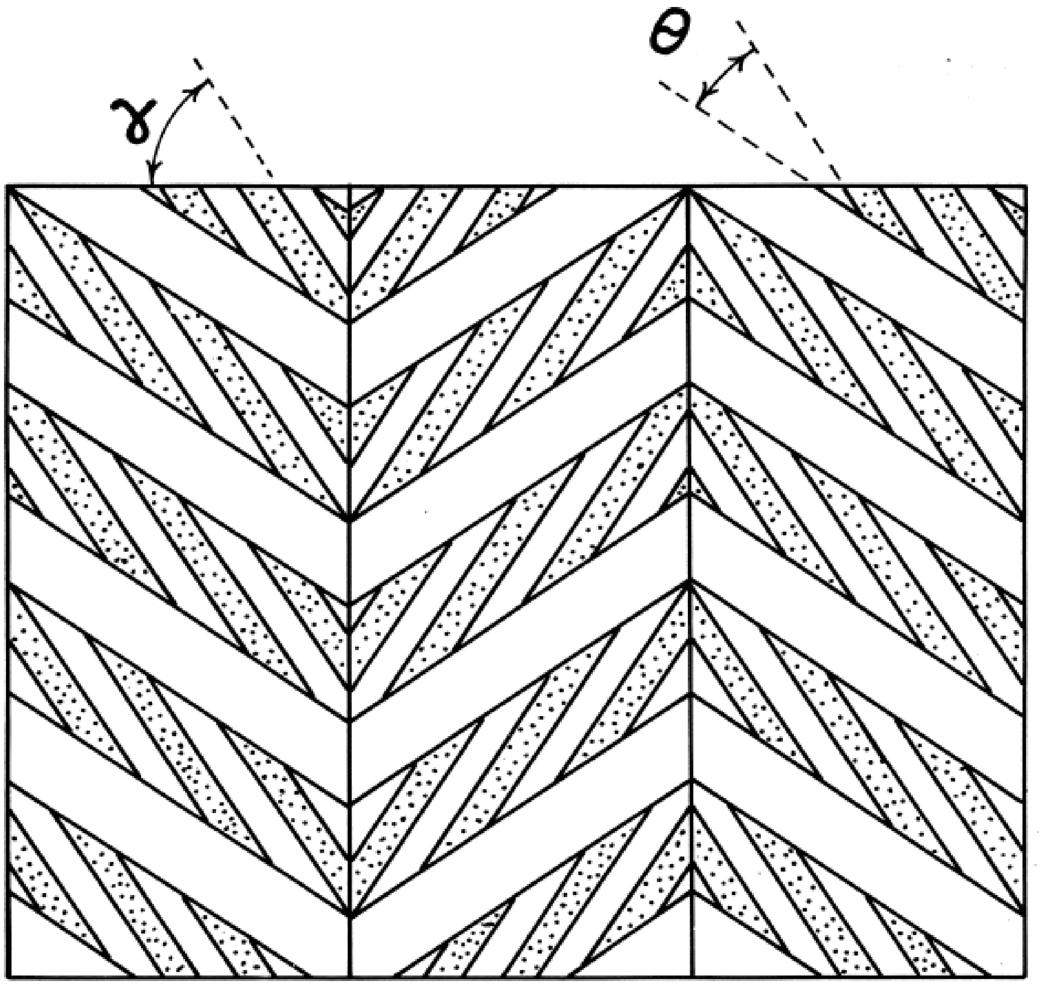


Compare with bounds of Cherkaev and Gibiansky (1993)

