Mechanical Metamaterials

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Walser 1999:

Macroscopic composites having a manmade, three-dimensional, periodic cellular architecture designed to produce an optimized combination, not available in nature, of *two or more responses* to specific excitation.

Browning and Wolf 2001:

Metamaterials are a new class of ordered composites that exhibit exceptional properties not readily observed in nature.

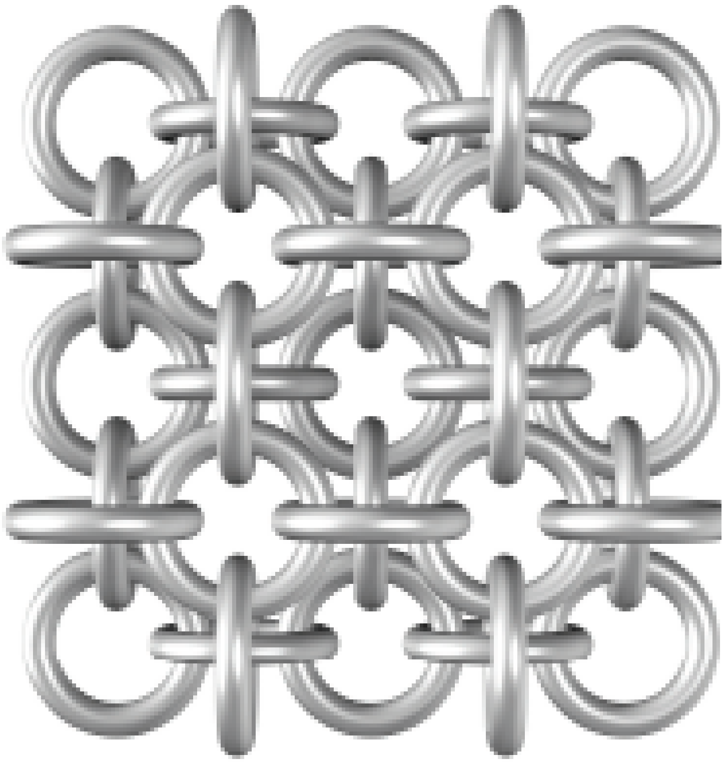
It's constantly a surprise to find what properties a composite can exhibit.

One interesting example:

In elementary physics textbooks one is told that in classical physics the sign of the Hall coefficient tells one the sign of the charge carrier.

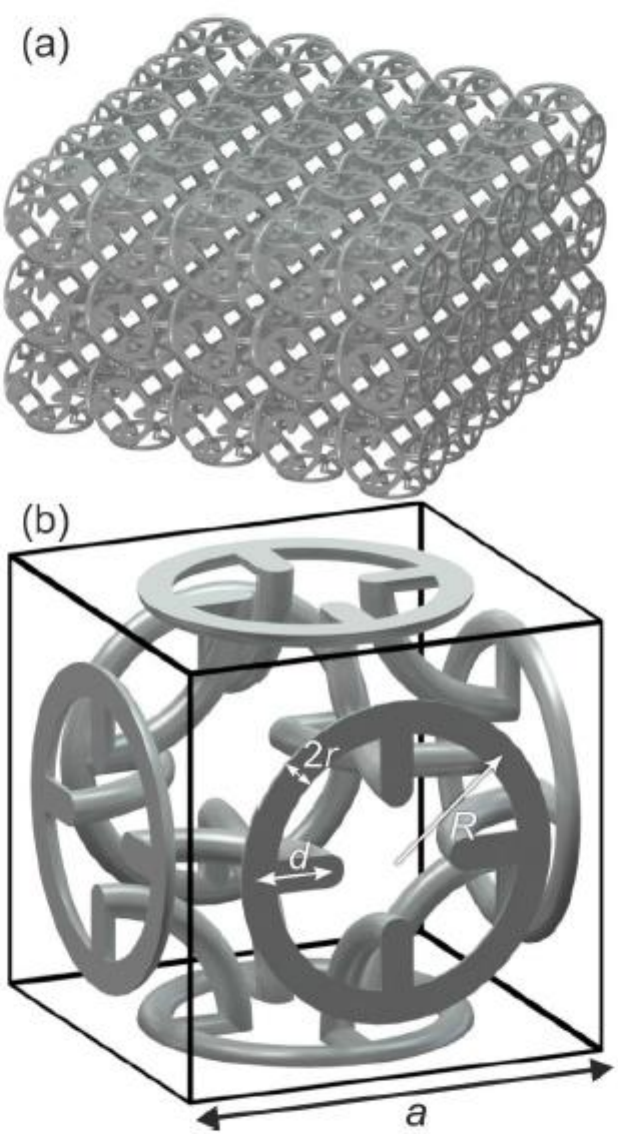
However there is a counterexample!

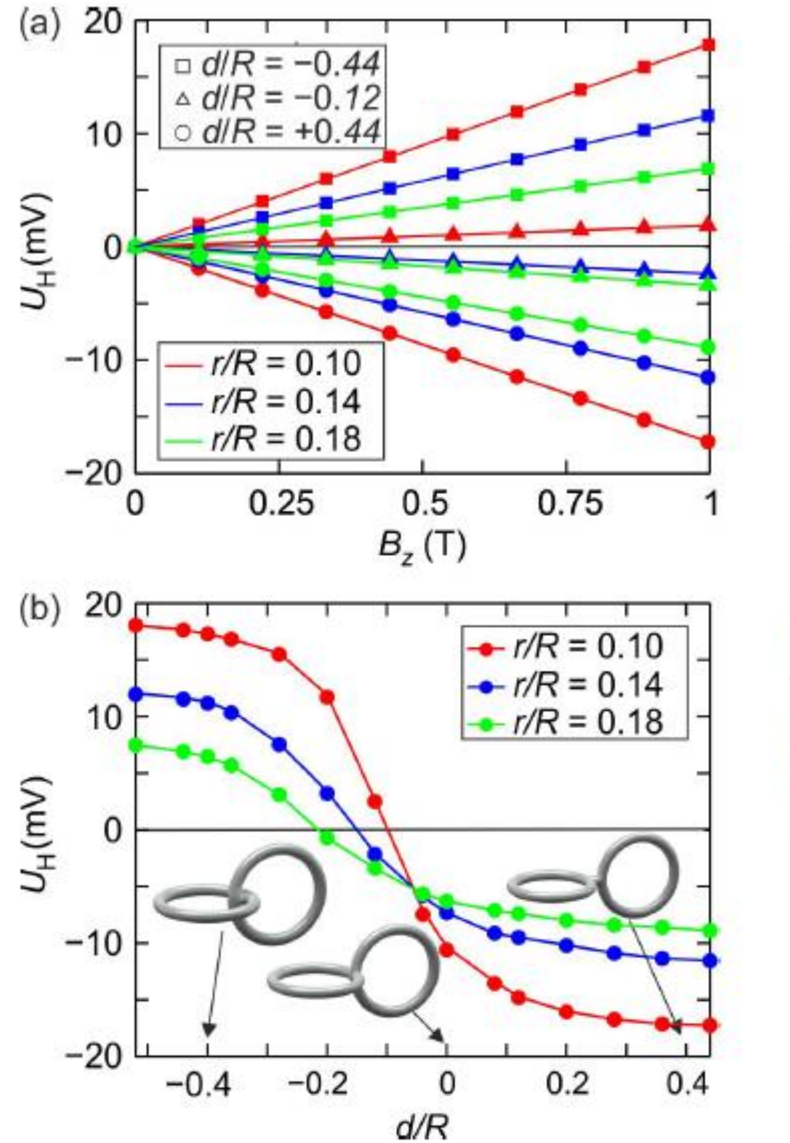
Geometry suggested by artist Dylon Whyte



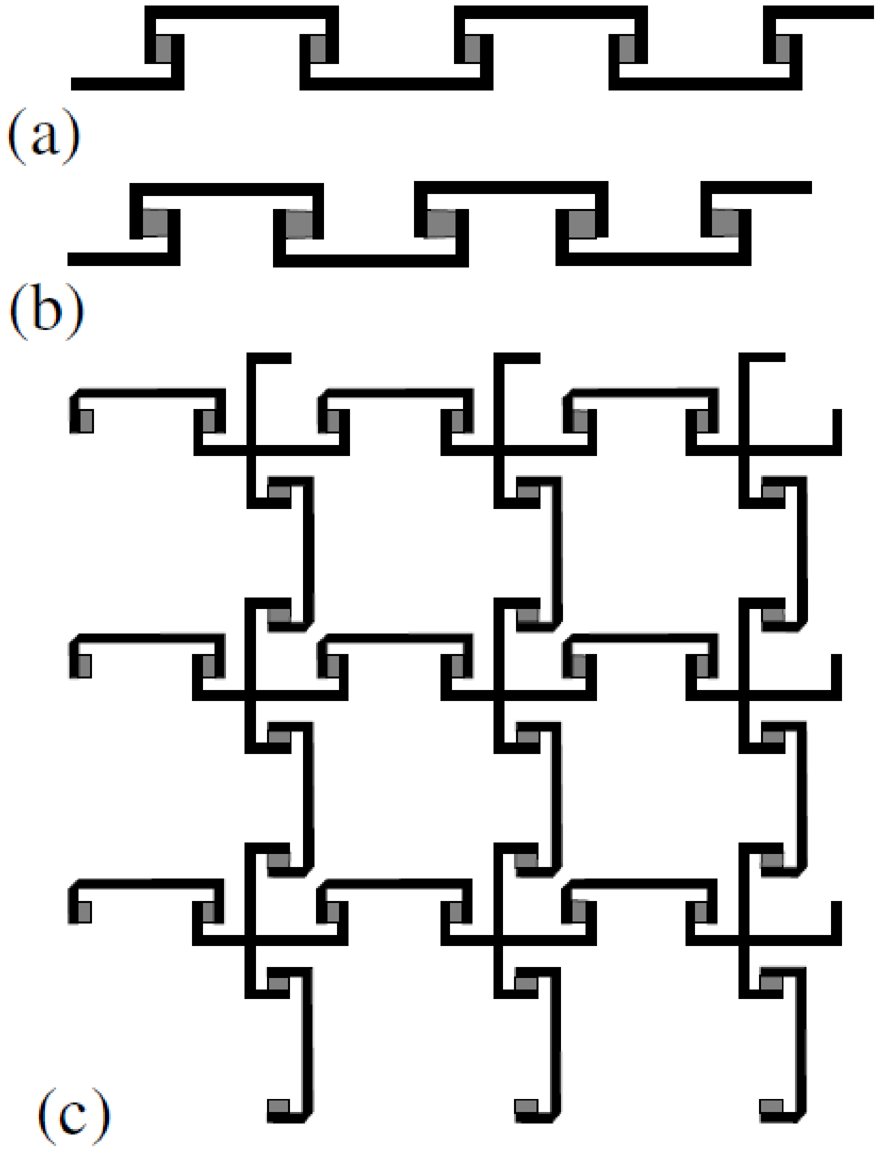
A material with cubic symmetry having a Hall Coefficient opposite to that of the constituents (with Marc Briane)

Simplification of Kadic et.al. (2015)





Another example: negative expansion from positive expansion

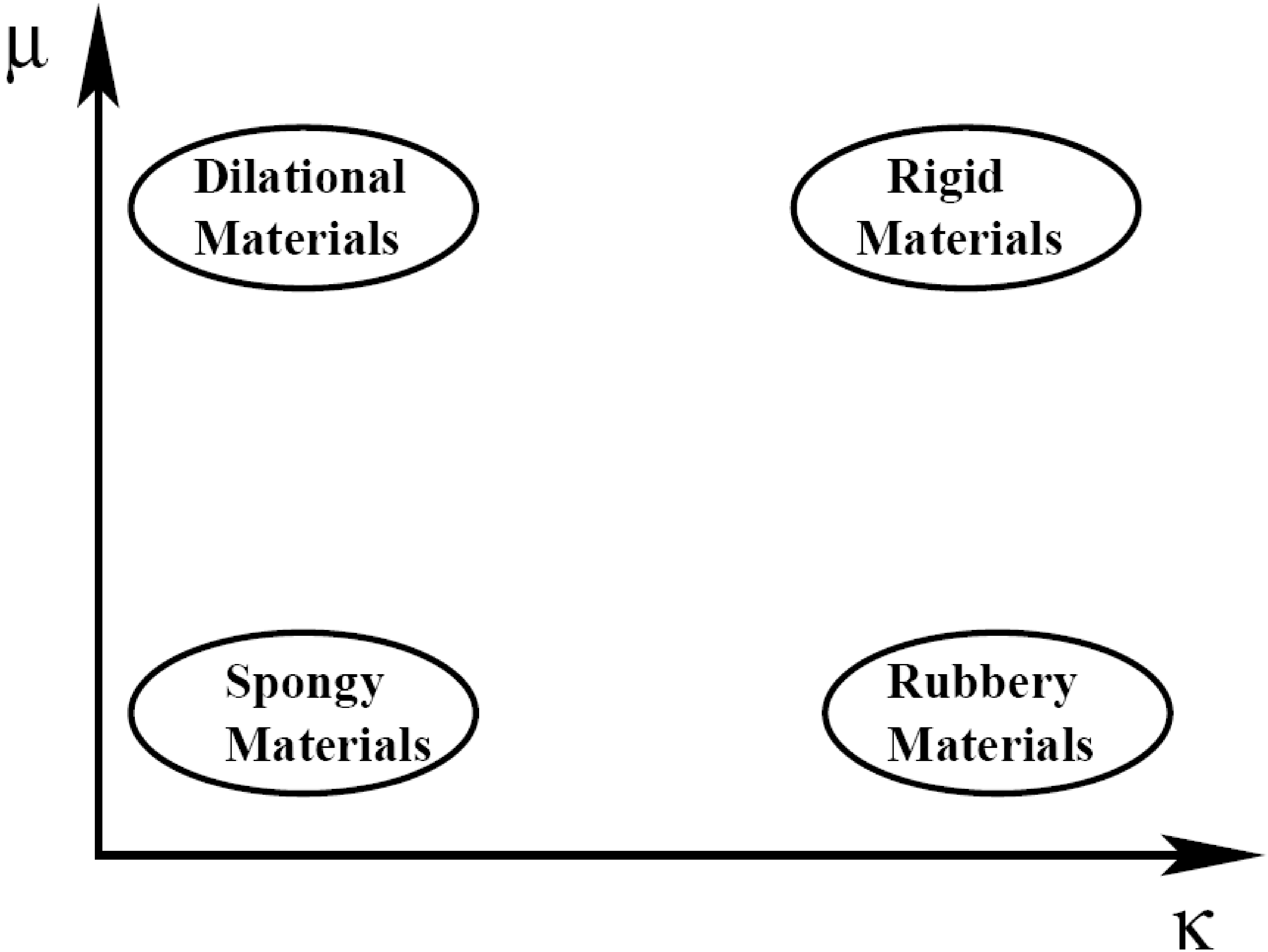


Original designs: Lakes (1996); Sigmund & Torquato (1996, 1997)

What linearly elastic materials can be realized?

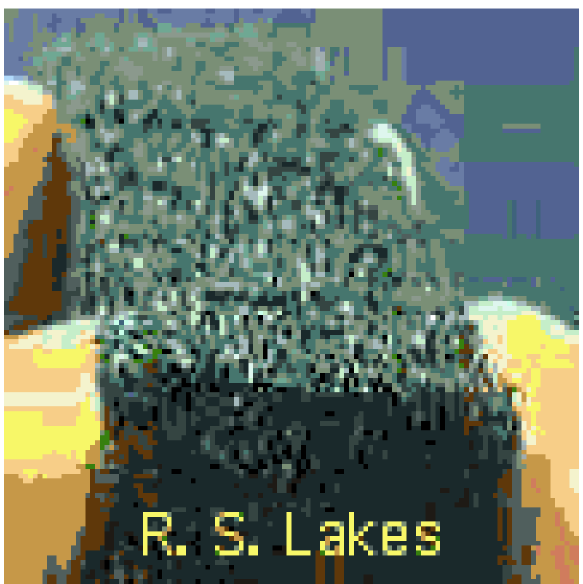
(joint with Andrej Cherkaev, 1995)

Landscape of isotropic materials



Experiment of R. Lakes (1987)

Normal Foam



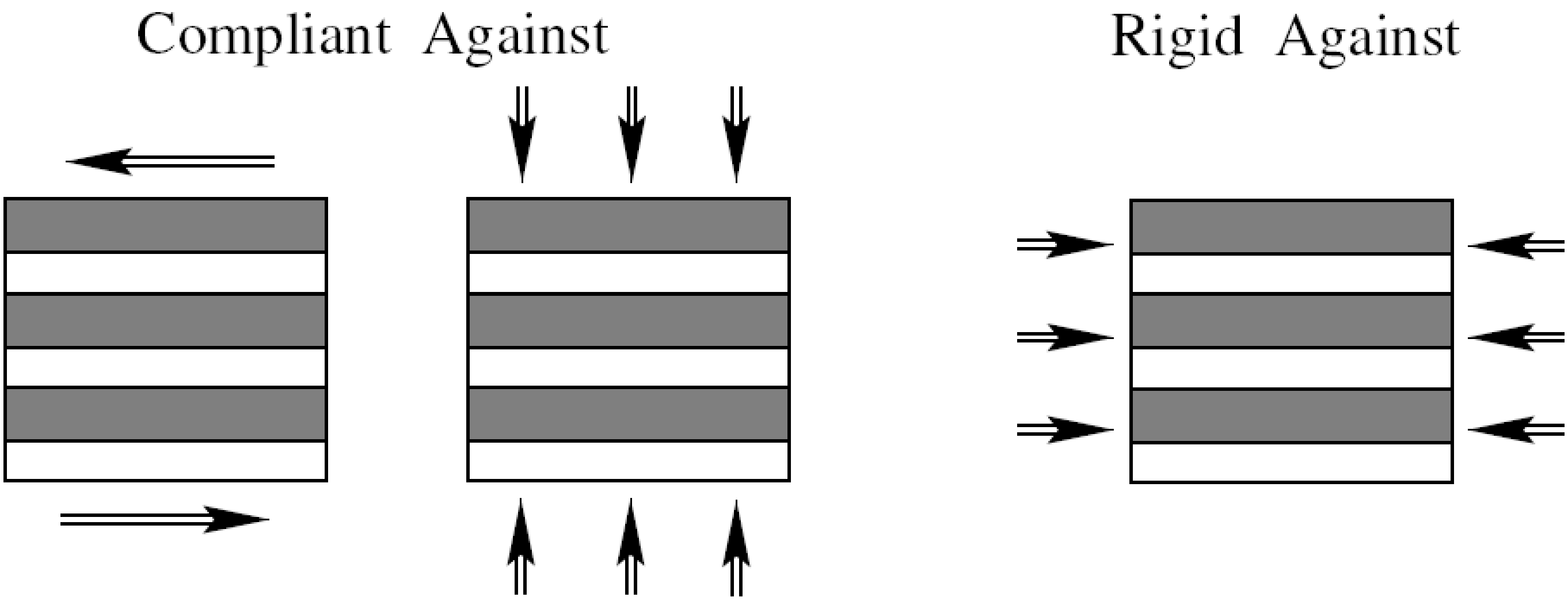
A material with Poisson's ratio close to -1 (a dilational material) is an example of a unimode extremal material.

It is compliant with respect to one strain (dilation) yet stiff with respect to all orthogonal loadings (pure shears)

The elasticity tensor has one eigenvalue which is small, and five eigenvalues which are large.

Can one obtain all other types of extremal materials?

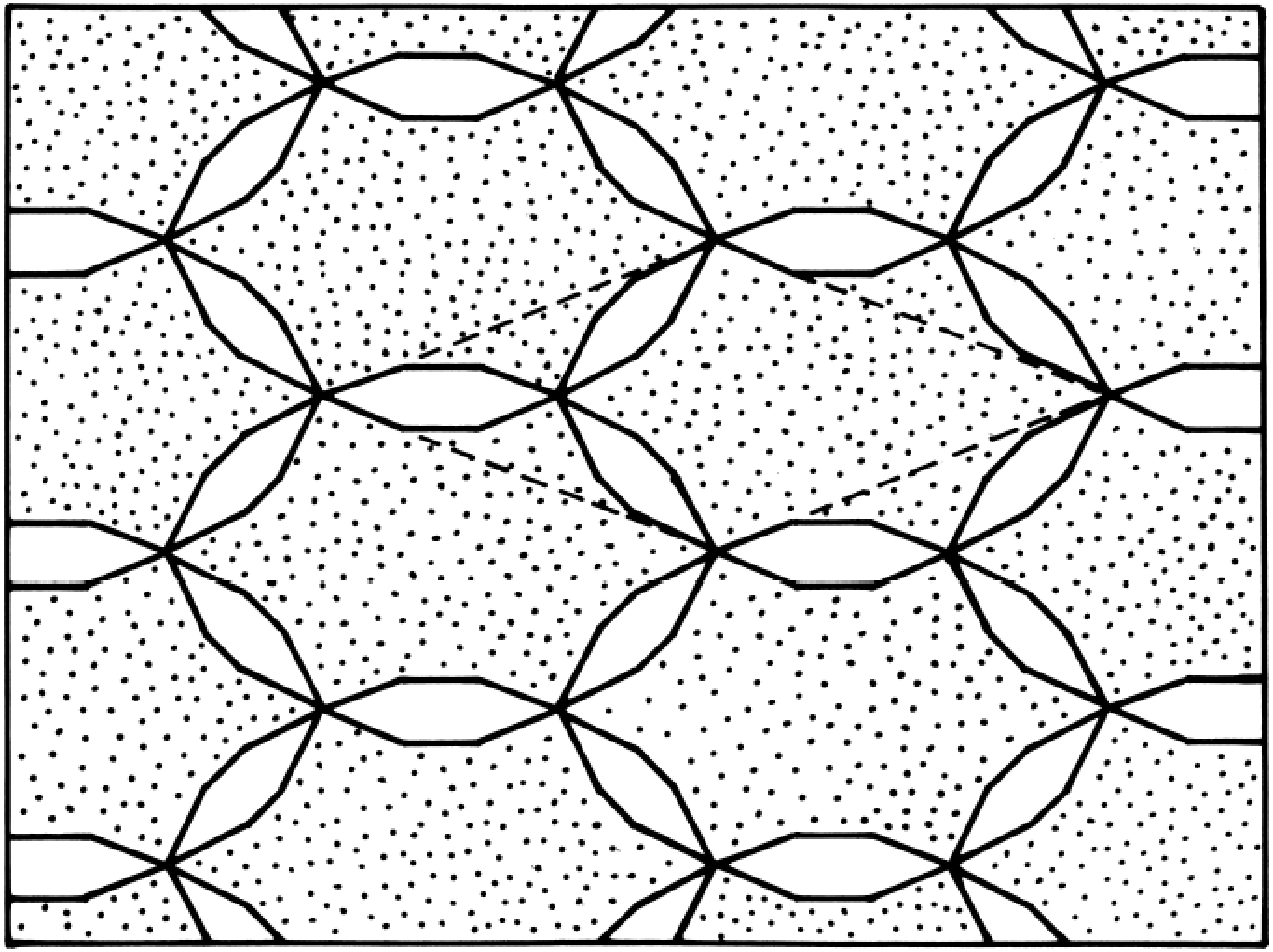
A two-dimensional laminate is a bimodal material