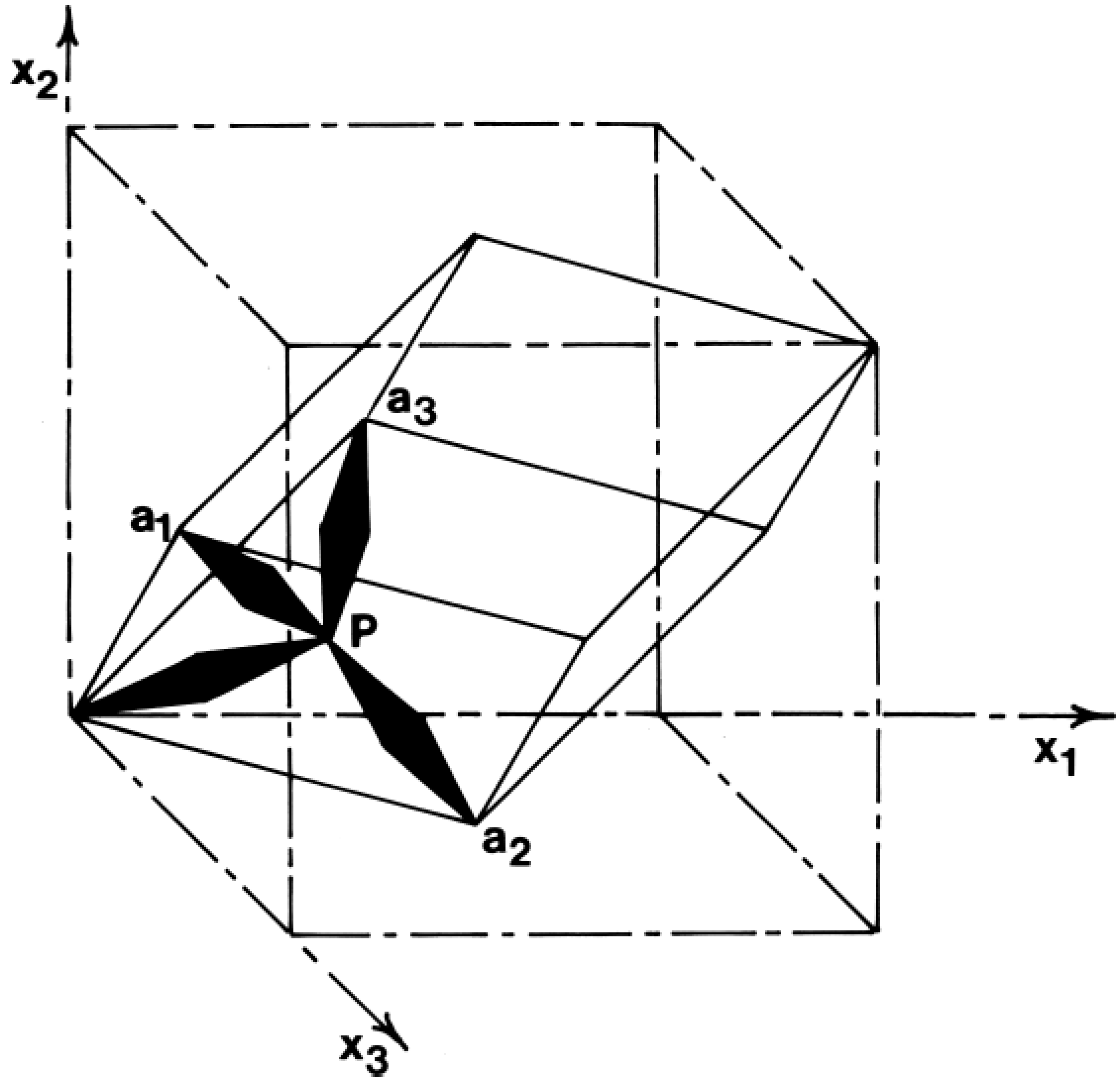


$$\kappa_1 = 2/r,\ \mu_1 = 1/r,\ \kappa_2 = 2, \text{ and } \mu_2 = 1$$

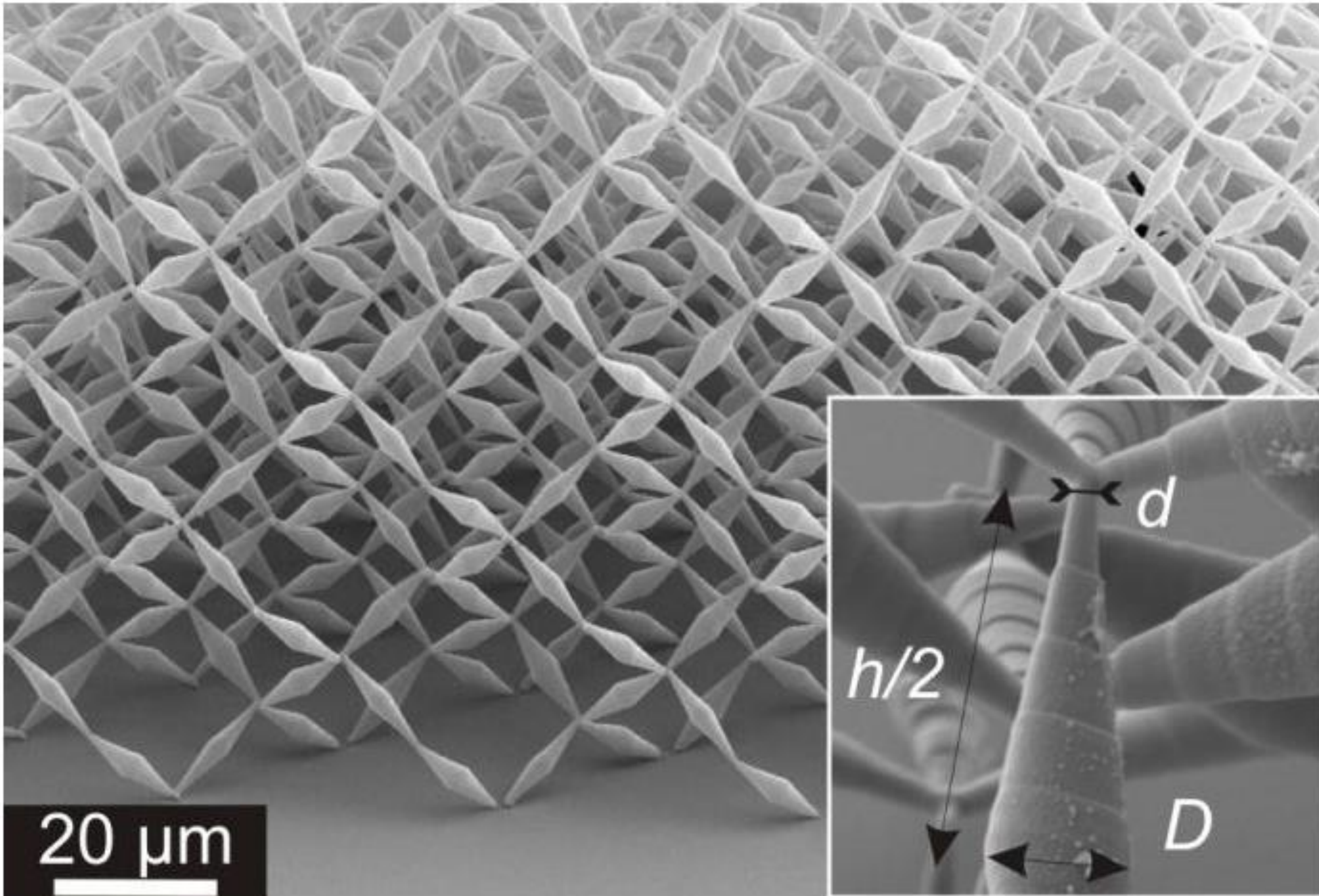
A unimode material which is compliant to any given loading

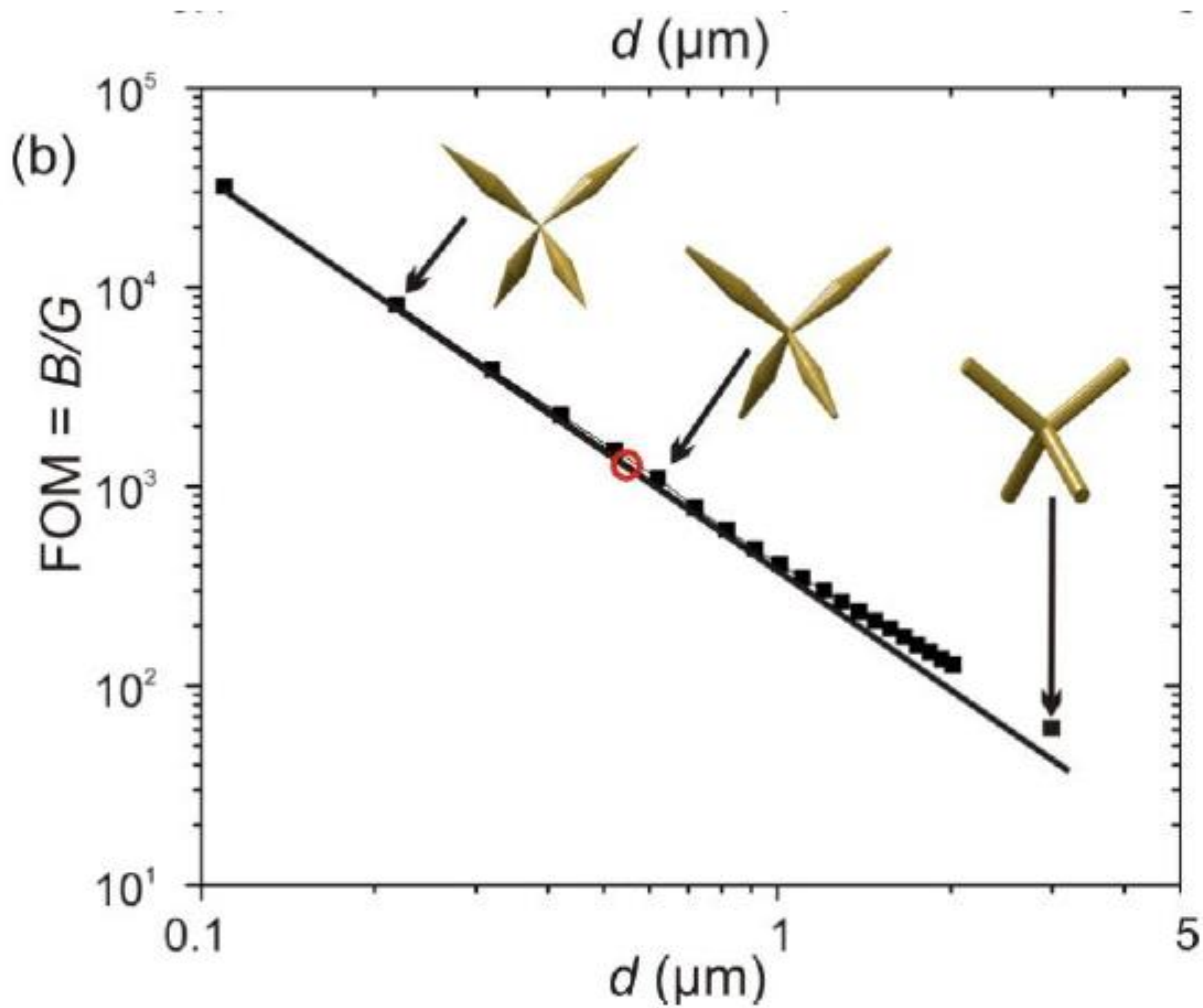


A three dimensional pentamode material which can support any prescribed loading

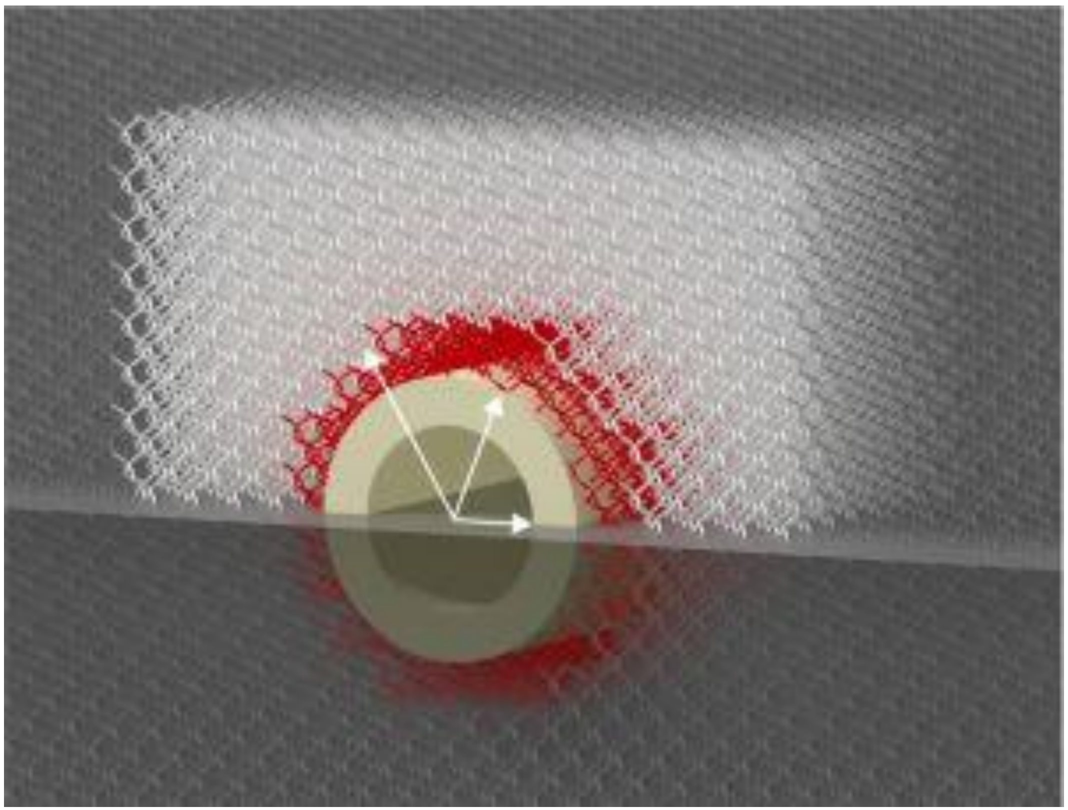