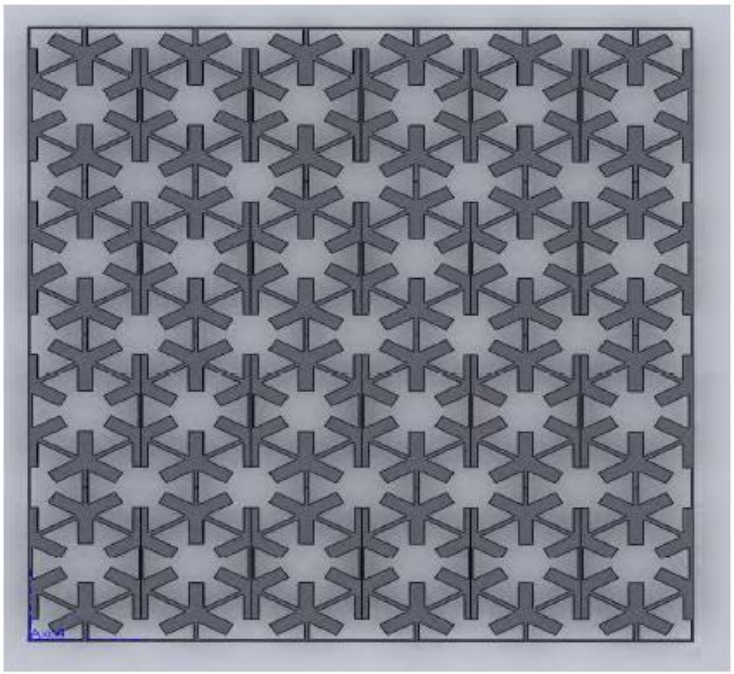
Two eigenvalues of the elasticity tensor are small

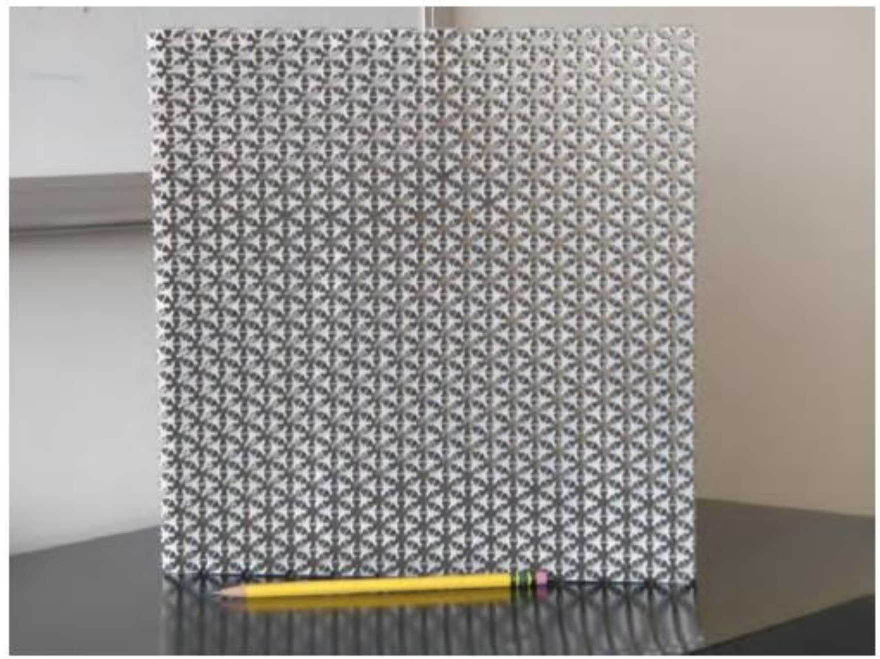
In three-dimensions such a laminate is a trimode extremal material

A bimode material which supports any biaxial loading with positive determinant



Two-dimensional Metal-water constructed by the group of Norris (2012)



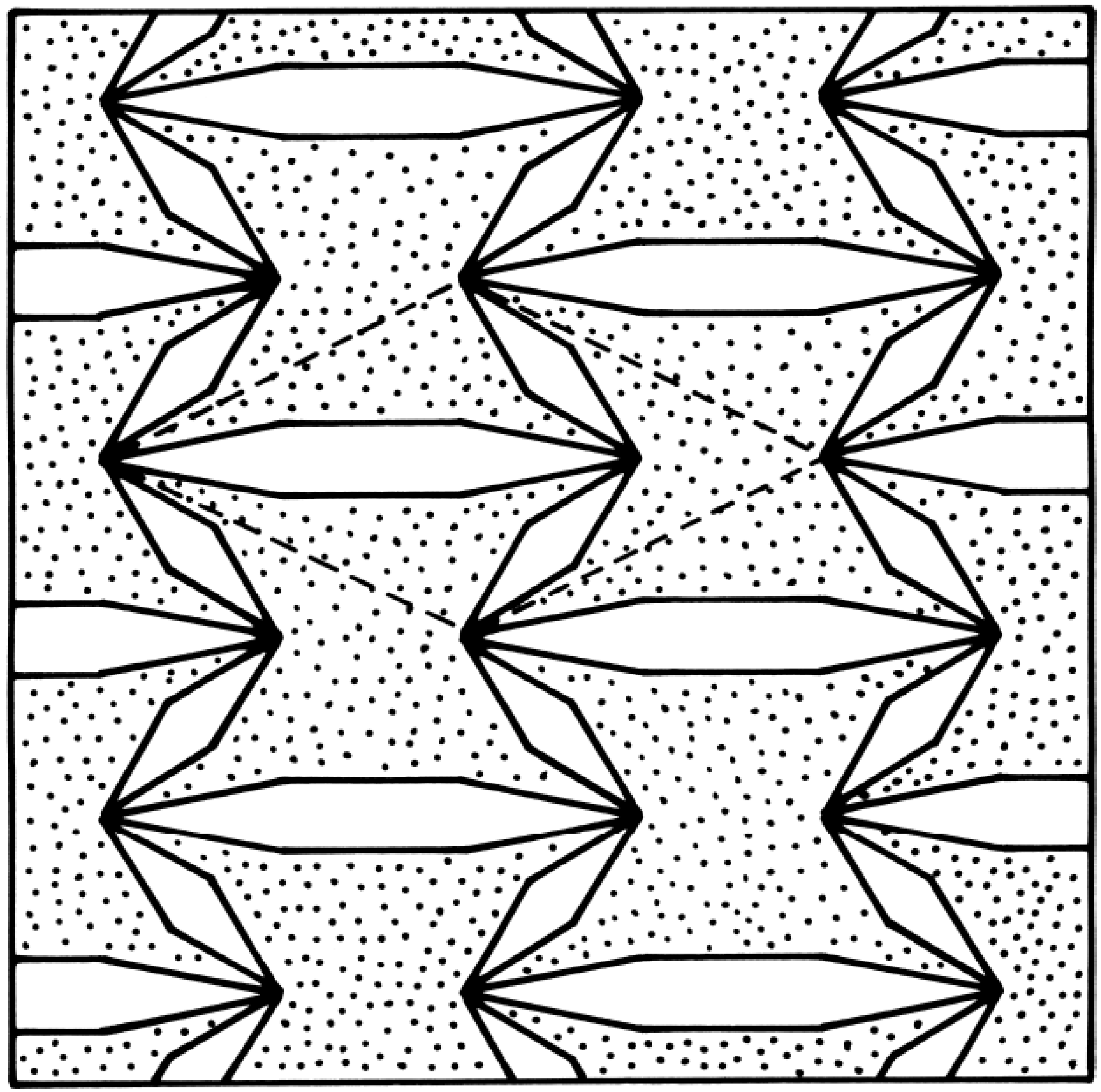


Bulk modulus = 2.25 Gpa

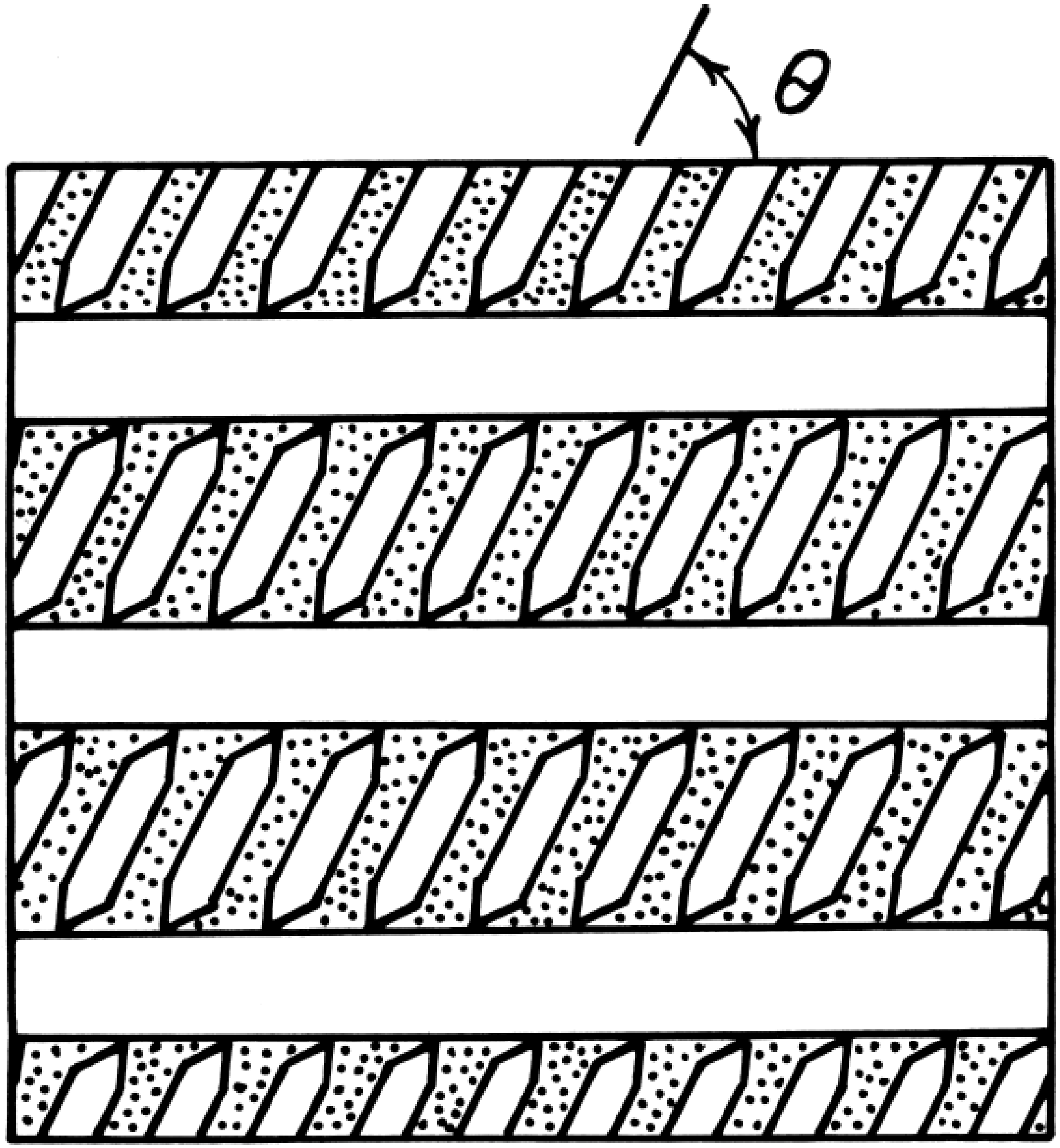
Density = 1000 kg/m^3

Shear modulus = 0.065 GPa

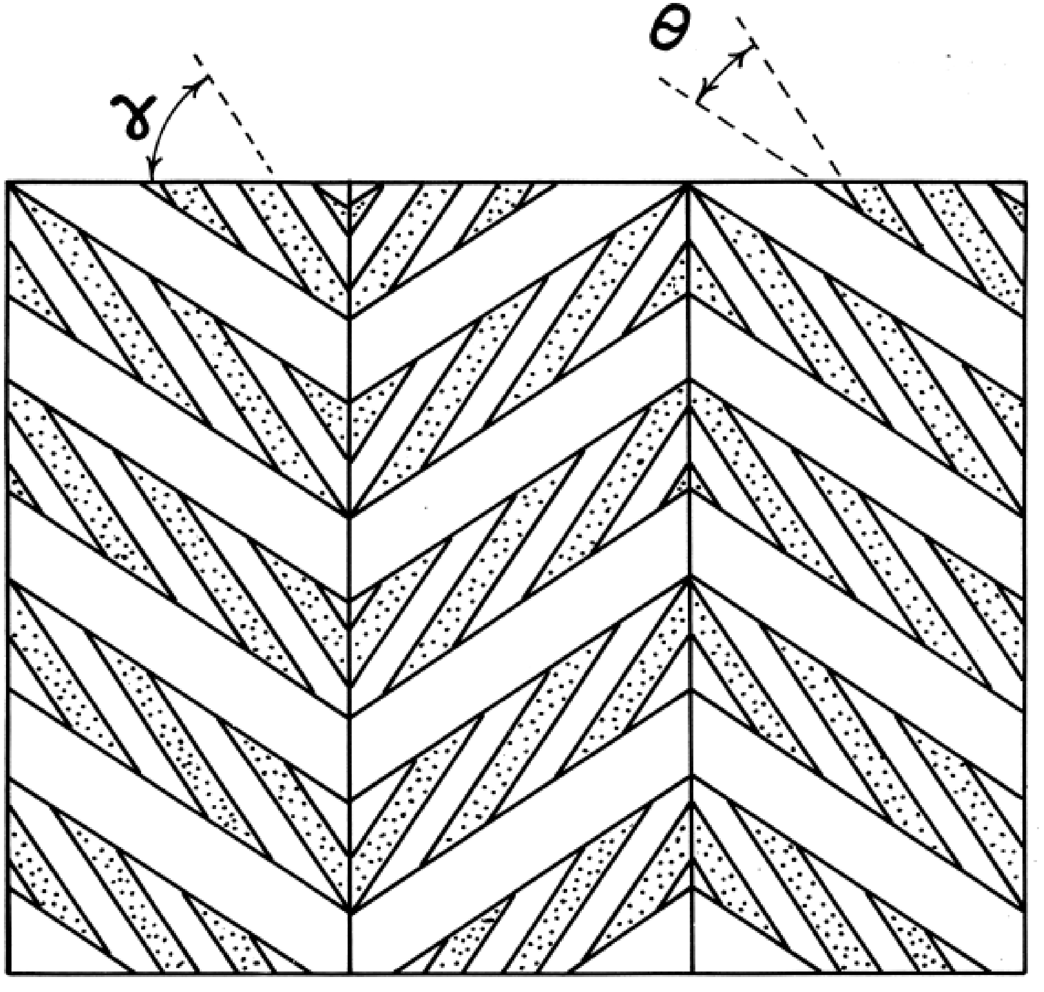
A bimode material which supports any biaxial loading with negative determinant



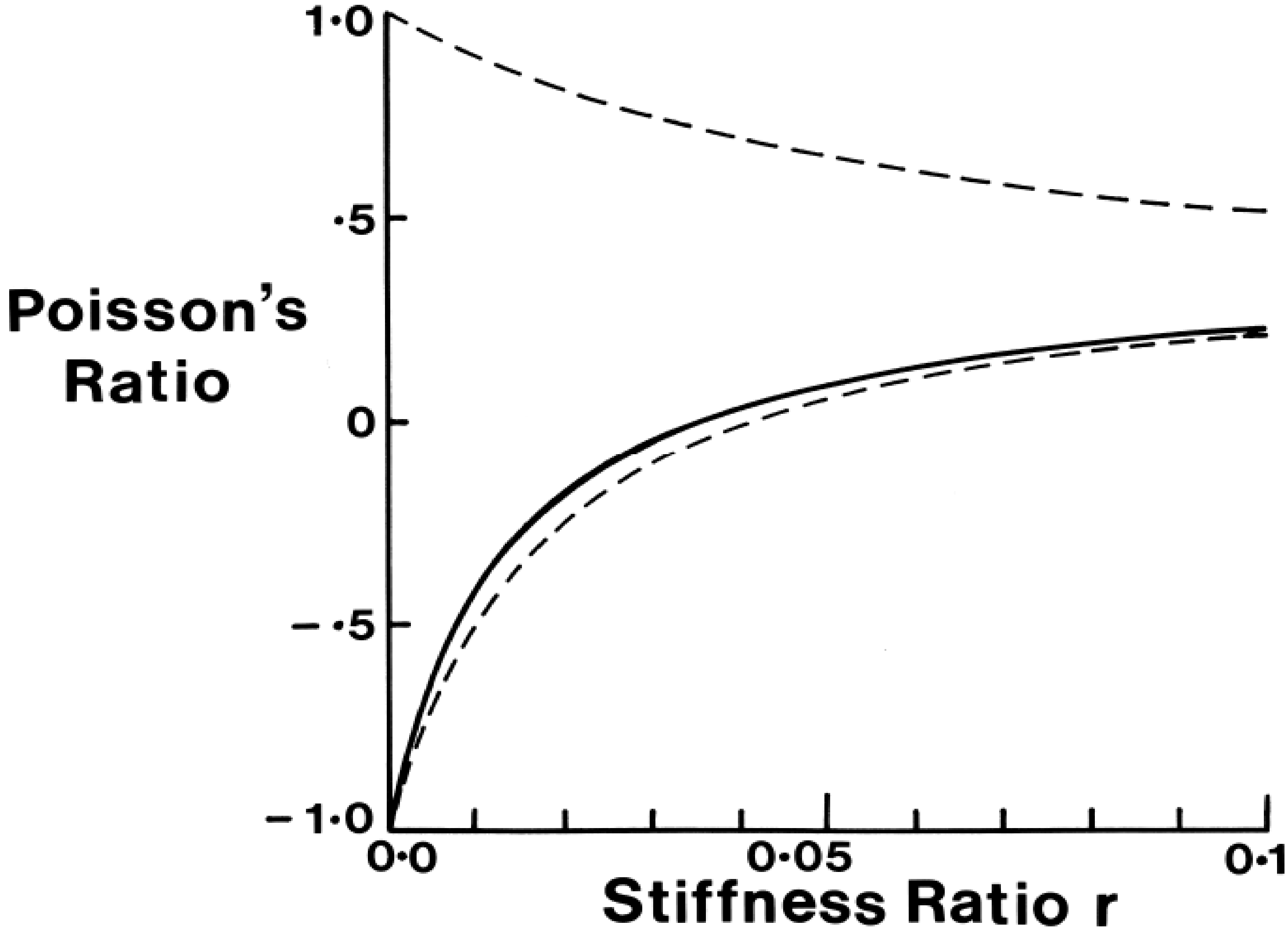
A unimode material which is compliant to any loading with negative determinant



A unimode material which is compliant to any given loading

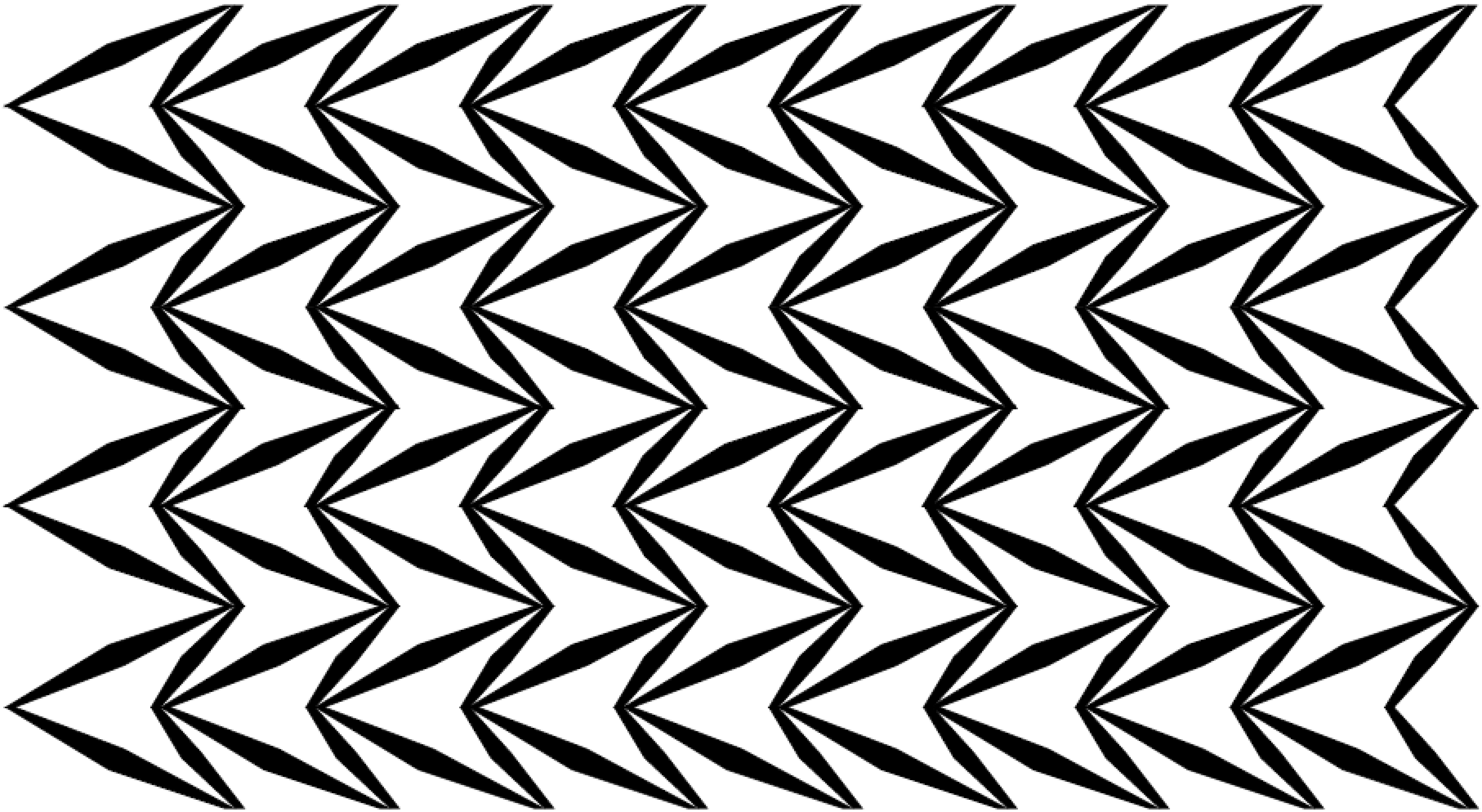


Compare with bounds of Cherkaev and Gibiansky (1993)

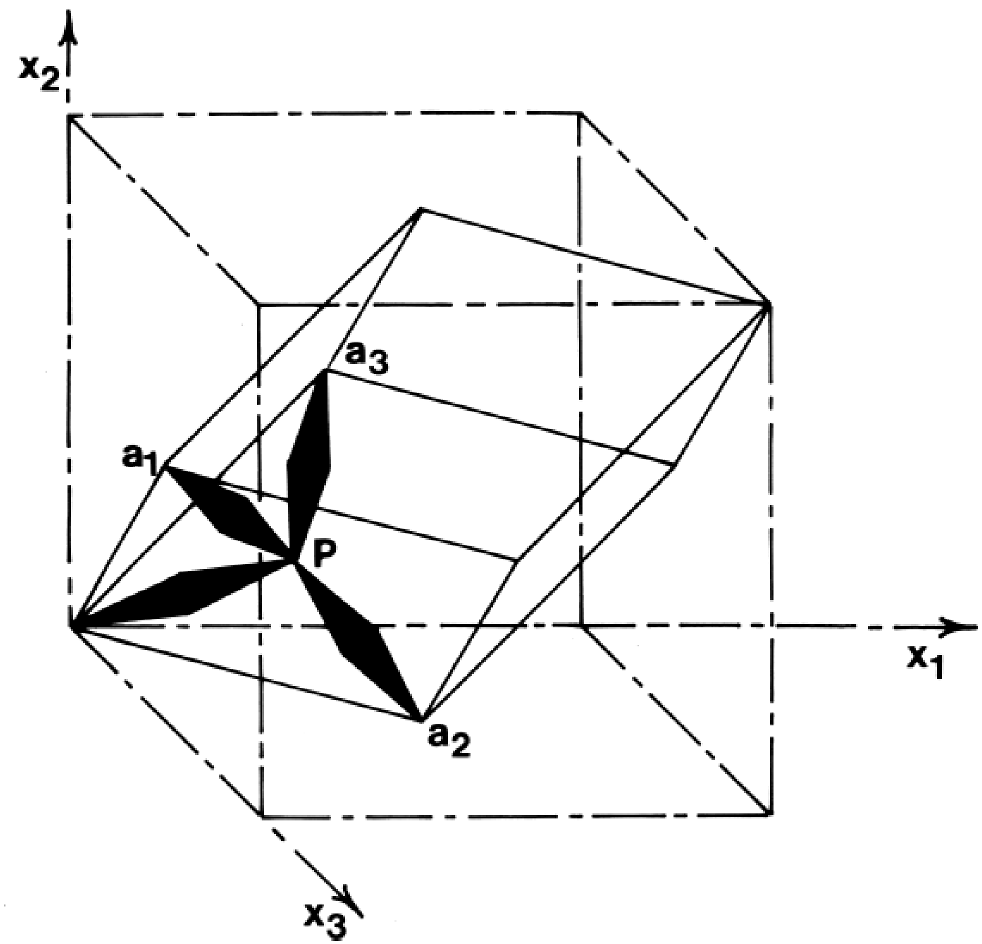


$$\kappa_1 = 2/r,\ \mu_1 = 1/r,\ \kappa_2 = 2, \text{ and } \mu_2 = 1$$

Related structure of Larsen, Sigmund and Bouwstra

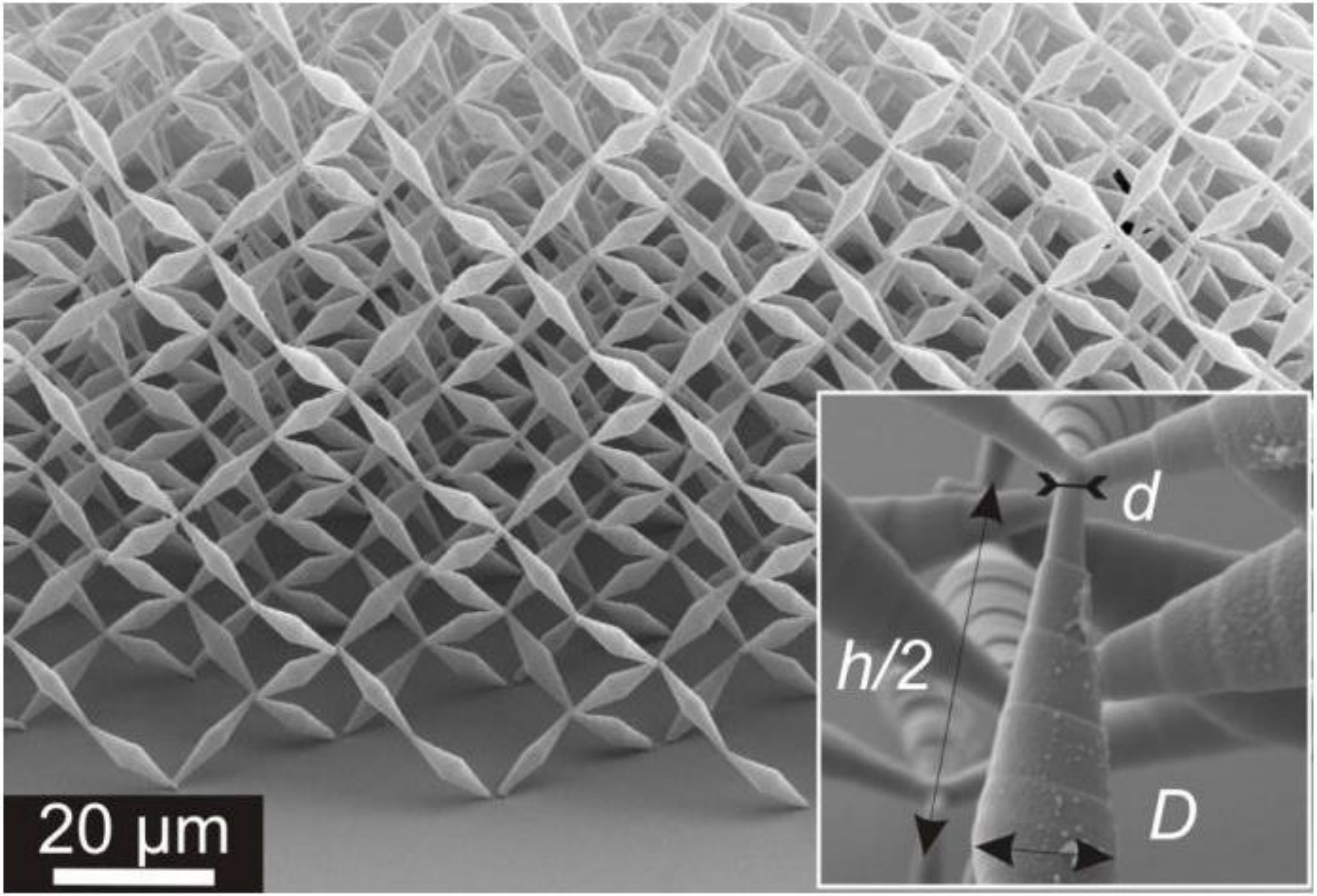


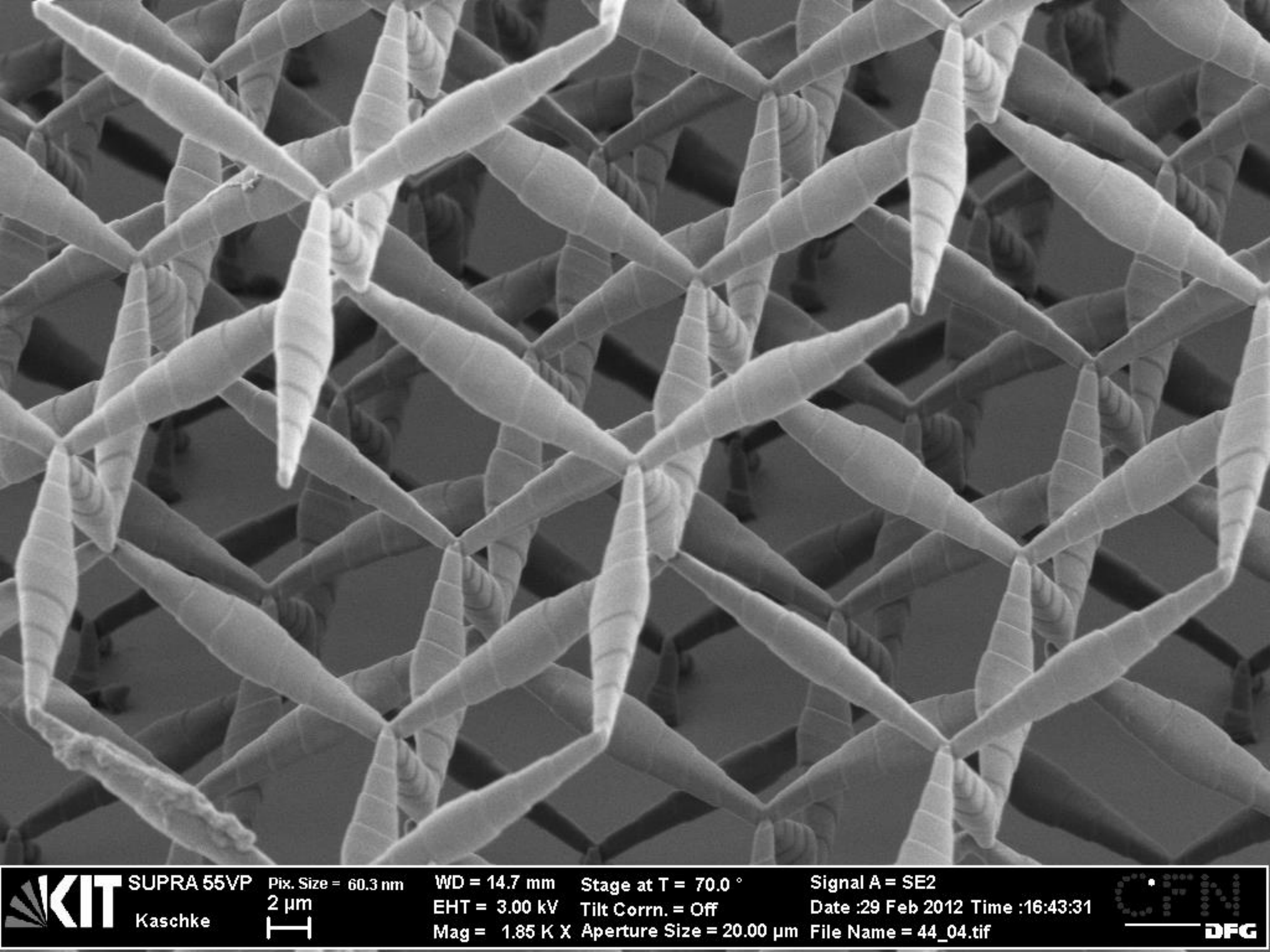
A three dimensional pentamode material which can support any prescribed loading



For hydrostatic loadings some other pentamode structures were found independently by Sigmund

Realization of Kadic et.al. 2012




KIT
SUPRA 55VP
Kaschke
Pix. Size = 60.3 nm
2 µm
WD = 14.7 mm
EHT = 3.00 kV
Mag = 1.85 K X
Stage at T = 70.0 °
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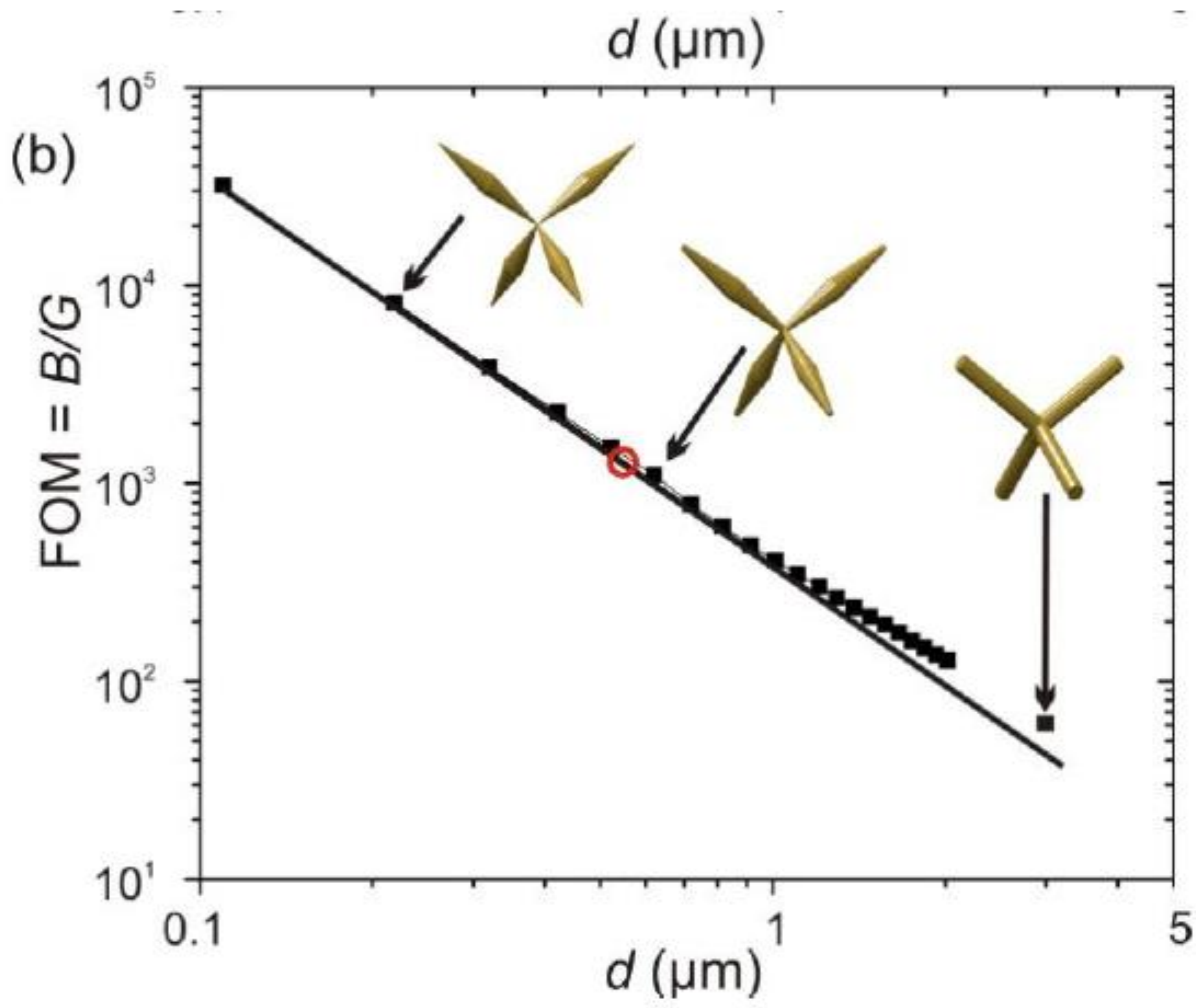
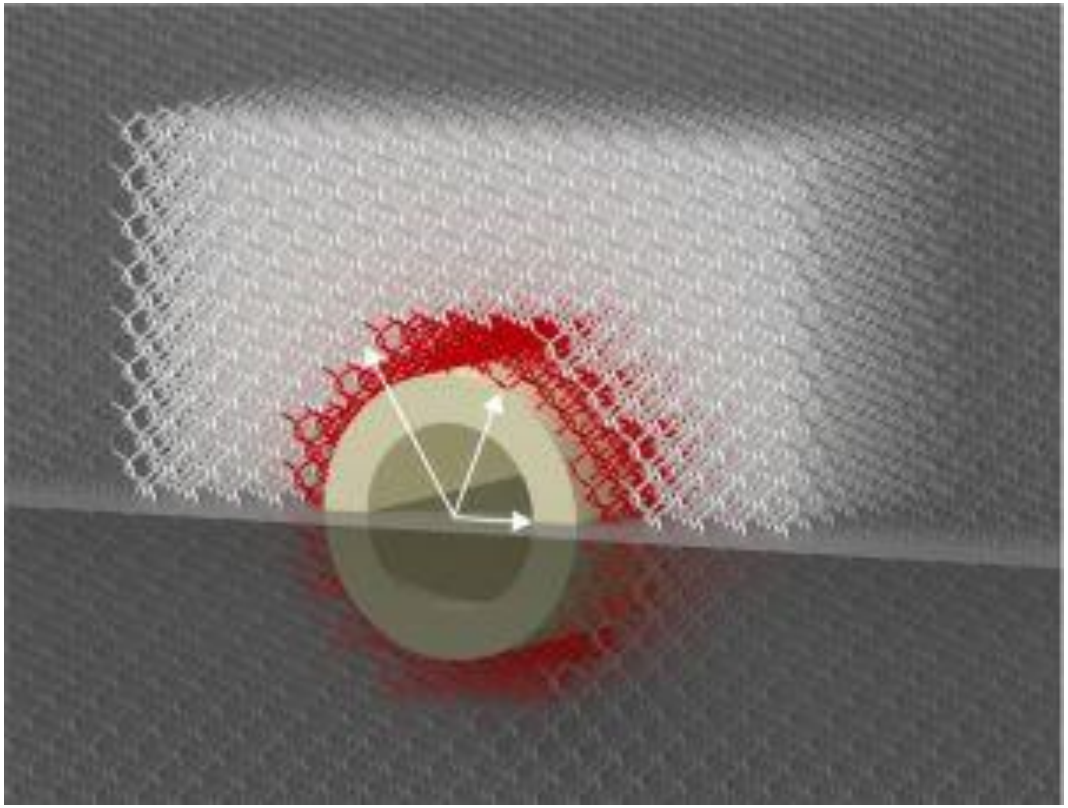
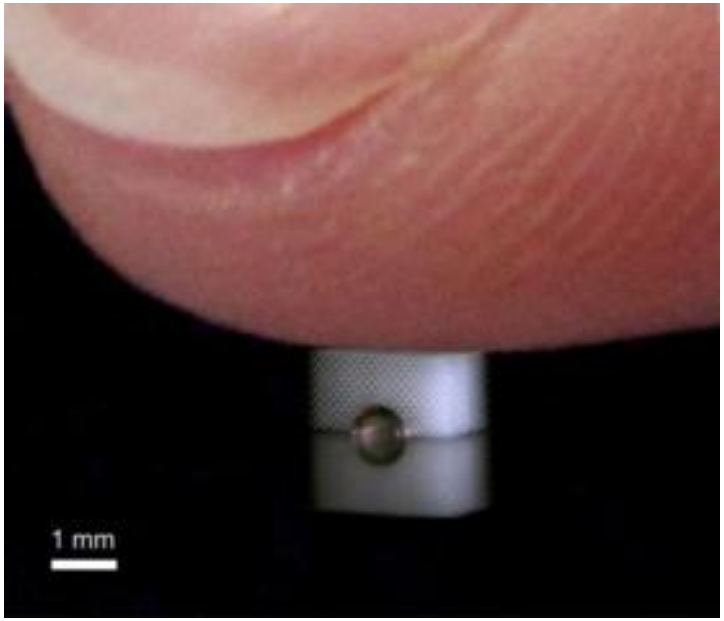
Aperture Size = 20.00 µm
Signal A = SE2
Date :29 Feb 2012 Time :16:43:31
File Name = 44_04.tif
CFN
DFG


d (μm)
10^5
(b)
10^4
FOM = B/G
10^3
10^2
10^1
0.1
1
5
d (μm)

Application of Pentamodes:

Cloak making an object "unfeelable": Buckmann et. al. (2014)





By superimposing appropriate pentamode material structures one can generate all possible unimode, bimode, trimode, and quadmode extremal materials.

Having obtained all possible extremal materials one can use them as building blocks and laminate them together to obtain a material with any desired 6 by 6 symmetric positive definite matrix as its elasticity tensor.
All elasticity tensors are realizable!

Camar Eddine and Seppecher (2003) have characterized all possible non-local responses

One can also get interesting dynamic effects
(joint with Marc Briane and John Willis)

An important parallel:

Maxwell's Equations:

$$\frac{\partial}{\partial x_i}\left(C_{ijk\ell}\frac{\partial E_\ell}{\partial x_k}\right) = \{\omega^2 \boldsymbol{\varepsilon}\mathbf{E}\}_j$$

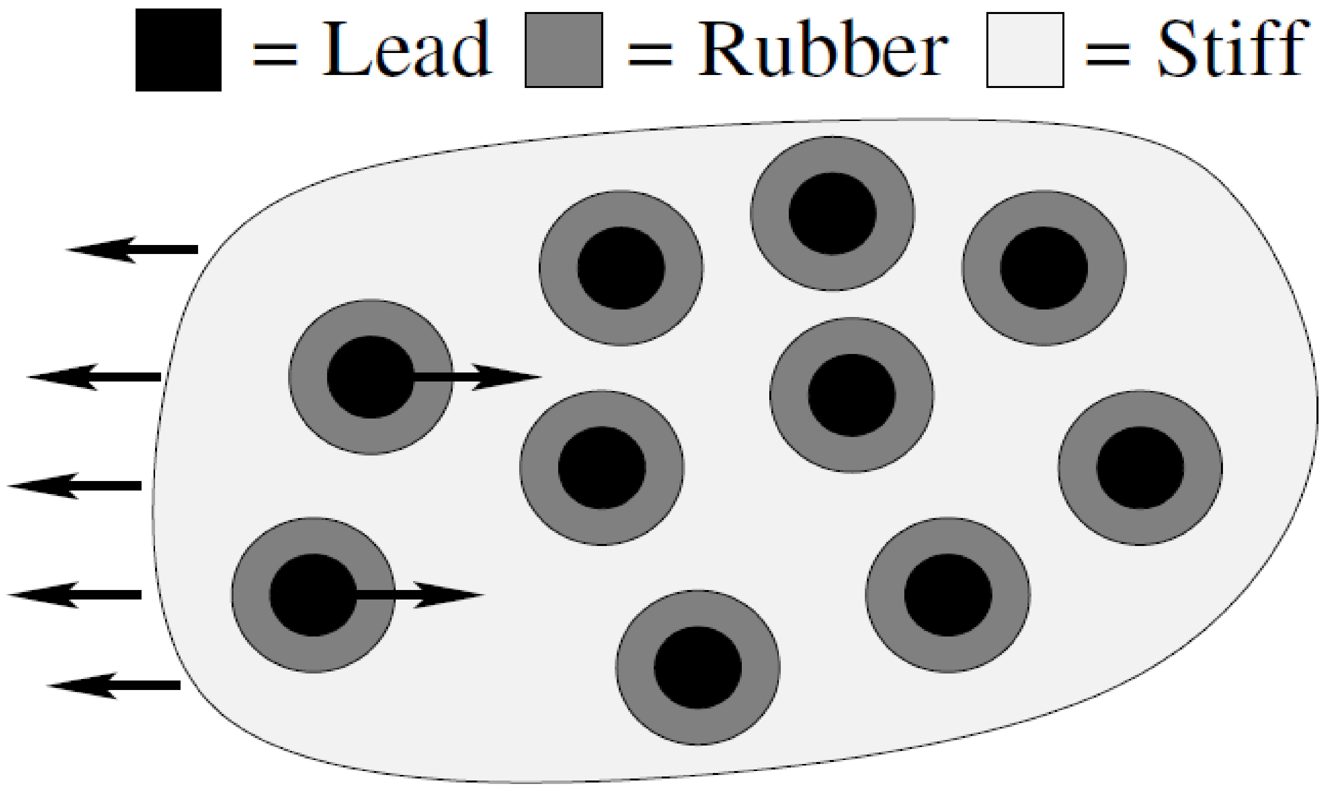
$$C_{ijk\ell} = e_{ijm}e_{k\ell n}\{\boldsymbol{\mu}^{-1}\}_{mn}$$

Continuum Elastodynamics:

$$\frac{\partial}{\partial x_i}\left(C_{ijk\ell}\frac{\partial u_\ell}{\partial x_k}\right) = -\{\omega^2 \boldsymbol{\rho}\mathbf{u}\}_j$$