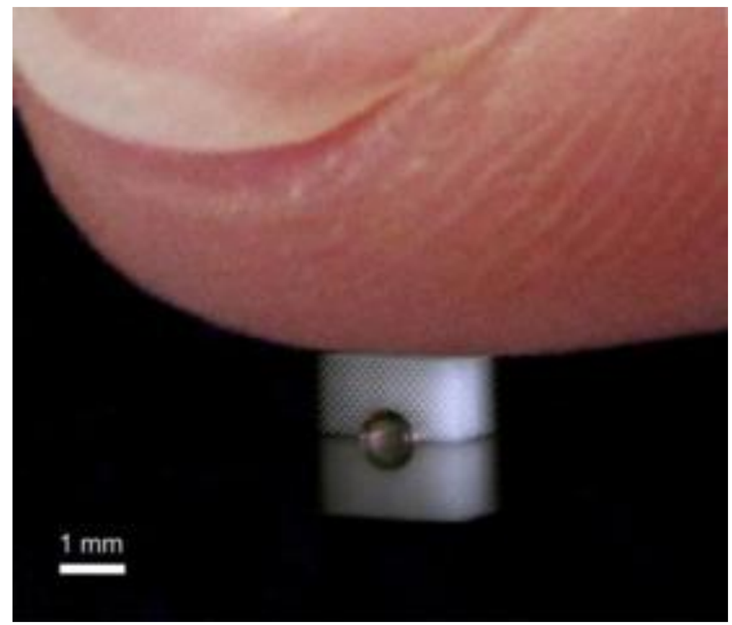
Realization of Kadic et.al. 2012




d (μm)
(b)
FOM = B/G
10^5
10^4
10^3
10^2
10^1
0.1
1
5
d (μm)

Cloak making an object "unfeelable": Buckmann et. al. (2014)