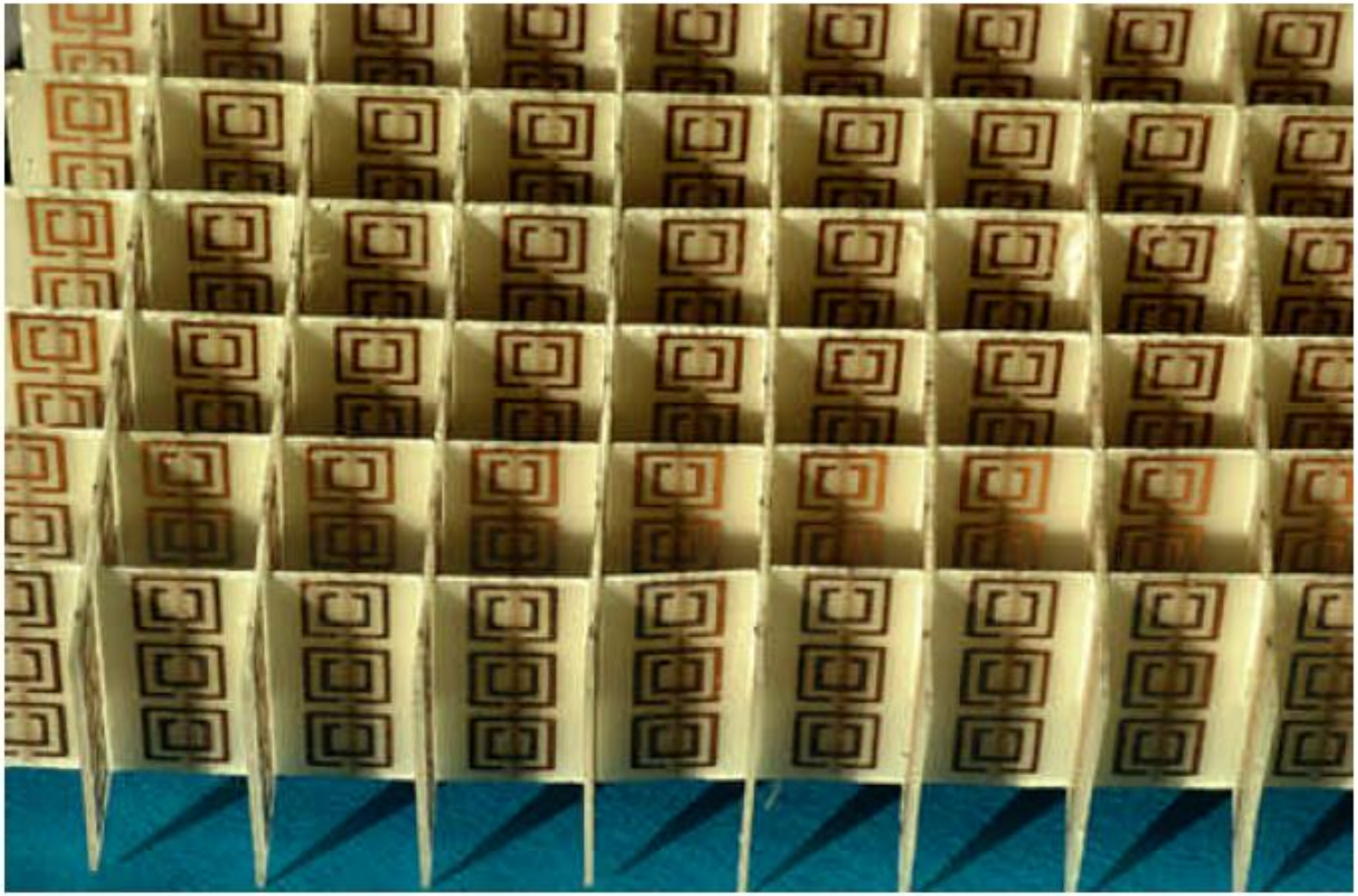
Suggests that $\boldsymbol{\varepsilon}(\omega)$ and $\boldsymbol{\rho}(\omega)$ might have similar properties

Sheng, Zhang, Liu, and Chan (2003) found that materials could exhibit a negative effective density over a range of frequencies



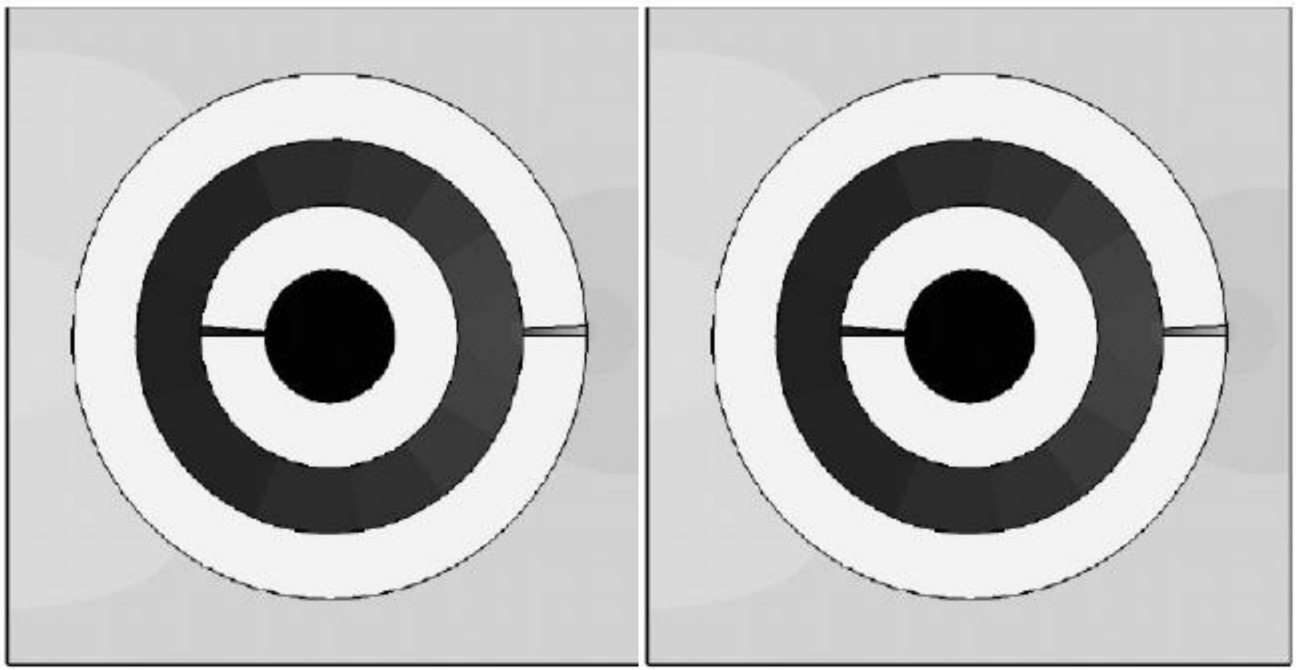
Mathematically the observation goes back to Zhikov (2000) also Bouchitte & Felbacq (2004)

There is a close connection between negative density and negative magnetic permeability



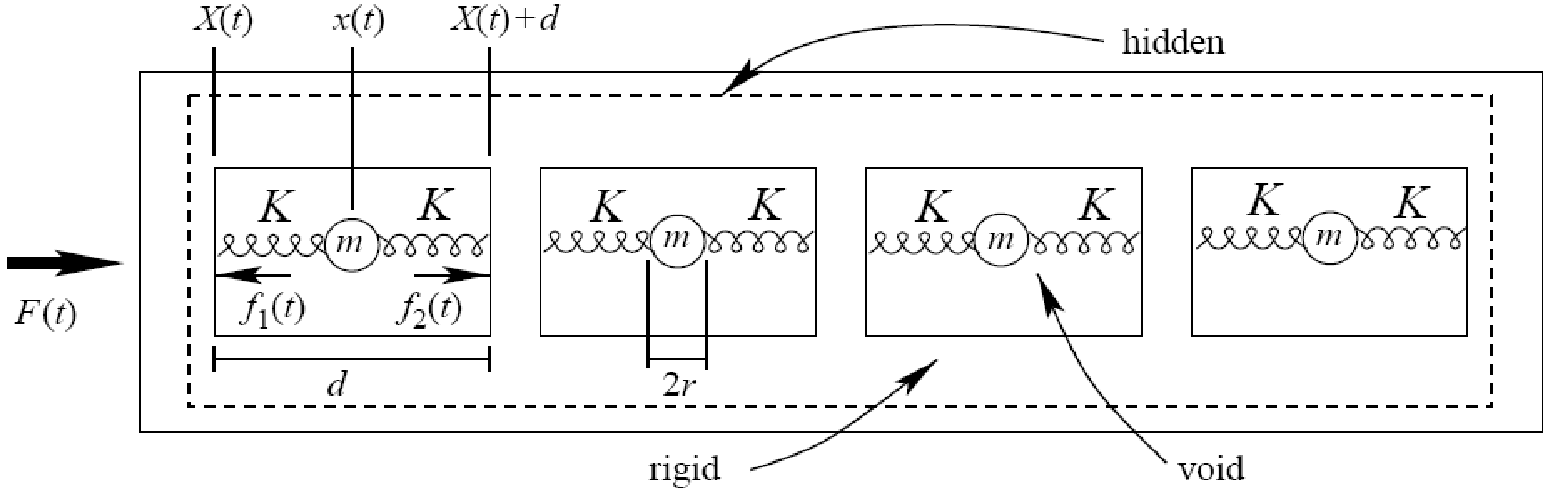
Split ring structure of David Smith

In two dimensions the Helmholtz equation describes both antiplane elastodynamics and TE (or TM) electrodynamics



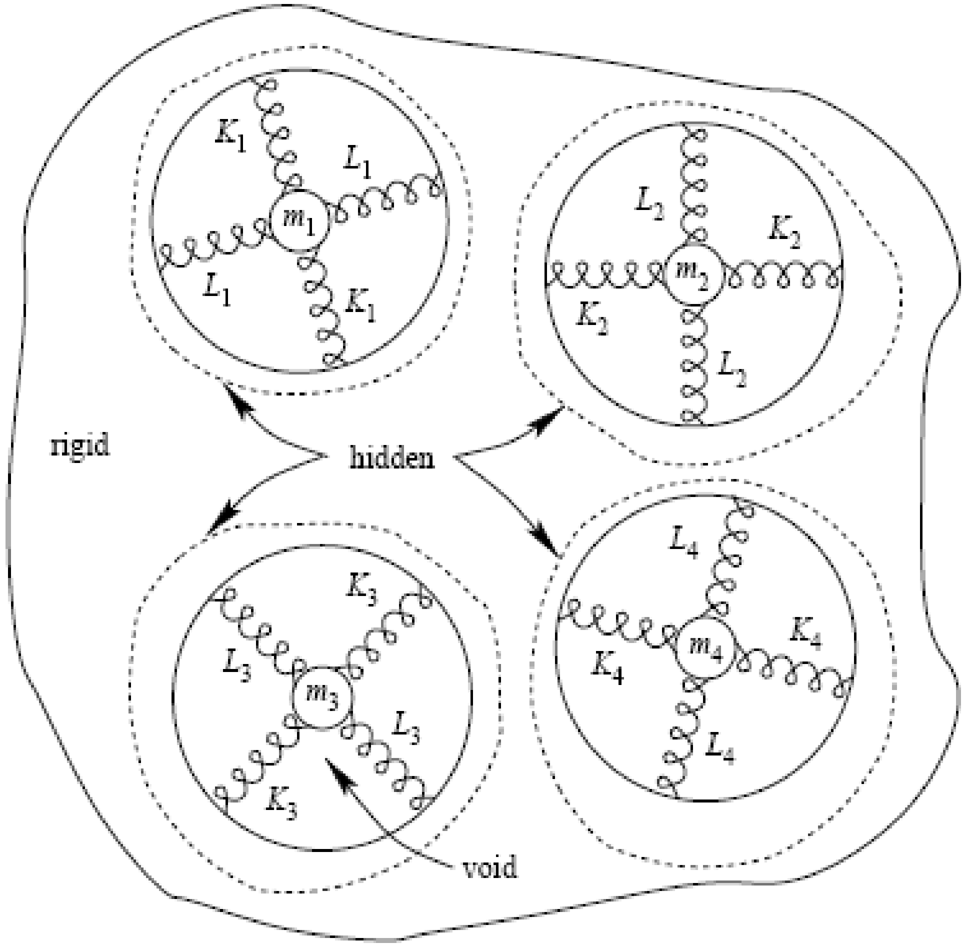
Split ring resonantor structure behaves as an acoustic band gap material (Movchan and Guenneau, 2004)

A simplified one-dimensional model:



$$\hat{P} = M\,\hat{V}, \quad \text{with} \quad M = M_0 + \frac{2Knm}{2K - m\omega^2}\,,$$

Seemingly rigid body



Eigenvectors of the effective mass density can rotate with frequency

Upshot:

For materials with microstructure (and at some level, everything has microstructure)

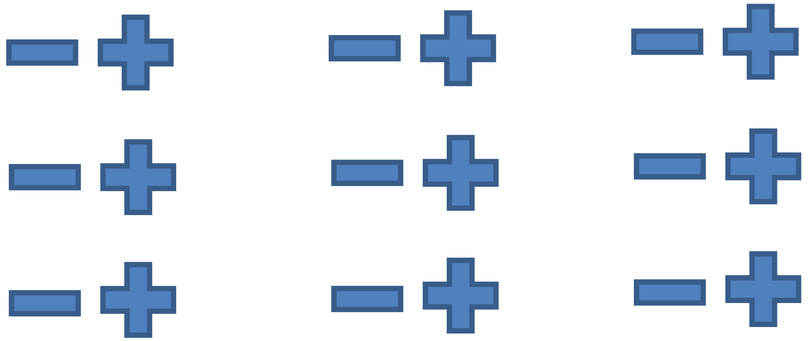
Newton's law

$$\mathbf{F} = m\mathbf{a} \text{ or } \mathbf{p} = m\mathbf{v}, \text{ and } \mathbf{F} = \partial\mathbf{p}/\partial t$$

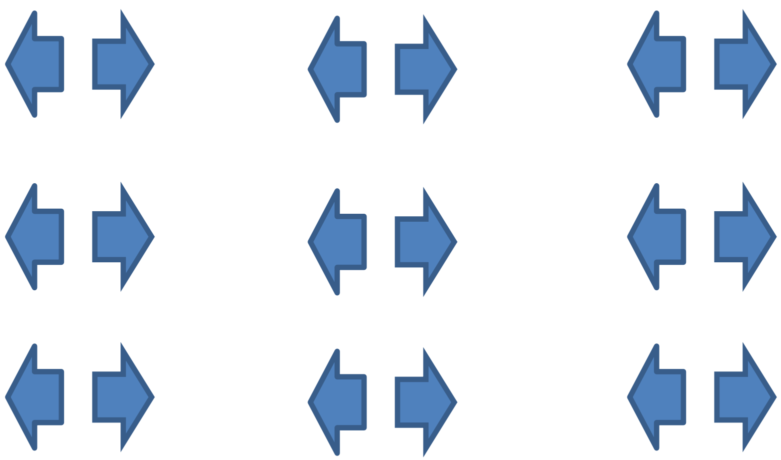
needs to be replaced by

$$\mathbf{p}(t) = \int_{t'=-\infty}^{t} \mathbf{K}(t-t')\mathbf{v}(t')dt', \quad \mathbf{F} = \partial\mathbf{p}/\partial t$$

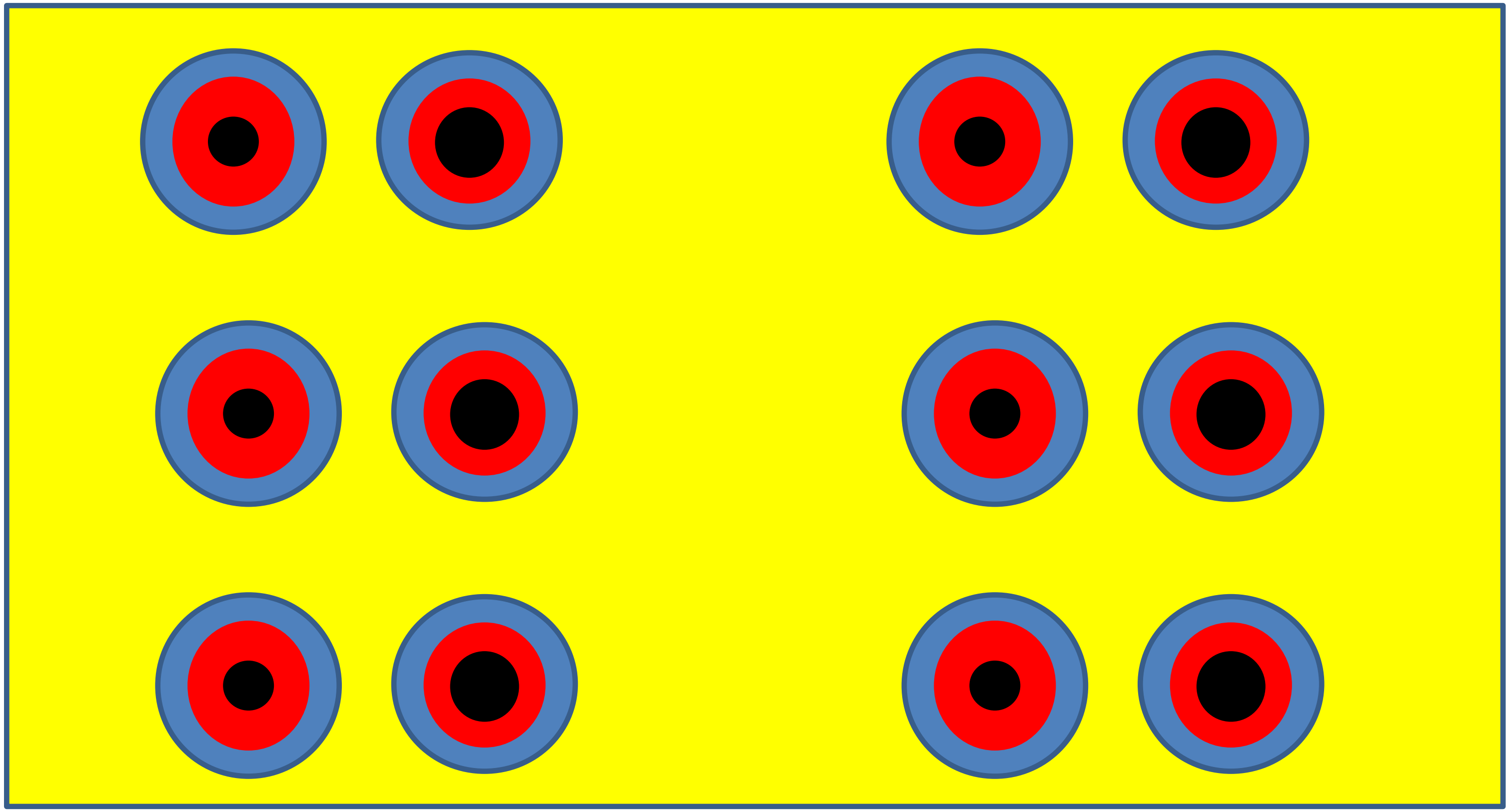
[On modification's to Newton's second law and linear continuum elastodynamics, with J.R. Willis]



Electric dipole array generates polarization field

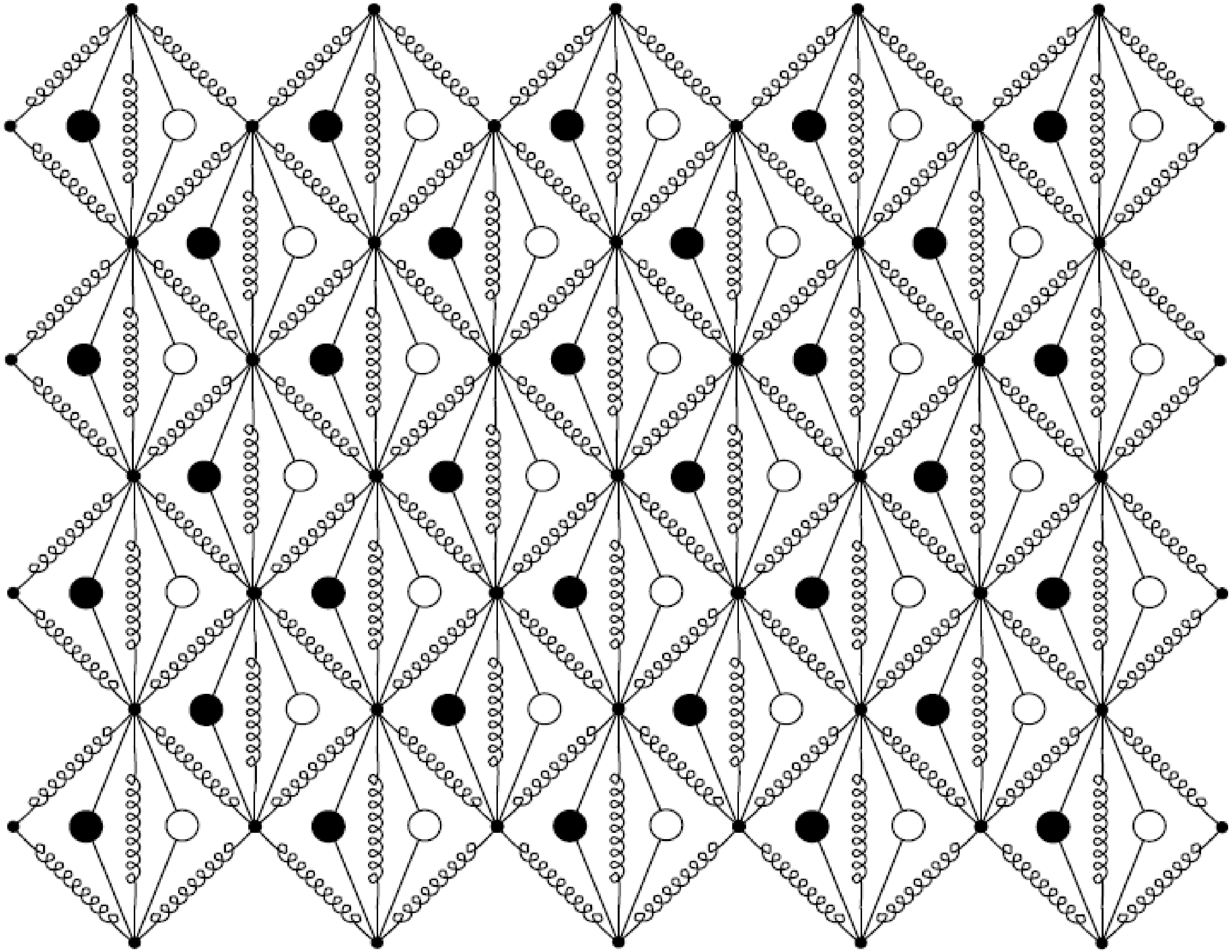


Force dipole array generates stress field



Yellow=Compliant, Blue=Stiff
Red=Rubber, Black=Lead

Time harmonic acceleration with no strain gives stress: Example of a Willis material



The Black circles have positive effective mass
The White circles have negative effective mass

Such materials may be useful for elastic cloaking

Linear elastic equations under a Galilean transformation

$$\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix} = \underbrace{\begin{pmatrix} -\boldsymbol{\mathcal{C}}(\mathbf{x}) & 0 \\ 0 & \rho(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} -\frac{1}{2}\left[\nabla \mathbf{v} + \nabla \mathbf{v}^T\right] \\ \frac{\partial \mathbf{v}}{\partial t} \end{pmatrix} \qquad x_4 = -t,$$

$$\overline{\nabla} = \begin{pmatrix} \nabla \\ \frac{\partial}{\partial x_4} \end{pmatrix} = \begin{pmatrix} \nabla \\ -\frac{\partial}{\partial t} \end{pmatrix}, \quad J_{ik} = -\frac{\partial \sigma_{ik}}{\partial t}, \quad \text{for } i,k = 1,2,3, \quad J_{4k} = -\{\nabla \cdot \boldsymbol{\sigma}\}_k,$$

$$\overline{\nabla} \cdot \mathbf{J} = 0, \quad \mathbf{J} = \mathbf{Z}\overline{\nabla}\mathbf{v}.$$ (looks a bit like conductivity)

Galilean transformation: $\overline{\mathbf{x}}' = \mathbf{A}\overline{\mathbf{x}}, \quad \text{with } \mathbf{A} = \begin{pmatrix} \mathbf{I} & \mathbf{w} \\ 0 & 1 \end{pmatrix},$