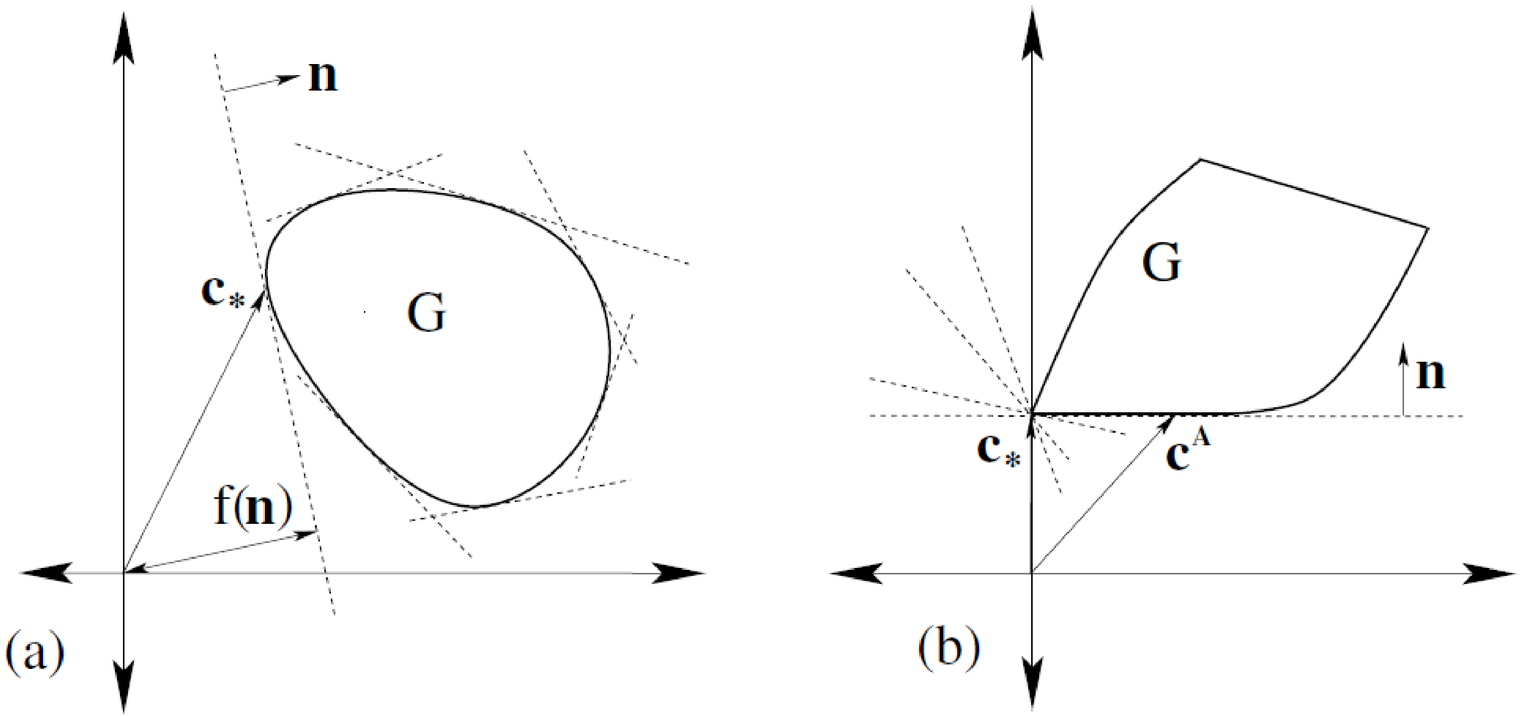
By superimposing appropriate pentamode material structures one can generate all possible elasticity tensors.

.All elasticity tensors are realizable!

Camar Eddine and Seppecher (2003) have characterized all possible non-local responses

Recall: A convex set G can be characterized by its Legendre transform:

$$f(\mathbf{n}) = \min_{\mathbf{c} \in G} \mathbf{n} \cdot \mathbf{c}.$$



G-closures are not convex sets but can be characterized by their W-transform

$$W_f(\mathbf{N},\mathbf{N}') = \min_{\mathbf{C}_* \in GU_f} (\mathbf{C}_*, \mathbf{N}) + (\mathbf{C}_*^{-1}, \mathbf{N}'),$$

$$(\mathbf{N},\mathbf{C}) = N_{ijk\ell} C_{ijk\ell}$$

$$\bigcap_{\substack{\mathbf{N},\mathbf{N}' \geq 0 \\ \mathbf{N}\mathbf{N}'=0}} \{\mathbf{C} : (\mathbf{C},\mathbf{N}) + (\mathbf{C}^{-1},\mathbf{N}') \geq W_f(\mathbf{N},\mathbf{N}')\} = GU_f.$$

W-transforms generalize the idea of Legendre transforms

$$\mathbf{N} = \sum_{i=1}^{2} \boldsymbol{\epsilon}_i^0 \otimes \boldsymbol{\epsilon}_i^0, \quad \mathbf{N}' = \sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 \otimes \boldsymbol{\sigma}_j^0,$$

Need to know the 7 energy functions

$$
\begin{aligned}
W_f^0(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\sigma}_4^0,\boldsymbol{\sigma}_5^0,\boldsymbol{\sigma}_6^0) &= \min_{\mathbf{C}_*\in GU_f} \sum_{j=1}^{6} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0, \\
W_f^1(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\sigma}_4^0,\boldsymbol{\sigma}_5^0,\boldsymbol{\epsilon}_1^0) &= \min_{\mathbf{C}_*\in GU_f} \left[\boldsymbol{\epsilon}_1^0 : \mathrm{C}_*\boldsymbol{\epsilon}_1^0 + \sum_{j=1}^{5} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0\right], \\
W_f^2(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\sigma}_4^0,\boldsymbol{\epsilon}_1^0,\boldsymbol{\epsilon}_2^0) &= \min_{\mathbf{C}_*\in GU_f} \left[\sum_{i=1}^{2}\boldsymbol{\epsilon}_i^0 : \mathrm{C}_*\boldsymbol{\epsilon}_i^0 + \sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0\right], \\
W_f^3(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\epsilon}_1^0,\boldsymbol{\epsilon}_2^0,\boldsymbol{\epsilon}_3^0) &= \min_{\mathbf{C}_*\in GU_f} \left[\sum_{i=1}^{3}\boldsymbol{\epsilon}_i^0 : \mathrm{C}_*\boldsymbol{\epsilon}_i^0 + \sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0\right], \\
W_f^4(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\epsilon}_1^0,\boldsymbol{\epsilon}_2^0,\boldsymbol{\epsilon}_3^0,\boldsymbol{\epsilon}_4^0) &= \min_{\mathbf{C}_*\in GU_f} \left[\sum_{i=1}^{4}\boldsymbol{\epsilon}_i^0 : \mathrm{C}_*\boldsymbol{\epsilon}_i^0 + \sum_{j=1}^{2} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1}\boldsymbol{\sigma}_j^0\right], \\
W_f^5(\boldsymbol{\sigma}_1^0,\boldsymbol{\epsilon}_1^0,\boldsymbol{\epsilon}_2^0,\boldsymbol{\epsilon}_3^0,\boldsymbol{\epsilon}_4^0,\boldsymbol{\epsilon}_5^0) &= \min_{\mathbf{C}_*\in GU_f} \left[\left(\sum_{i=1}^{5}\boldsymbol{\epsilon}_i^0 : \mathrm{C}_*\boldsymbol{\epsilon}_i^0\right) + \boldsymbol{\sigma}_1^0 : \mathrm{C}_*^{-1}\boldsymbol{\sigma}_1^0\right], \\
W_f^6(\boldsymbol{\epsilon}_1^0,\boldsymbol{\epsilon}_2^0,\boldsymbol{\epsilon}_3^0,\boldsymbol{\epsilon}_4^0,\boldsymbol{\epsilon}_5^0,\boldsymbol{\epsilon}_6^0) &= \min_{\mathbf{C}_*\in GU_f} \sum_{i=1}^{6} \boldsymbol{\epsilon}_i^0 : \mathrm{C}_*\boldsymbol{\epsilon}_i^0.
\end{aligned}
$$

Orthogonality conditions

$$(\boldsymbol{\epsilon}_i^0, \boldsymbol{\sigma}_j^0) = 0, \qquad (\boldsymbol{\epsilon}_i^0, \boldsymbol{\epsilon}_k^0) = 0, \qquad (\boldsymbol{\sigma}_j^0, \boldsymbol{\sigma}_\ell^0) = 0$$

$$\text{for all } i, j, k, \ell \;\; \text{with } i \neq j, \; i \neq k, \; j \neq \ell.$$

Result of Avellaneda (1987): If $\mathbf{C}_1 \geq \mathbf{C}_2$ then

$$W_f^0(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\sigma}_6^0) = \min_{\mathbf{C}_* \in GU_f} \sum_{j=1}^{6} \boldsymbol{\sigma}_j^0 : \mathbf{C}_*^{-1} \boldsymbol{\sigma}_j^0,$$

$$W_f^6(\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0, \boldsymbol{\epsilon}_6^0) = \min_{\mathbf{C}_* \in GU_f} \sum_{i=1}^{6} \boldsymbol{\epsilon}_i^0 : \mathbf{C}_* \boldsymbol{\epsilon}_i^0.$$

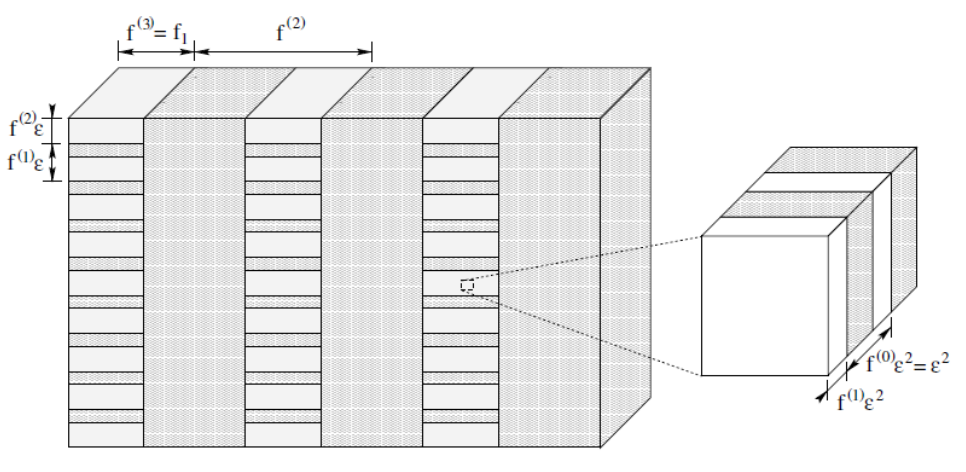
can be easily computed

They are attained by sequentially layered laminates, and we call the material which attains the minimum in

$$W_f^0(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\sigma}_4^0,\boldsymbol{\sigma}_5^0,\boldsymbol{\sigma}_6^0) = \min_{\mathbf{C}_*\in GU_f}\sum_{j=1}^{6}\boldsymbol{\sigma}_j^0 : \mathbf{C}_*^{-1}\boldsymbol{\sigma}_j^0,$$

the Avellaneda material, with elasticity tensor

$$\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0,\boldsymbol{\sigma}_2^0,\boldsymbol{\sigma}_3^0,\boldsymbol{\sigma}_4^0,\boldsymbol{\sigma}_5^0,\boldsymbol{\sigma}_6^0)$$



Maxwell (1873)

Obvious bounds:

$$
\begin{aligned}
\sum_{j=1}^{5} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^1(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\epsilon}_1^0), \\
\sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^2(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0), \\
\sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0), \\
\sum_{j=1}^{2} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^4(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0), \\
\boldsymbol{\sigma}_1^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, 0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_1^0 &\leq W_f^5(\boldsymbol{\sigma}_1^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0), \\
0 &\leq W_f^6(\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0, \boldsymbol{\epsilon}_6^0).
\end{aligned}
$$