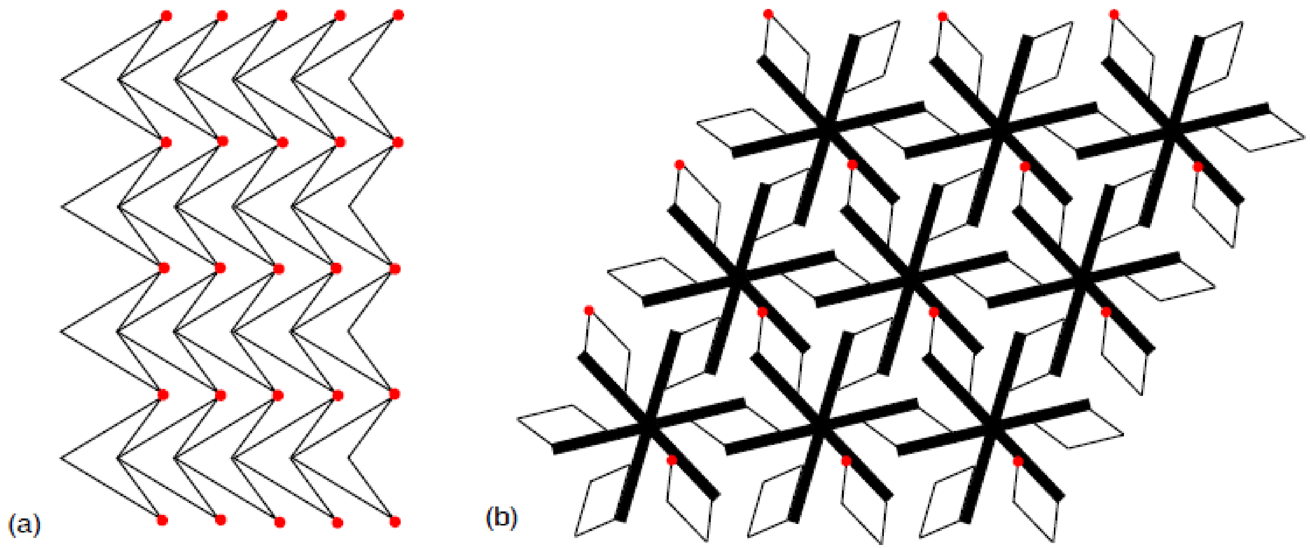
$$\begin{aligned}
\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}'}{\partial t'} \\ \nabla' \cdot \boldsymbol{\sigma}' \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\mathcal{I}} & \mathbf{w}\mathbf{I} \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{w}(\nabla \cdot \boldsymbol{\sigma})^T \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix}, \\
\begin{pmatrix} -\nabla' \mathbf{v}' \\ \frac{\partial \mathbf{v}'}{\partial t'} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\mathcal{I}} & 0 \\ \mathbf{I}\mathbf{w}^T & \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} -\nabla \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} \end{pmatrix} = \begin{pmatrix} -\nabla \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{w}^T \nabla \mathbf{v} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathbf{Z}'(\overline{\mathbf{x}}') &= \begin{pmatrix} \boldsymbol{\mathcal{I}} & \mathbf{w}\mathbf{I} \\ 0 & \mathbf{I} \end{pmatrix} \mathbf{Z}(\mathbf{x}) \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{I}\mathbf{w}^T & \mathbf{I} \end{pmatrix} \\
&= \begin{pmatrix} -\boldsymbol{\mathcal{C}}(\mathbf{x}) + \mathbf{w}\rho(\mathbf{x})\mathbf{w}^T & \mathbf{w}\rho(\mathbf{x}) \\ \rho(\mathbf{x})\mathbf{w}^T & \rho(\mathbf{x}) \end{pmatrix}.
\end{aligned}$$

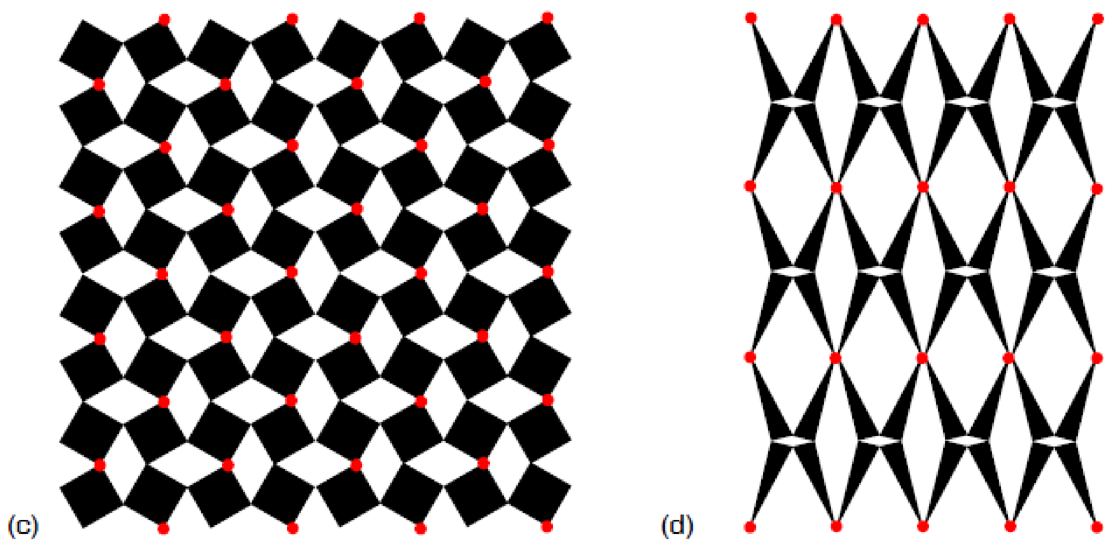
Has Willis type couplings!
Also a non-symmetric stress

How do you define unimode, bimode, trimode etc. in the non-linear case?

Examples of nonlinear 2d unimode materials

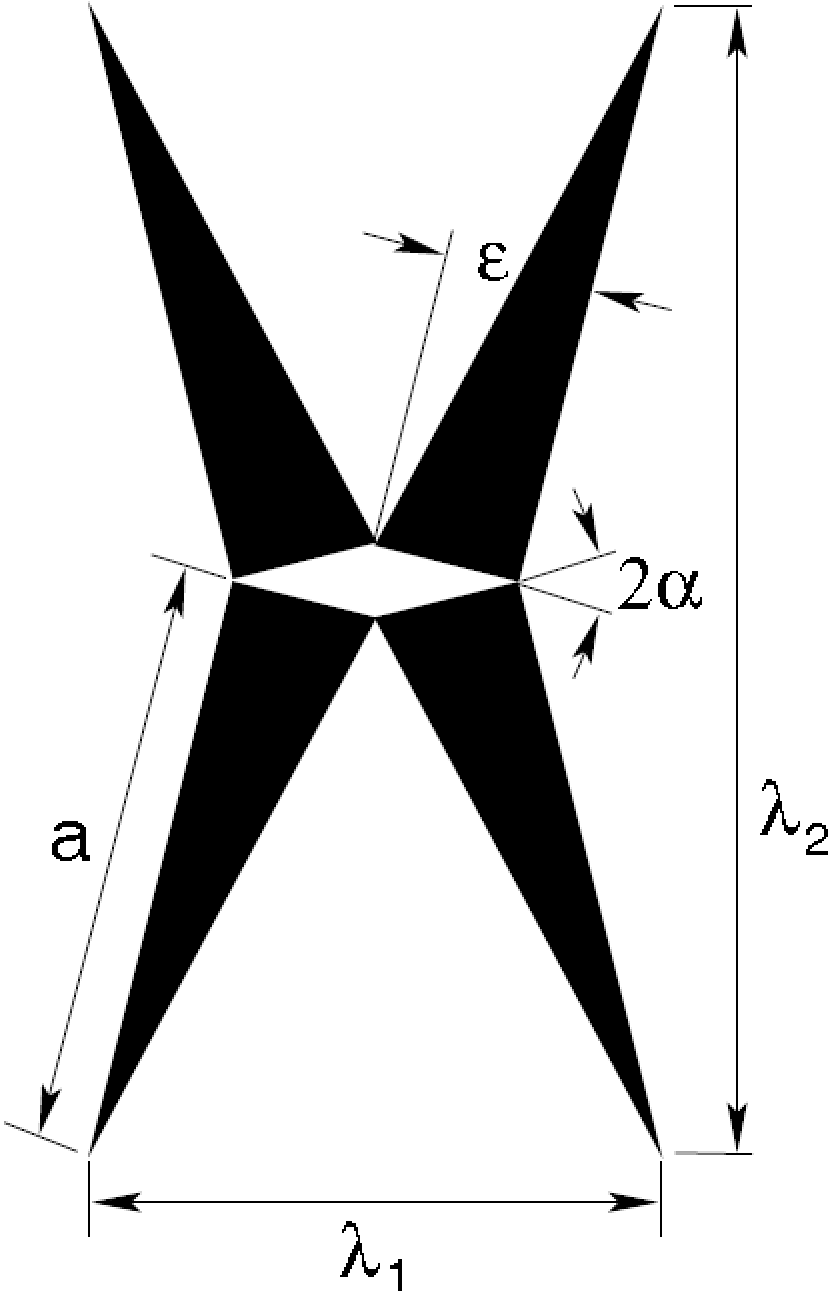


Larsen et. al. Milton



Grima and Evans

The Expander



(λ_1, λ_2) lies on the ellipse

$$(a\lambda_1 - \varepsilon\lambda_2)^2 = a^2(4a^2 - \lambda_2^2)$$

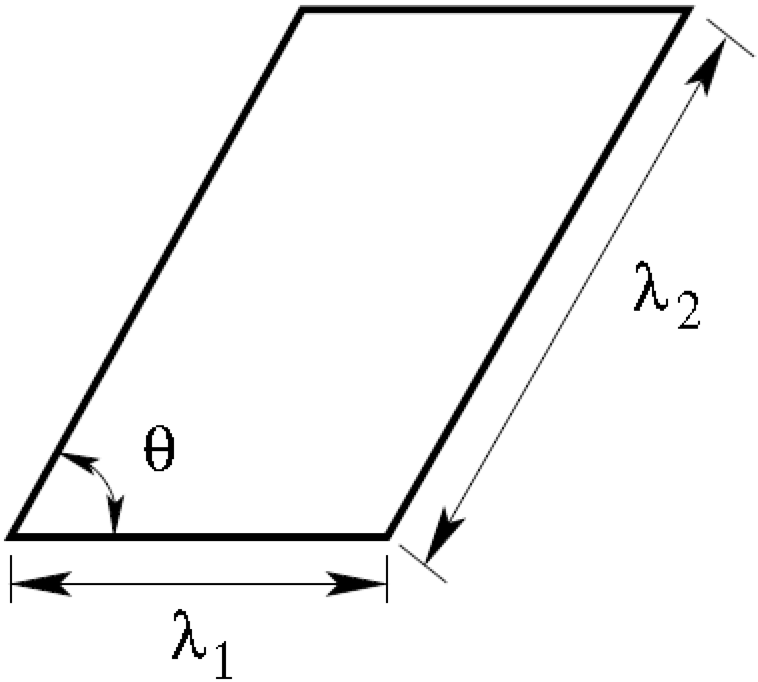
So what functions

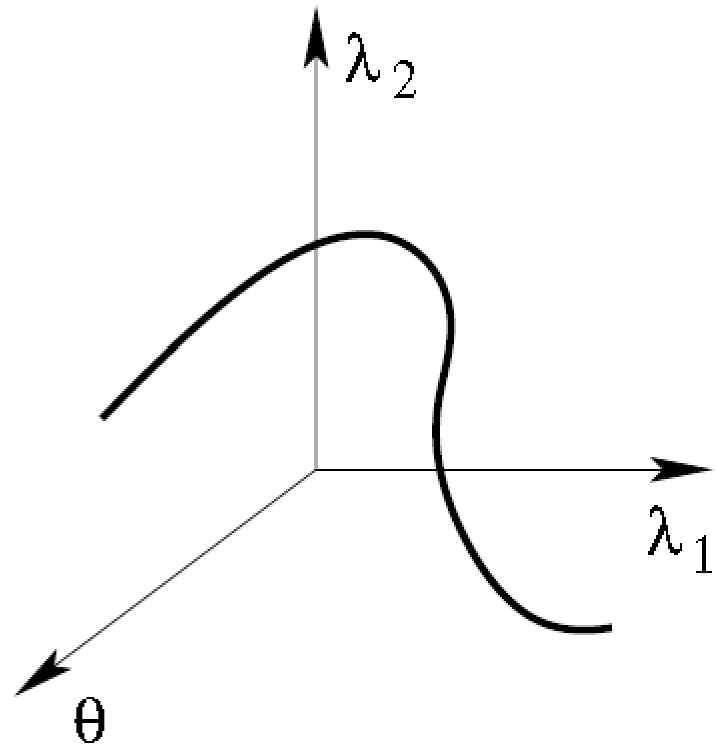
$$\lambda_2 = f(\lambda_1)$$

are realizable?

Main Result: Everything

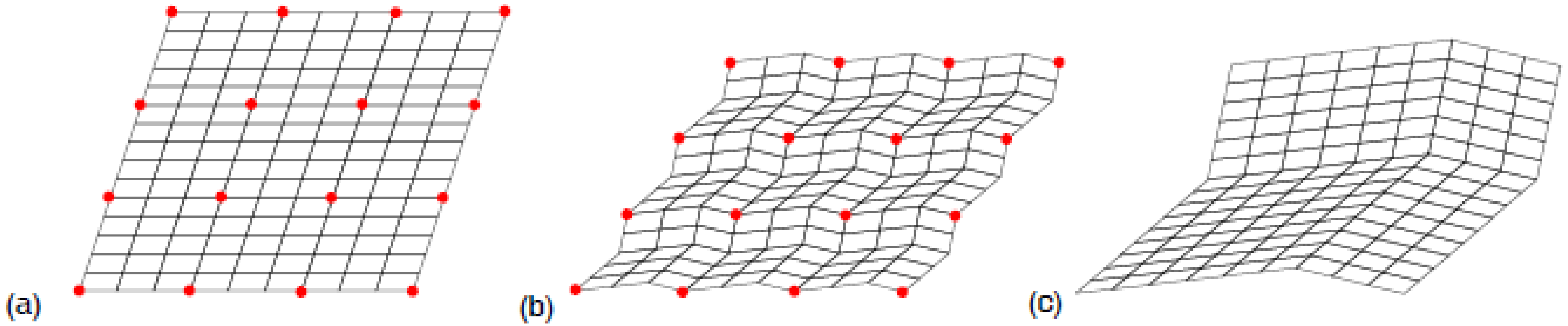
Unimode:





What trajectories $\lambda_1(t) = \lambda_2(t) = \theta(t)$ are realizable?

In a bimode material there is a surface of realizable motions.



A parallelogram array of bars is a non-linear non-affine trimode material

MAIN RESULT FOR AFFINE UNIMODE MATERIALS

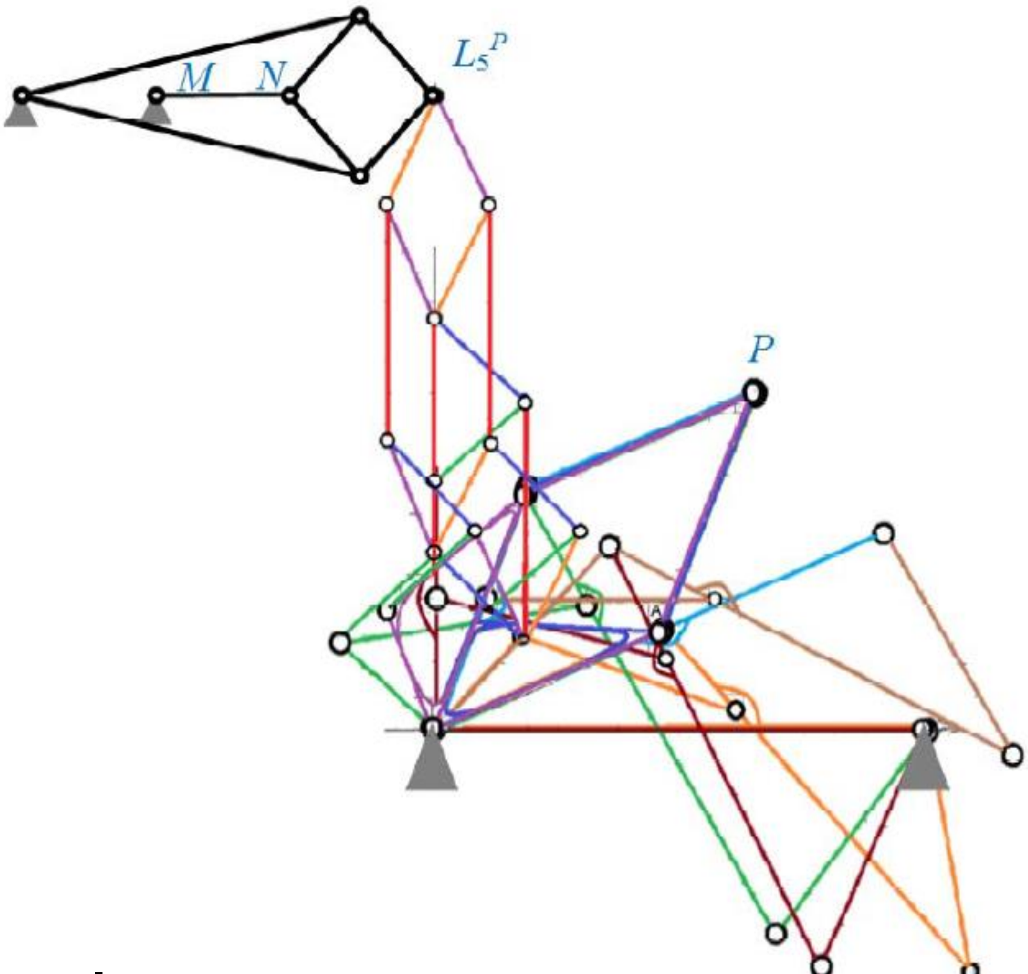
What trajectories are realizable in deformation space?

Answer: Anything! (so long as the deformation remains non-degenerate along the trajectory)

True both for two and three dimensional materials

USES A HIGHLY MULTISCALE CONSTRUCTION

In some sense its an extension to materials of Kempe's famous 1876 universality theorem, proved in 2002 by Kapovich and Millson