Main result: in many cases these bounds are sharp

Theorem

$$\begin{aligned}
\lim_{\delta\to 0} W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0) &= \sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0, \\
\lim_{\delta\to 0} W_f^4(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0) &= \sum_{j=1}^{2} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0, \\
\lim_{\delta\to 0} W_f^5(\boldsymbol{\sigma}_1^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0) &= \boldsymbol{\sigma}_1^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, 0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_1^0, \\
\lim_{\delta\to 0} W_f^6(\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0, \boldsymbol{\epsilon}_6^0) &= 0.
\end{aligned}$$

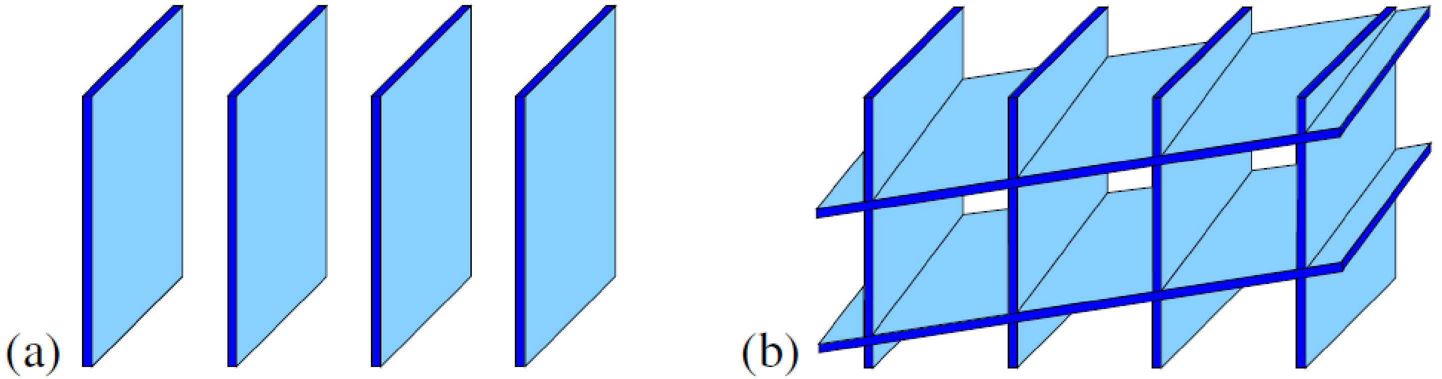
When ϵ_1^0 has one zero eigenvalue, and the other eigenvalues of opposite signs,

$$W_f^1(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \epsilon_1^0) = \sum_{j=1}^{5} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, 0)]^{-1} \boldsymbol{\sigma}_j^0$$

When $\det(\boldsymbol{\epsilon}_1^0 + t\boldsymbol{\epsilon}_2^0) = 0$ has at least two roots and $\boldsymbol{\epsilon}(t) = \boldsymbol{\epsilon}_1^0 + t\boldsymbol{\epsilon}_2^0$ is never positive or negative definite

$$W_f^2(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \epsilon_1^0, \epsilon_2^0) = \sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0$$

Idea of proof: Insert into the Avellaneda material a thin walled structure with sets of parallel walls:



Inside the walls put the appropriate multimode material e.g. to show

$$\lim_{\delta\to 0} W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0) = \sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0$$

Look in the three dimensional space spanned by $\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0$ and search for three independent symmetrized rank-one matrices

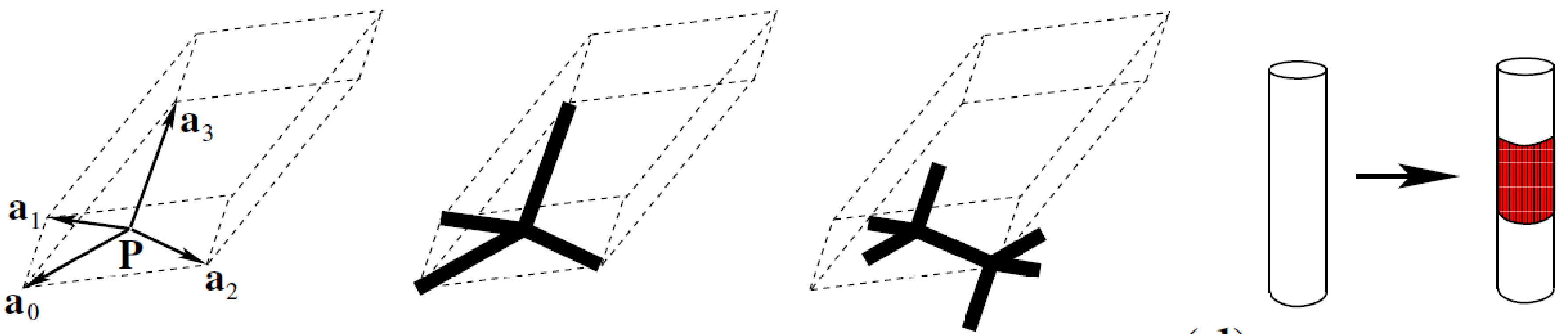
$$\boldsymbol{\epsilon}^{(k)} = (\mathbf{a}_k \mathbf{n}_k^T + \mathbf{n}_k \mathbf{a}_k^T)/2,$$

that form a basis for the space (algebraic problem)