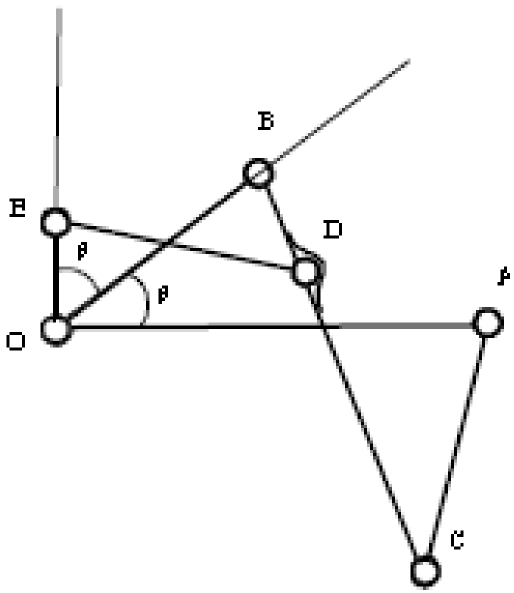
P Traces

$$(x-y)\,(x+y+1/\sqrt{2}) = 0$$

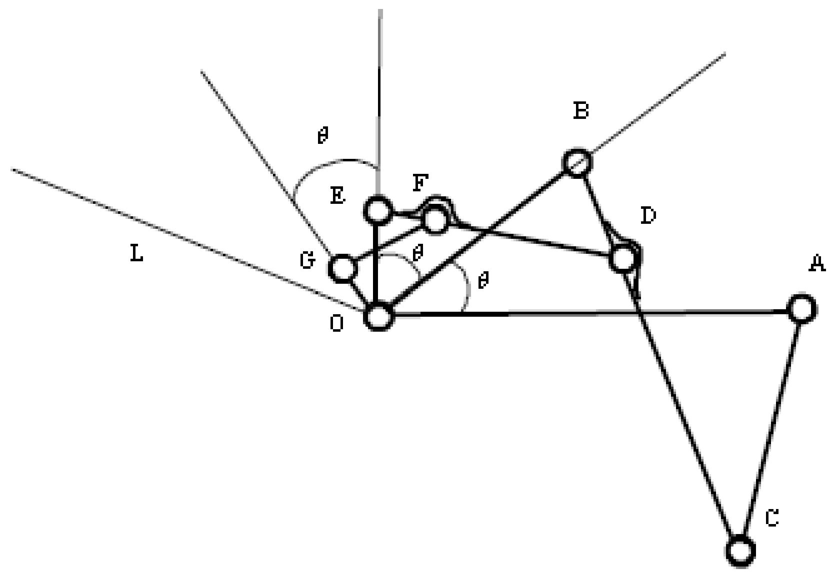
Example of Saxena

Rods can cross

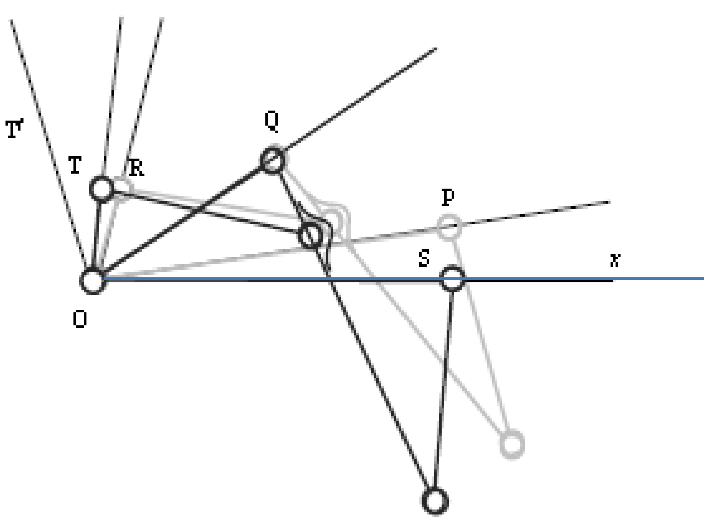
Reversor



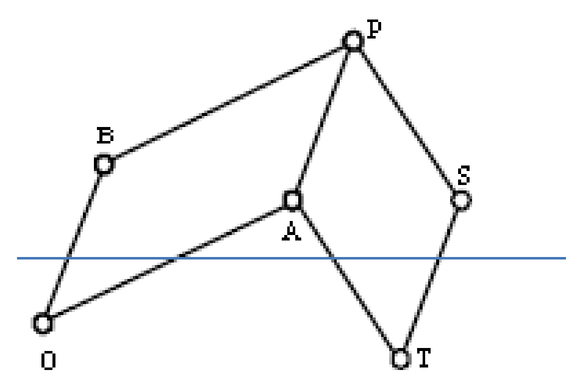
Multiplicactor

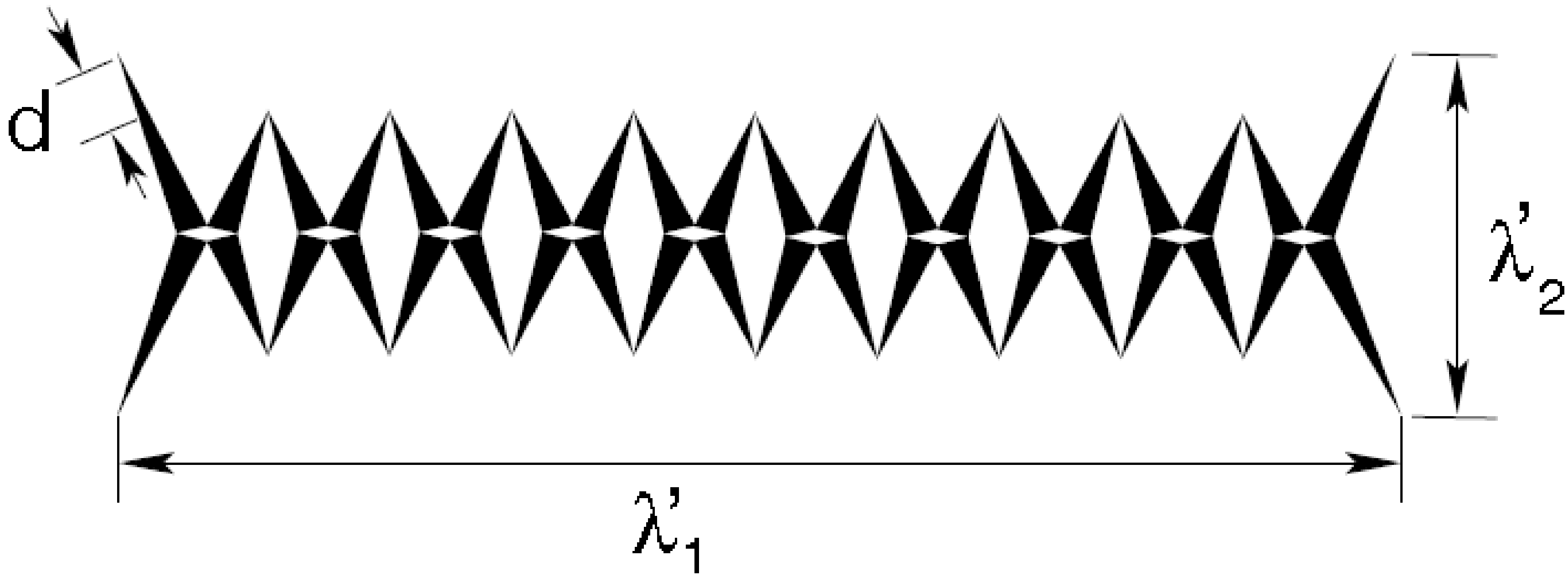


Additor



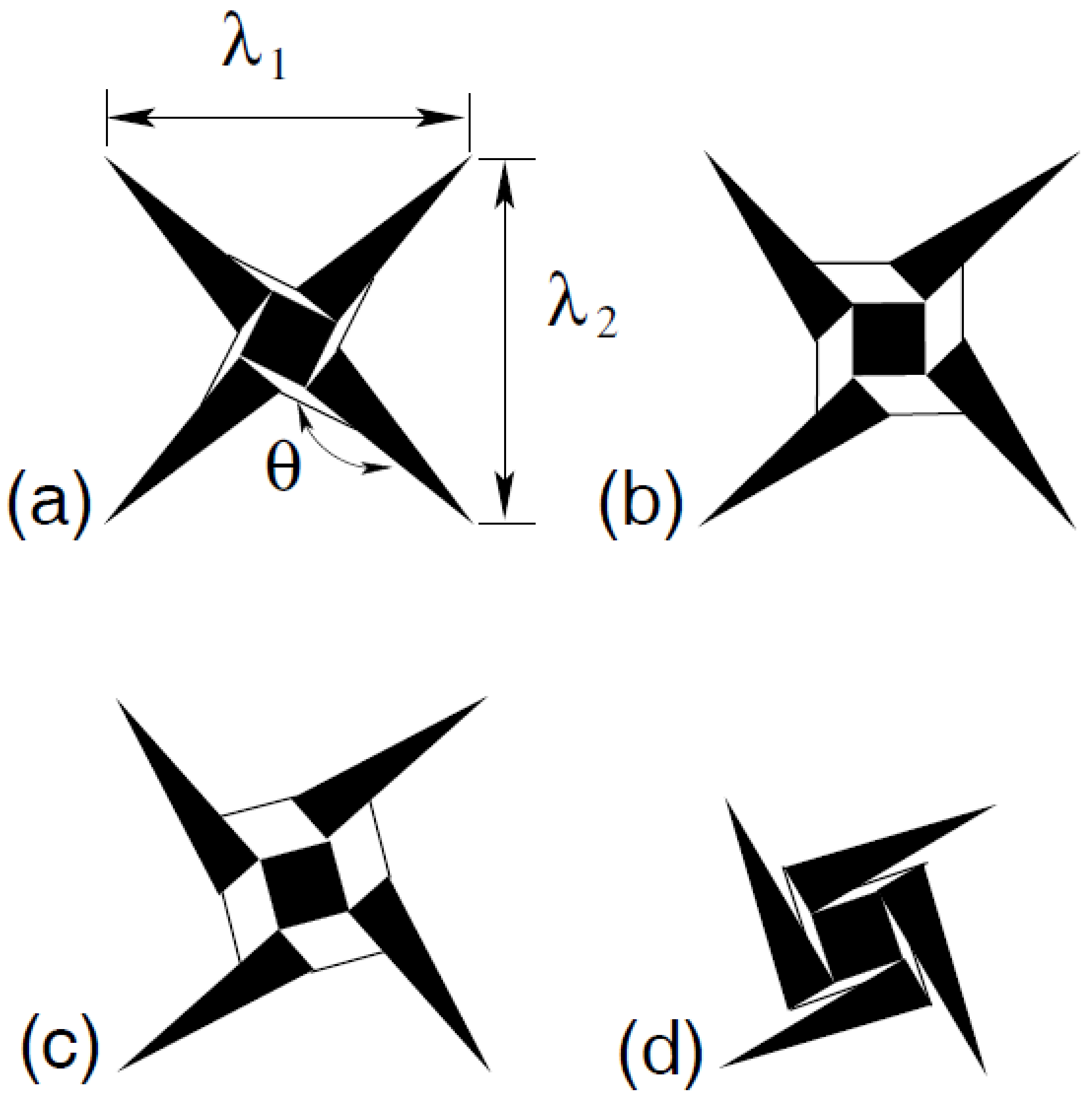
Translator



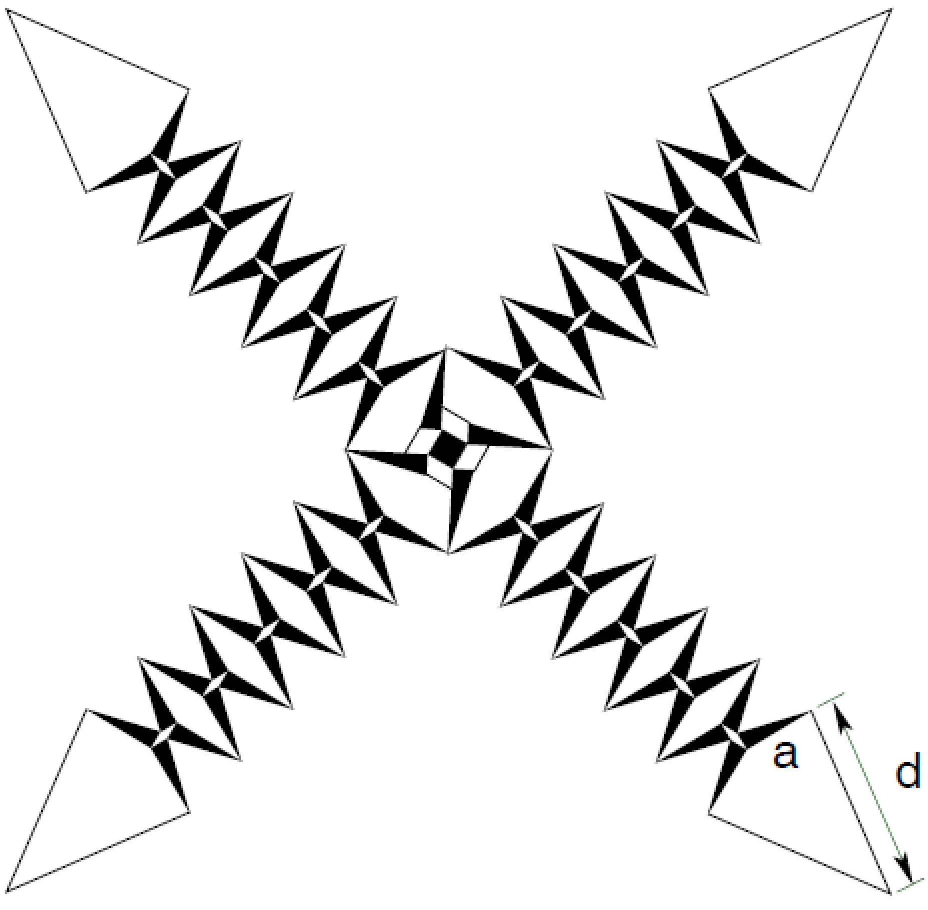


Ideal Expander: $\lambda_2' = c$ is approx realizable

A Dilator



A Dilator with arbitrarily large flexibility window

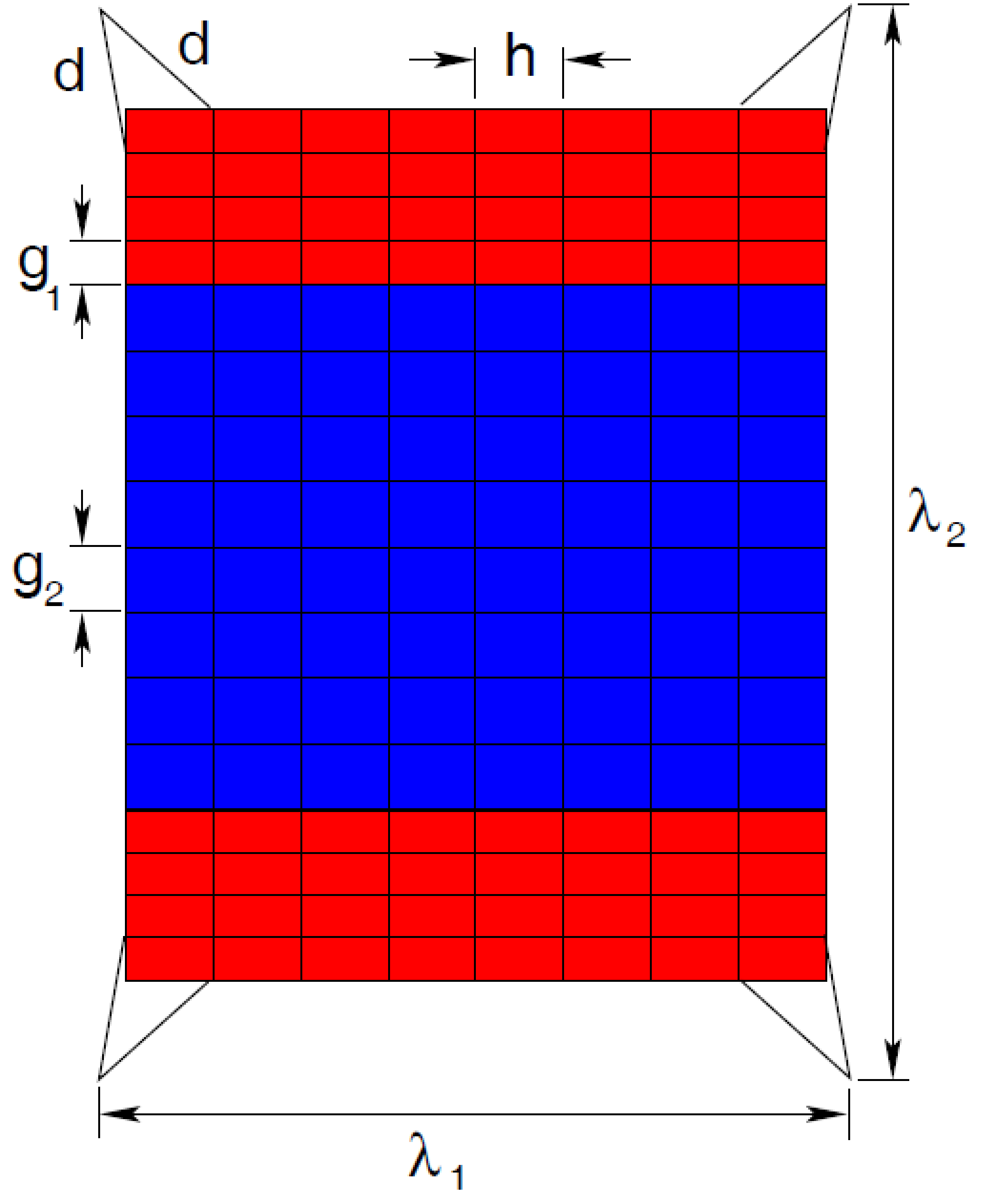


A pea can be made as large as a house

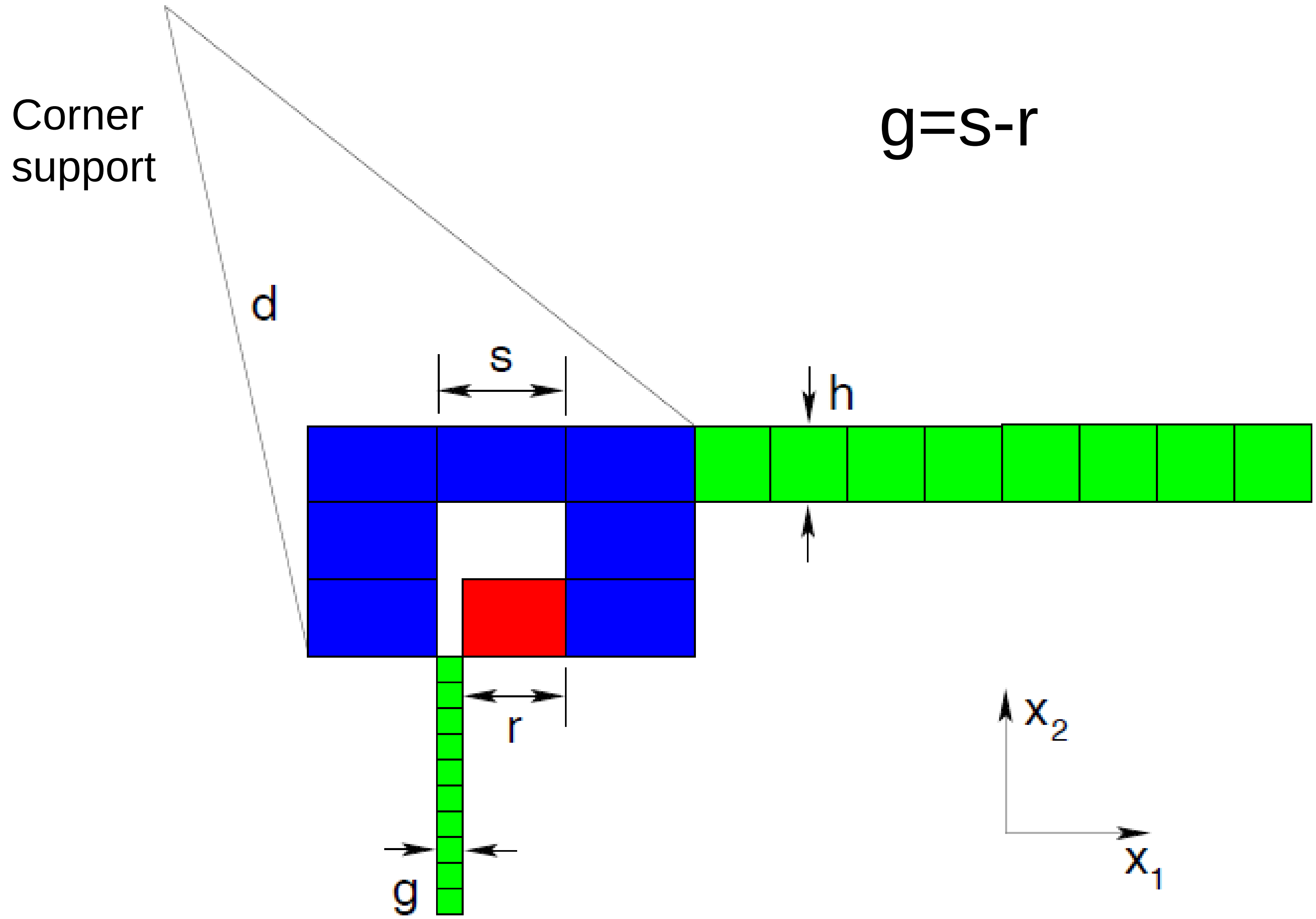
The Adder

Cell of periodicity

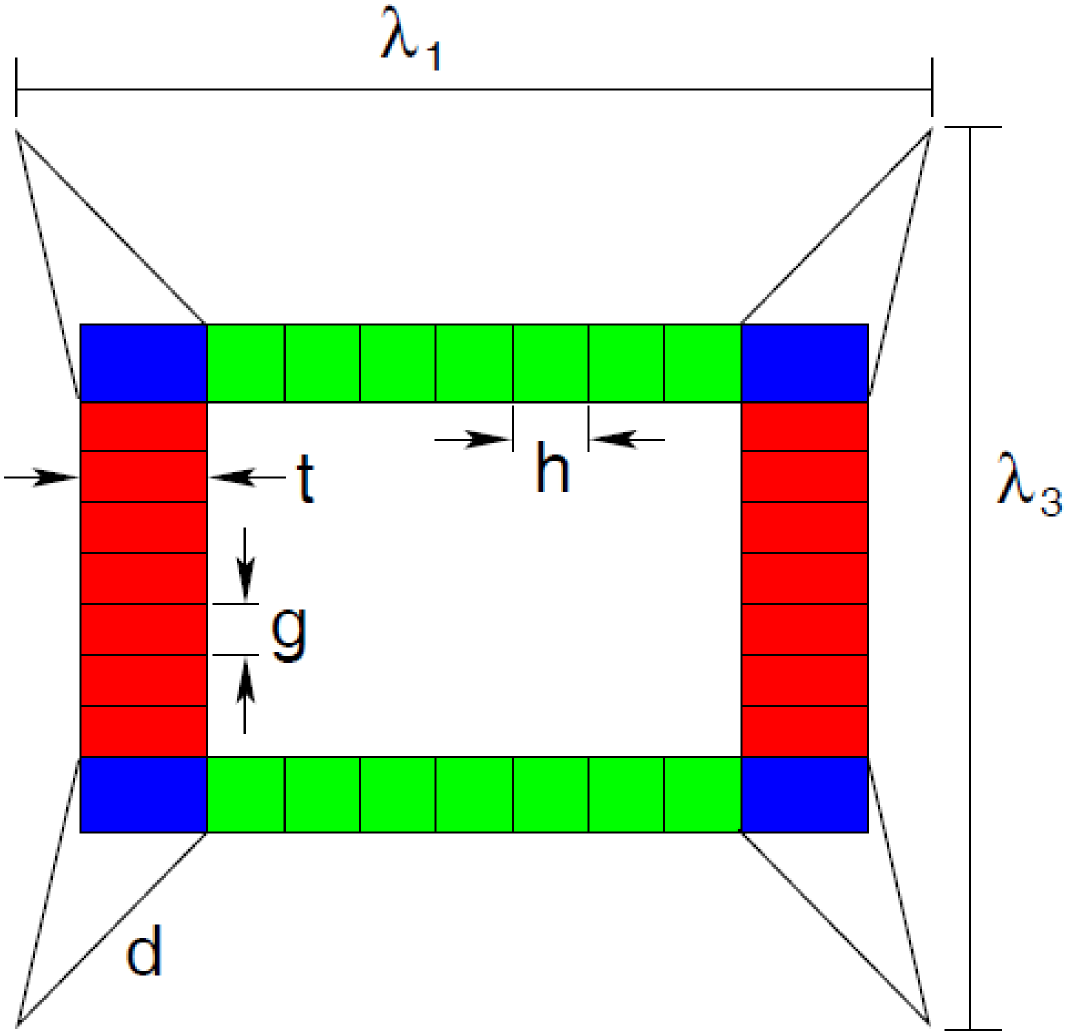
Corner structures are supports that in the limit have vanishingly small contribution



The Subtractor: structure at the corner of a cell of periodicity. Green: square dilator cells



The composer: unit cell of periodicity



Blue:

$$t = f_1(h)$$

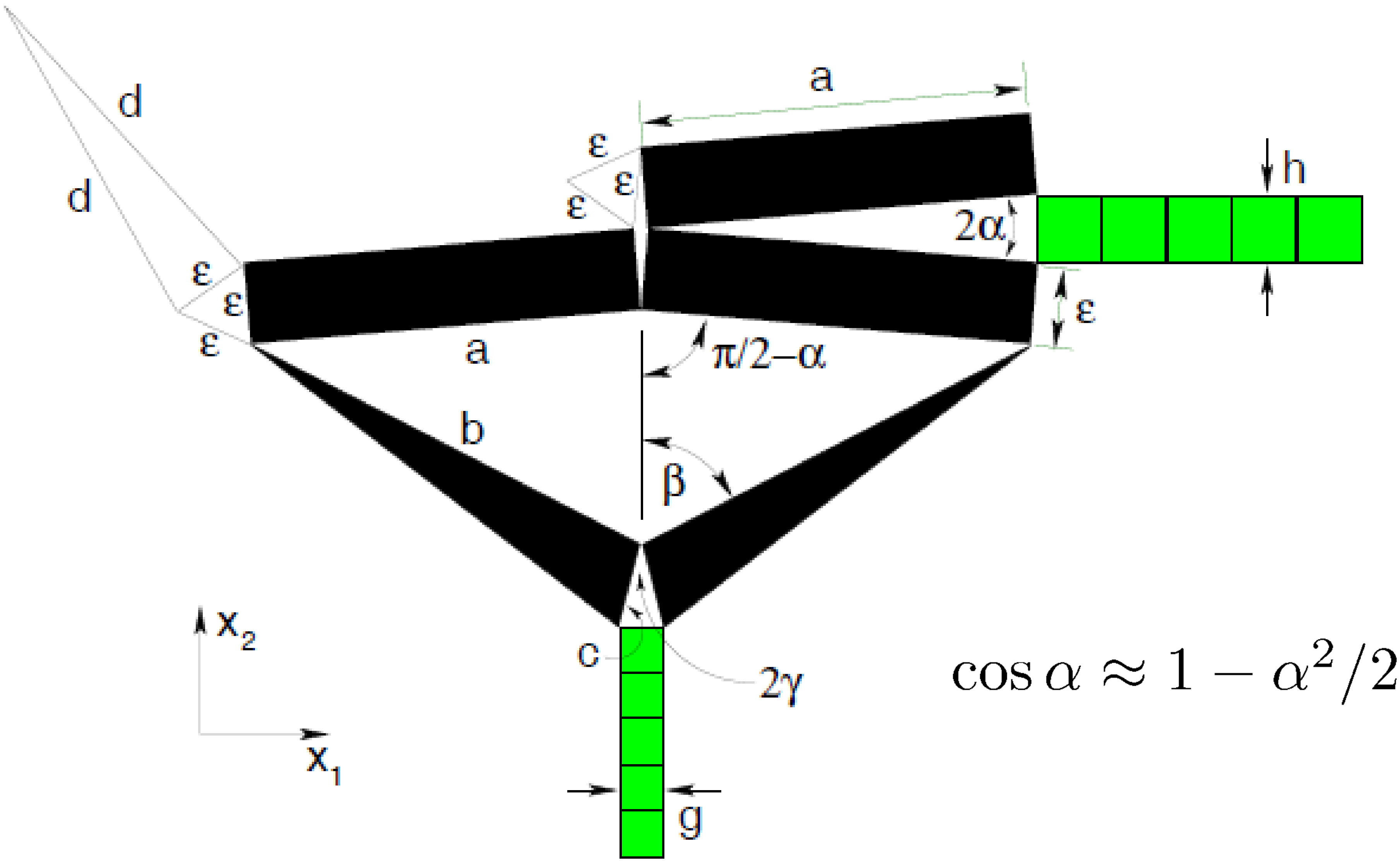
Red:

$$g = f_2(t)$$

Therefore:

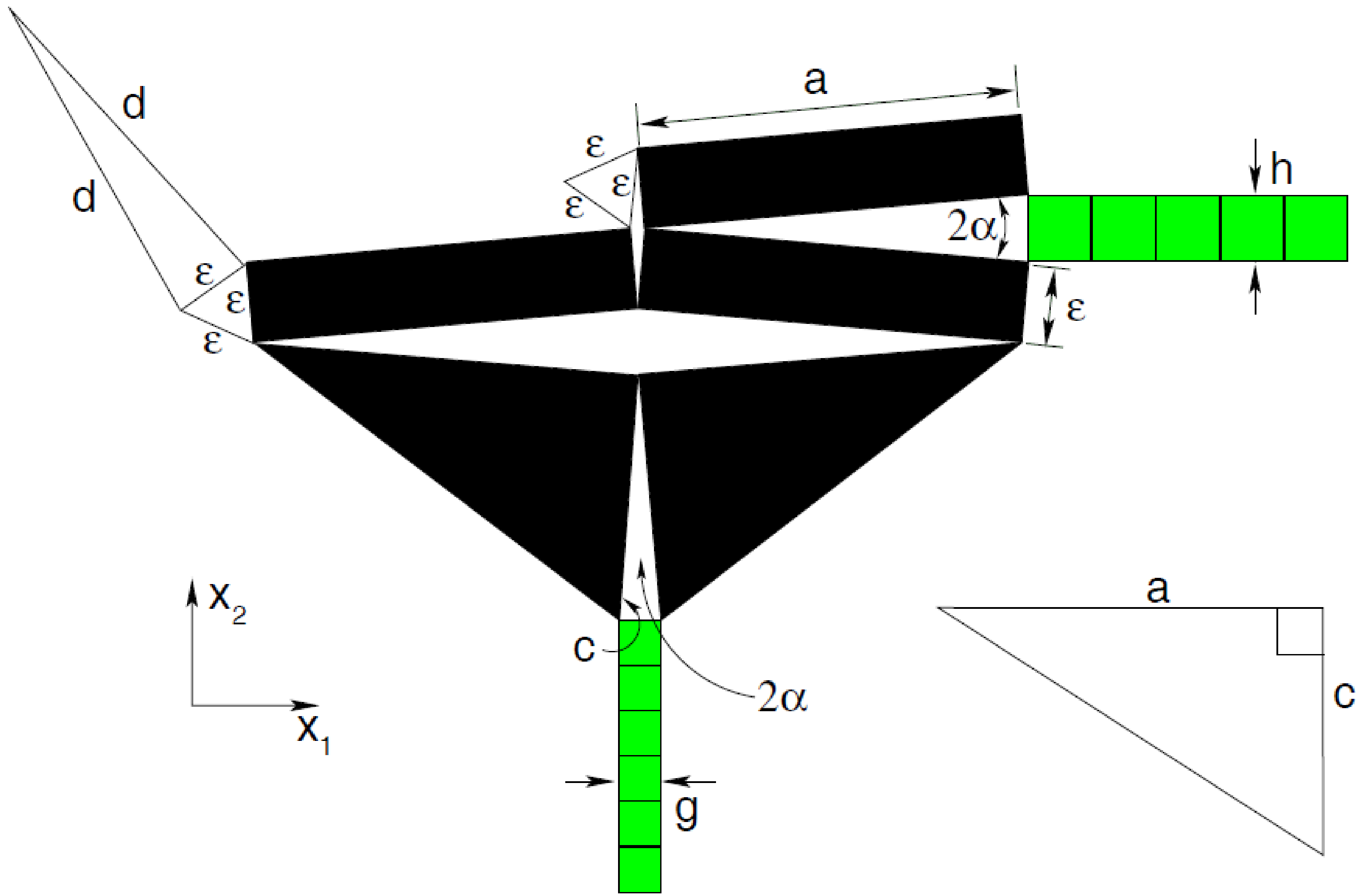
$$g = f_2(f_1(h))$$

The Squarer: vertical expansion the square of horizontal expansion



$$\cos\alpha \approx 1 - \alpha^2/2$$

Multiplier by a constant



$$g/(2c) = \sin\alpha = h/(2a)$$

Realizing any polynomial

$$\lambda_2 = p(\lambda_1) = a_0 + a_1\lambda_1 + a_2\lambda_1^2 + a_3\lambda_1^3 + \ldots + a_n\lambda_1^n$$

that is positive on the interval of λ_1 of interest.

Proof by induction, suppose its true for $n = 2m$. Can realize

$$\lambda_1^{2m+2} = (\lambda_1^{m+1})^2 \qquad (\lambda_1 + 1)^{2m+2} = \lambda_1^{2m+2} + (2m+2)\lambda_1^{2m+1} + g(\lambda_1)$$

$g(\lambda_1)$ is a polynomial of degree $2m$

given any polynomial $q(\lambda_1)$ of degree $2m+2$ or less.

$$q(\lambda_1) = c_1\lambda_1^{2m+2} + c_2(\lambda_1 + 1)^{2m+2} + r(\lambda_1)$$

there exists a sufficiently large constant $c > 0$ such that

$$c + c_1\lambda_1^{2m+2} \qquad c + c_2(\lambda_1 + 1)^{2m+2} \qquad c + r(\lambda_1)$$

are each realizable, and so too is their sum $s(\lambda_1)$ in terms of which

$$q(\lambda_1) = s(\lambda_1) - 3c$$

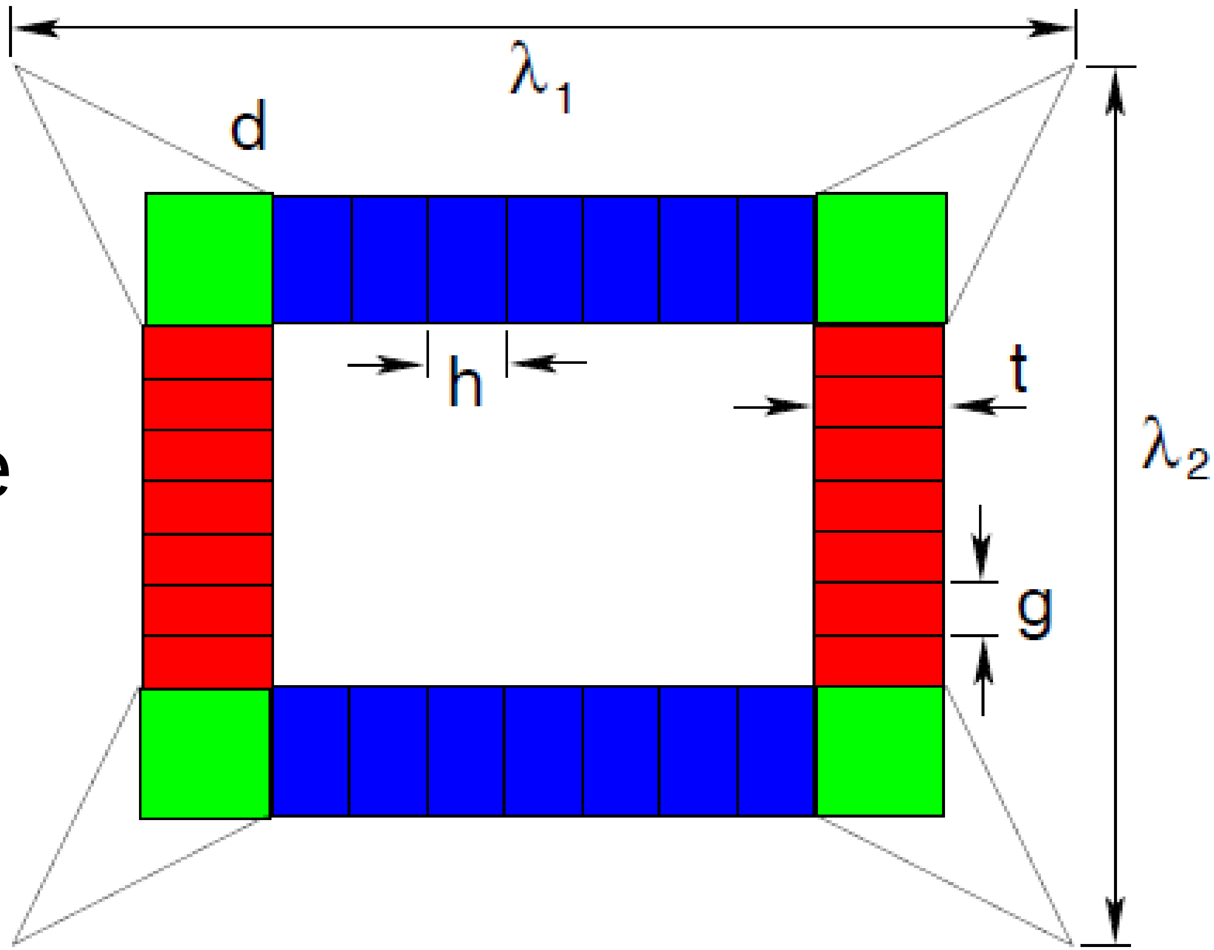
which is the difference of two realizable functions, and hence realizable if it is positive on the interval of interest.

Realizing any function $\lambda_2 = f(\lambda_1)$ which is positive on an interval $\mathbb{I}$ of λ_1. By the Weierstrass approximation theorem

$$\max_{\lambda_1 \in \mathbf{I}} |f(\lambda_1) - p(\lambda_1)| < \epsilon$$

for some polynomial $p(\lambda_1)$

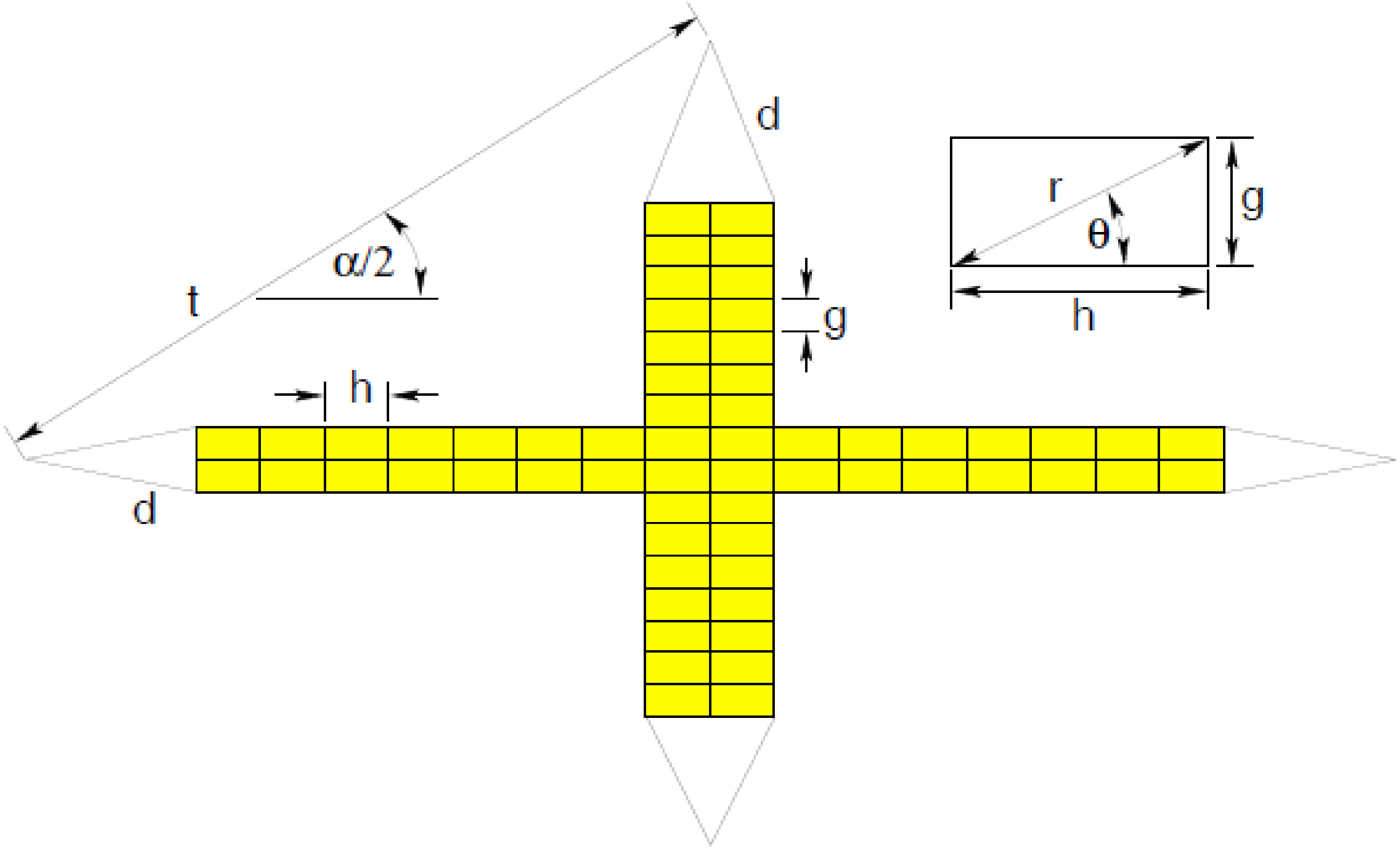
Realizing an arbitrary orthotropic material



Green: square dilator cells

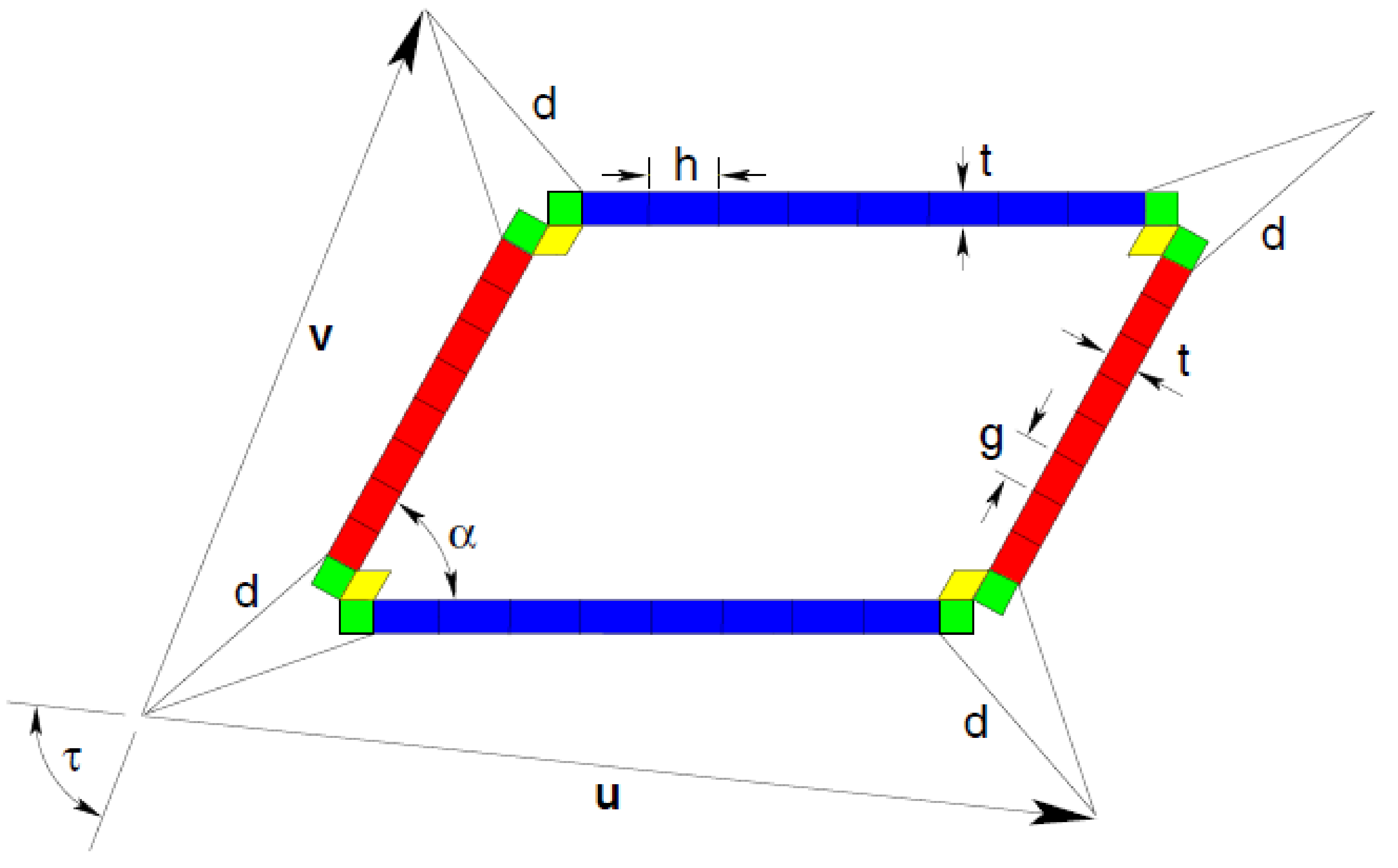
Hence $(\lambda_1, \lambda_2) = (f_1(t), f_2(t))$ is realizable

An angle adjuster



Angle α can be any desired function of t

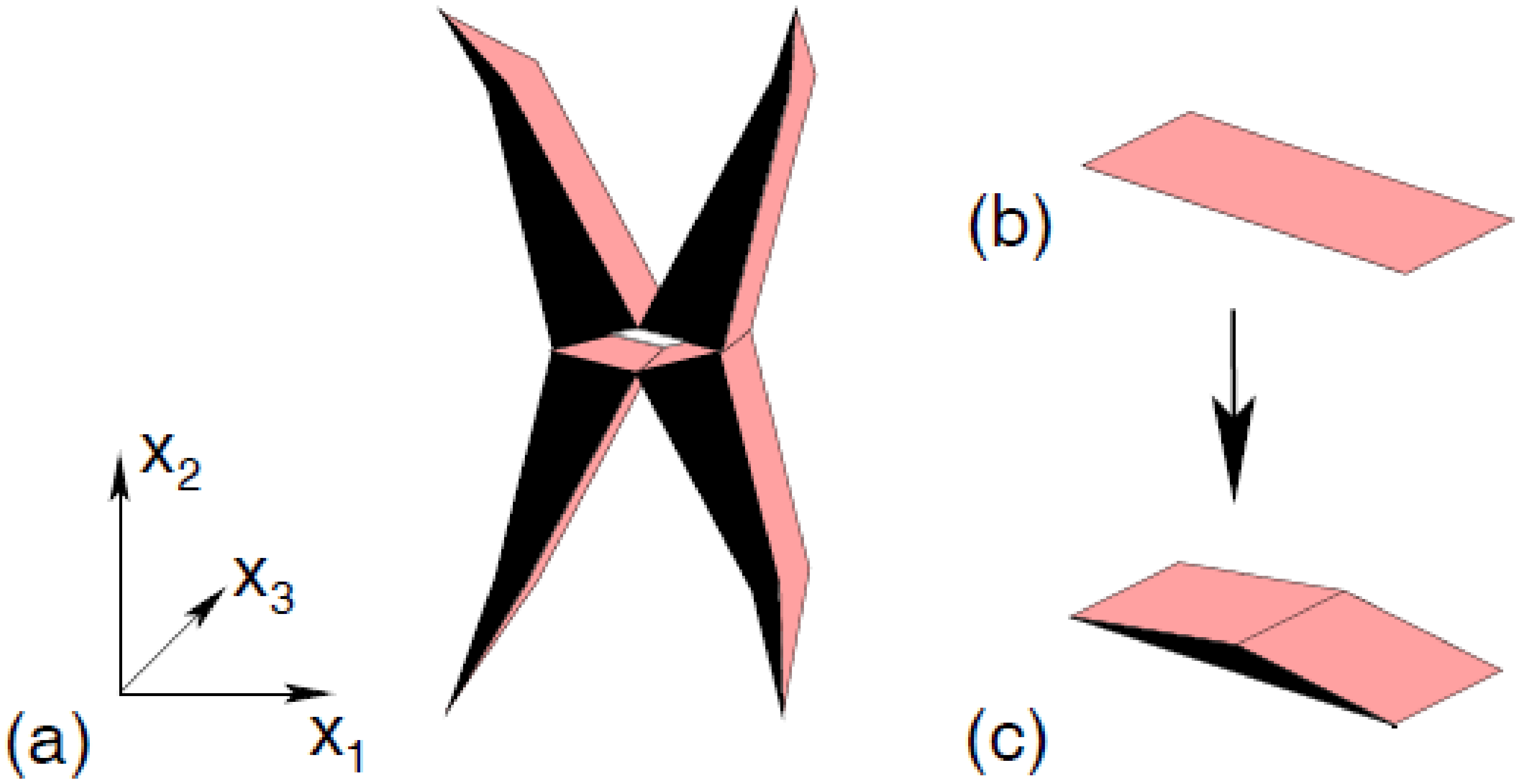
Realizability of an arbitrary oblique material
Unit cell of periodicity:



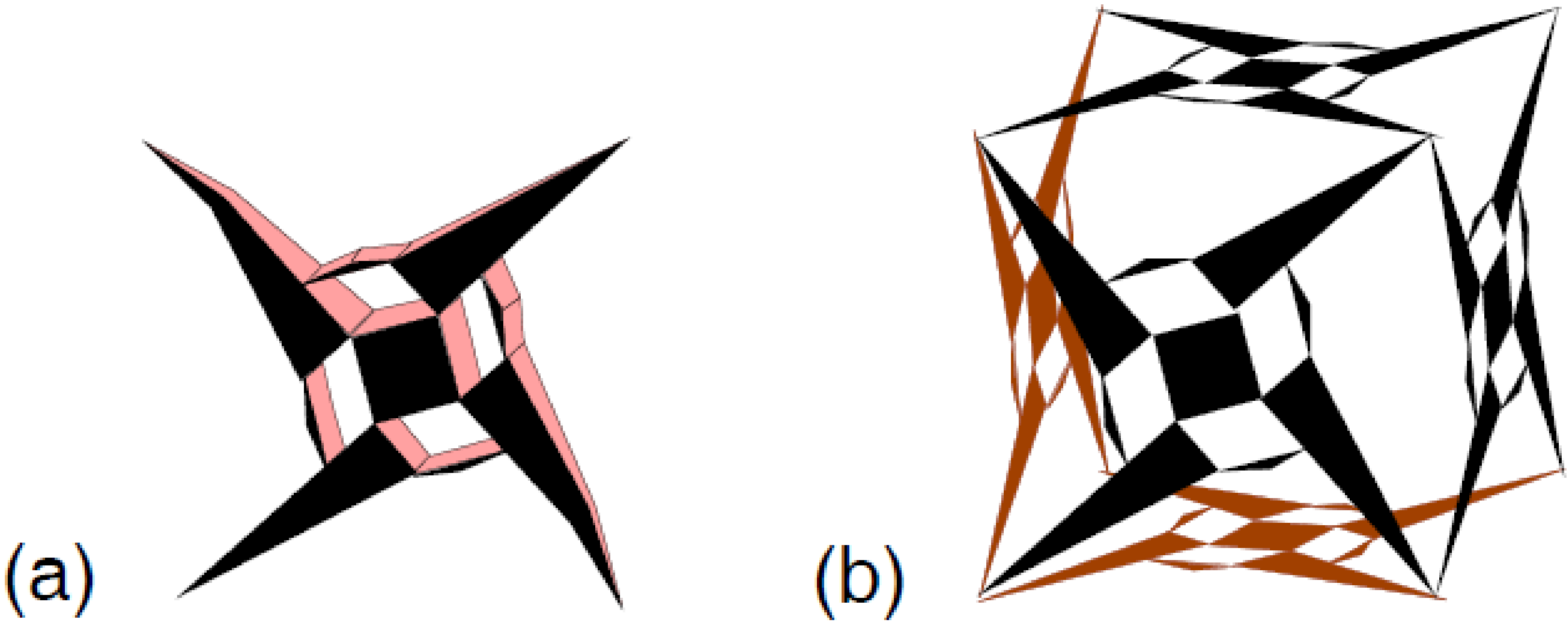
Green: dilator cells; Yellow: angle adjusters

What about three-dimensions?