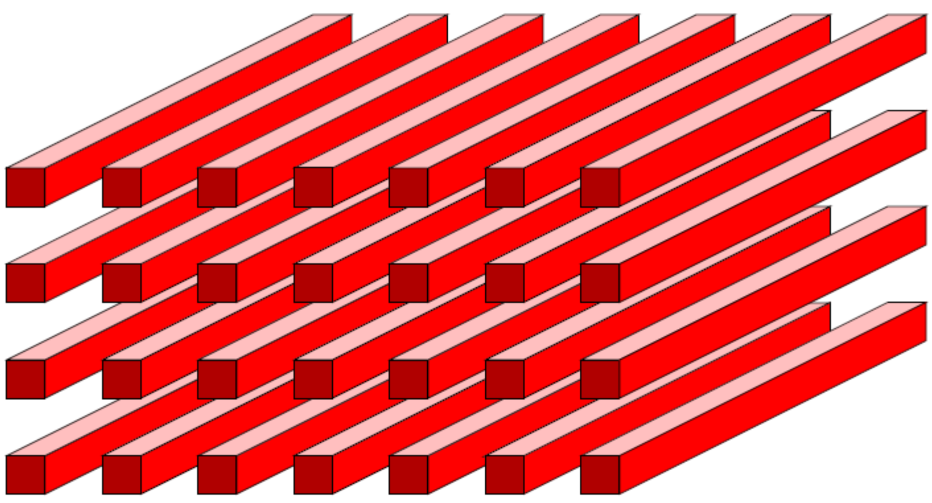
The $\mathbf{n}_k$ give the directions of the walls and the strain in them is a multiple of $\boldsymbol{\epsilon}^{(k)}$

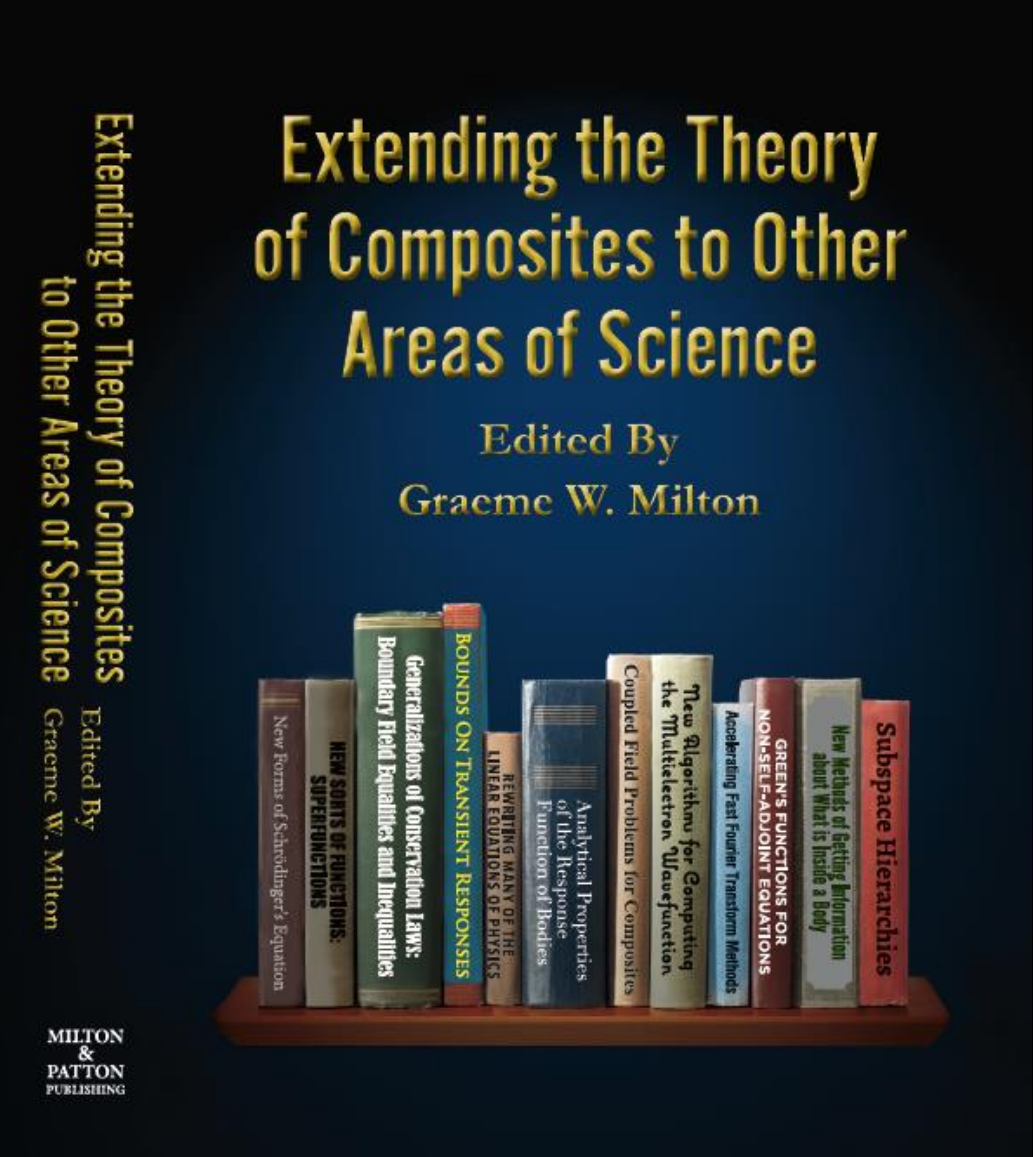
Walls support $\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0$ compliant to $\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0$

Modifying the pentamodes:



(a) (b) (c) (d)





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Thank you!

Thank you!

Thank you!

Thank you!

Thank you!