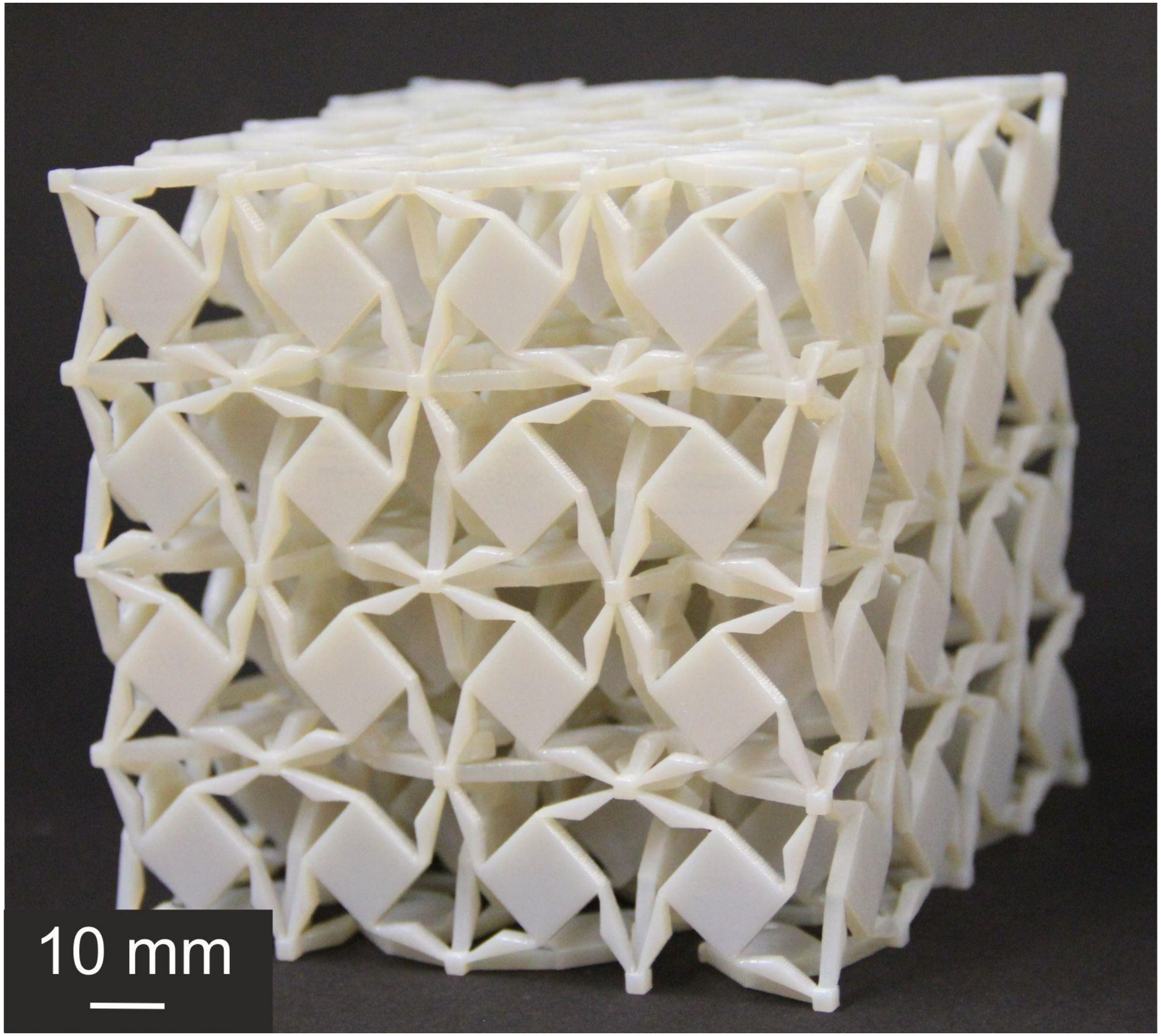
Three Dimensions: From Cells to Panels



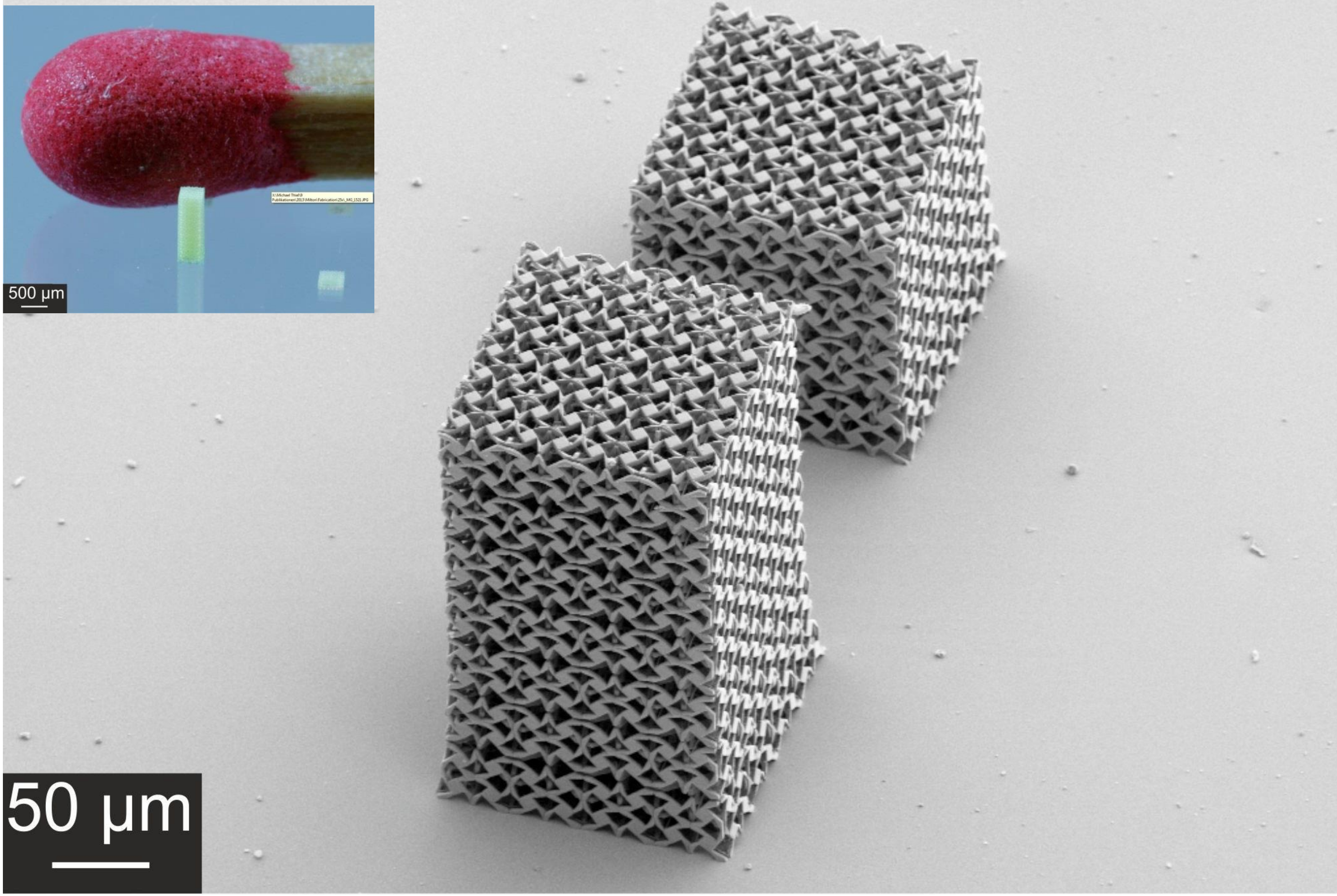
Three Dimensional Dilator



with Buckmann, Kadic, Thiel, Schittny, Wegener

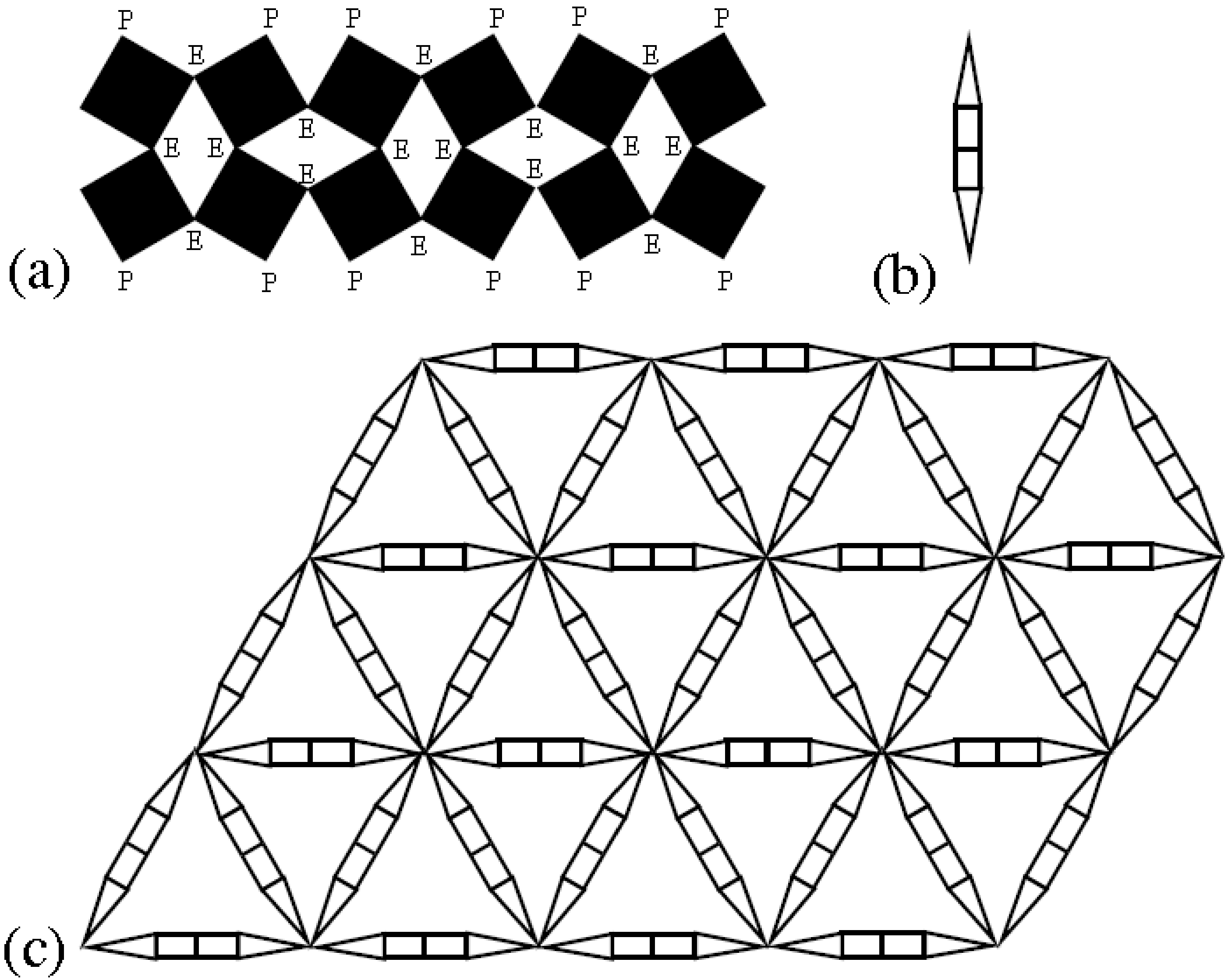


with Buckmann, Kadic, Thiel, Schittny, Wegener



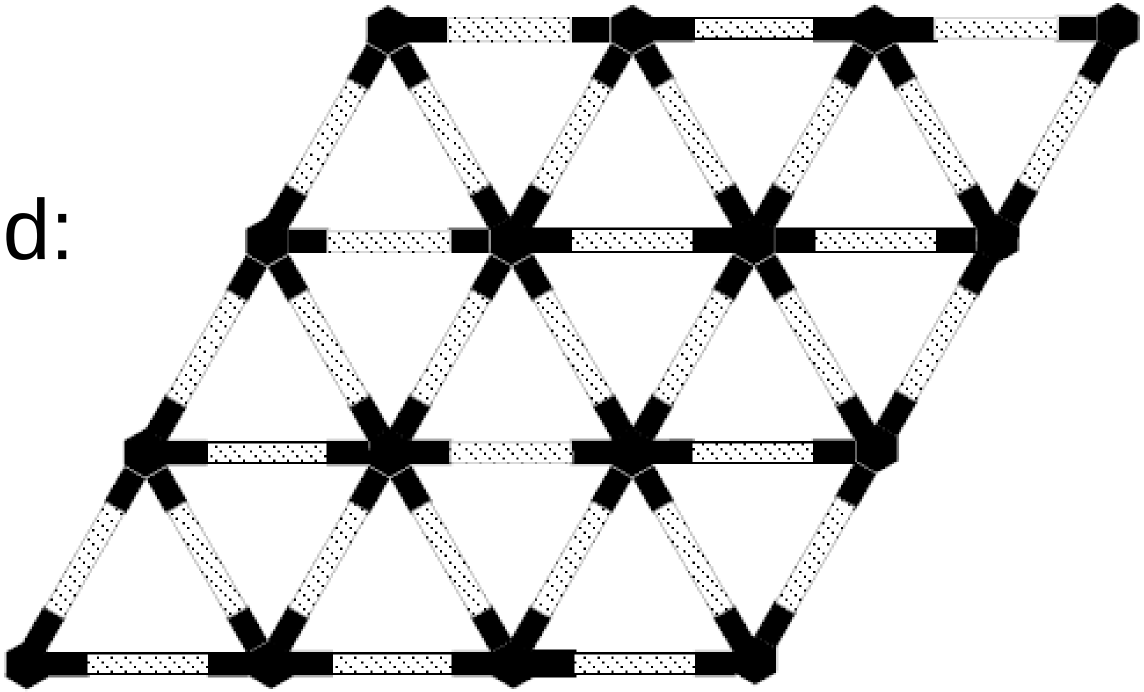
with Buckmann, Kadic, Thiel, Schittny, Wegener

Another 3d dilational material

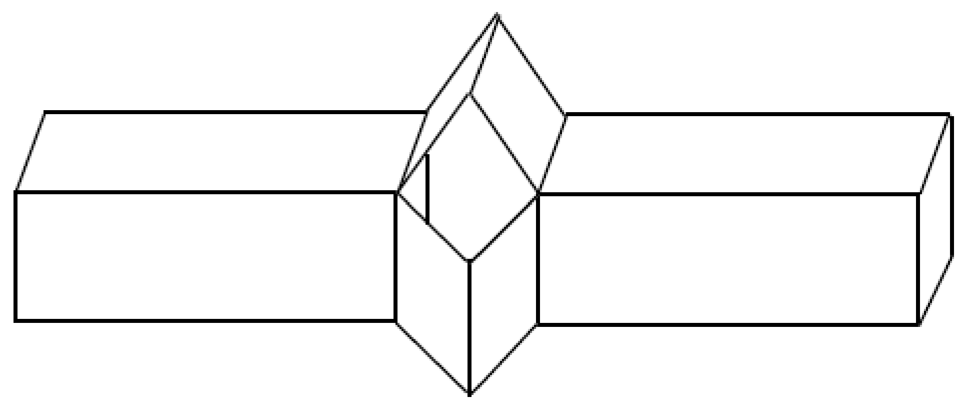


Yet another idea for 3d dilational materials

In 2d:



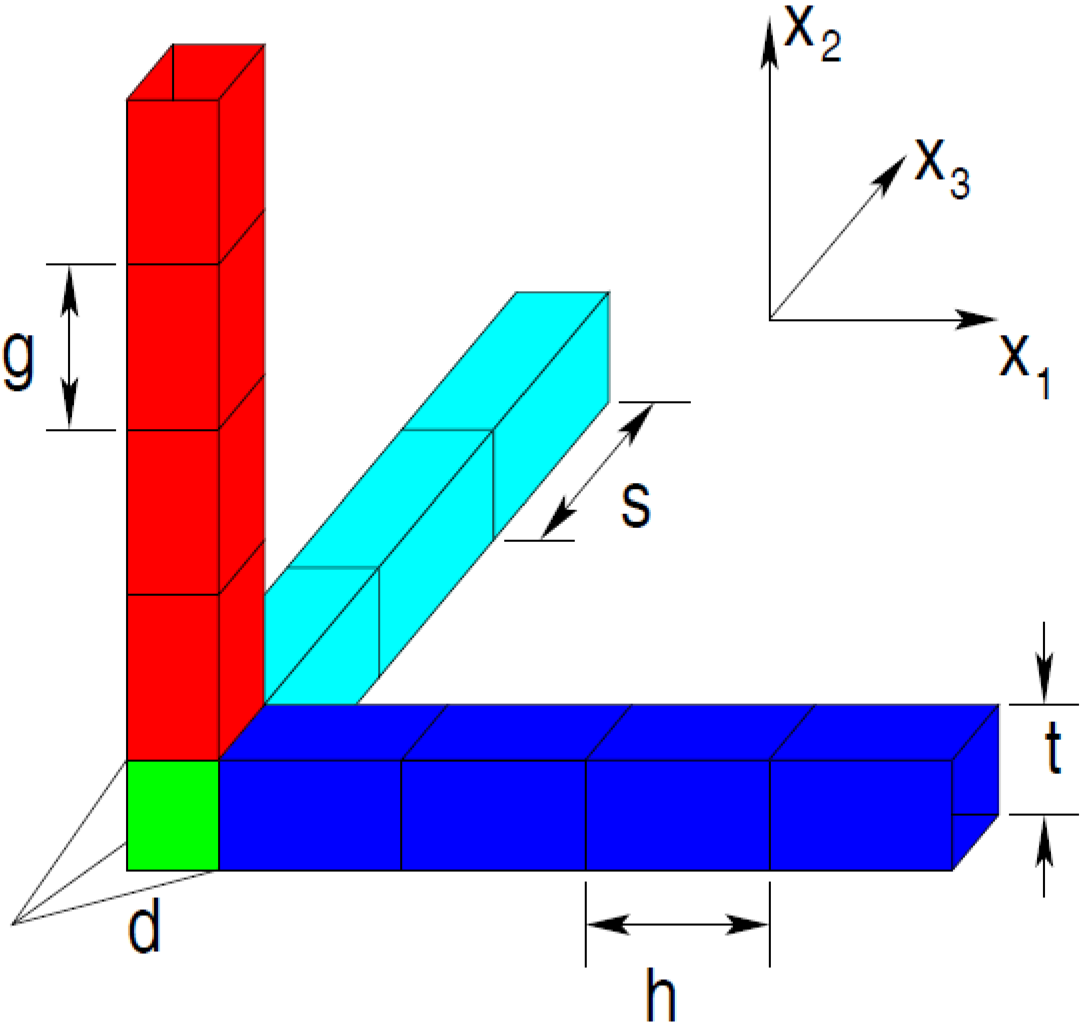
In 3d use a Sarrus linkage



Realizing an arbitrary orthotropic response

Tubes with the two-dimensional structures on the faces of the tube.

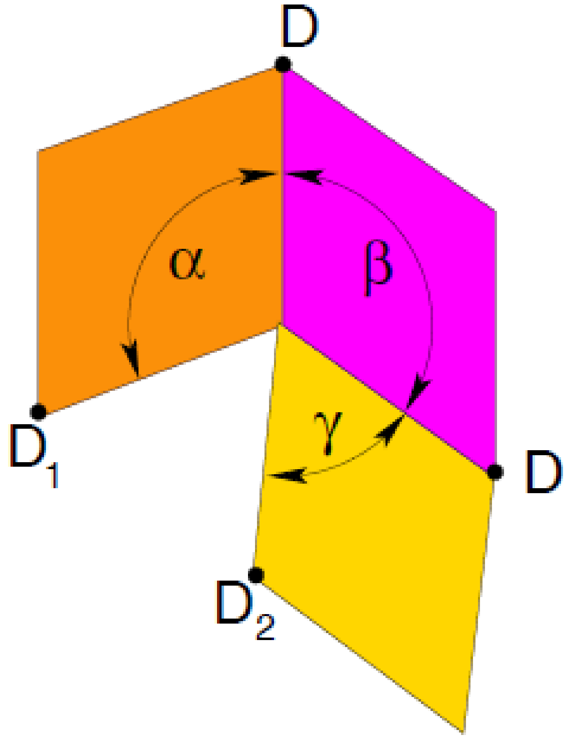
Green dilator cell at the corner

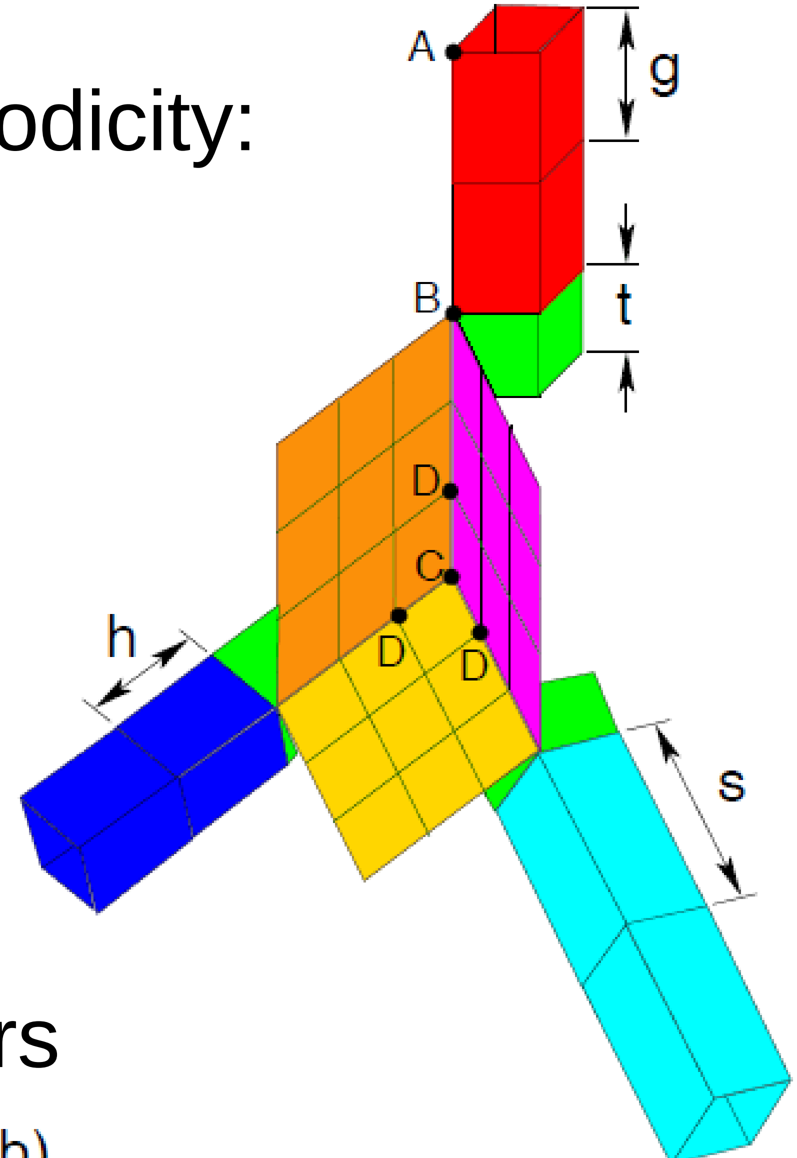


$$(\lambda_1, \lambda_2, \lambda_3) = (f_1(t), f_2(t), f_3(t))$$ is realizable

Realizing an arbitrary triclinic response

Corner of a cell of periodicity:





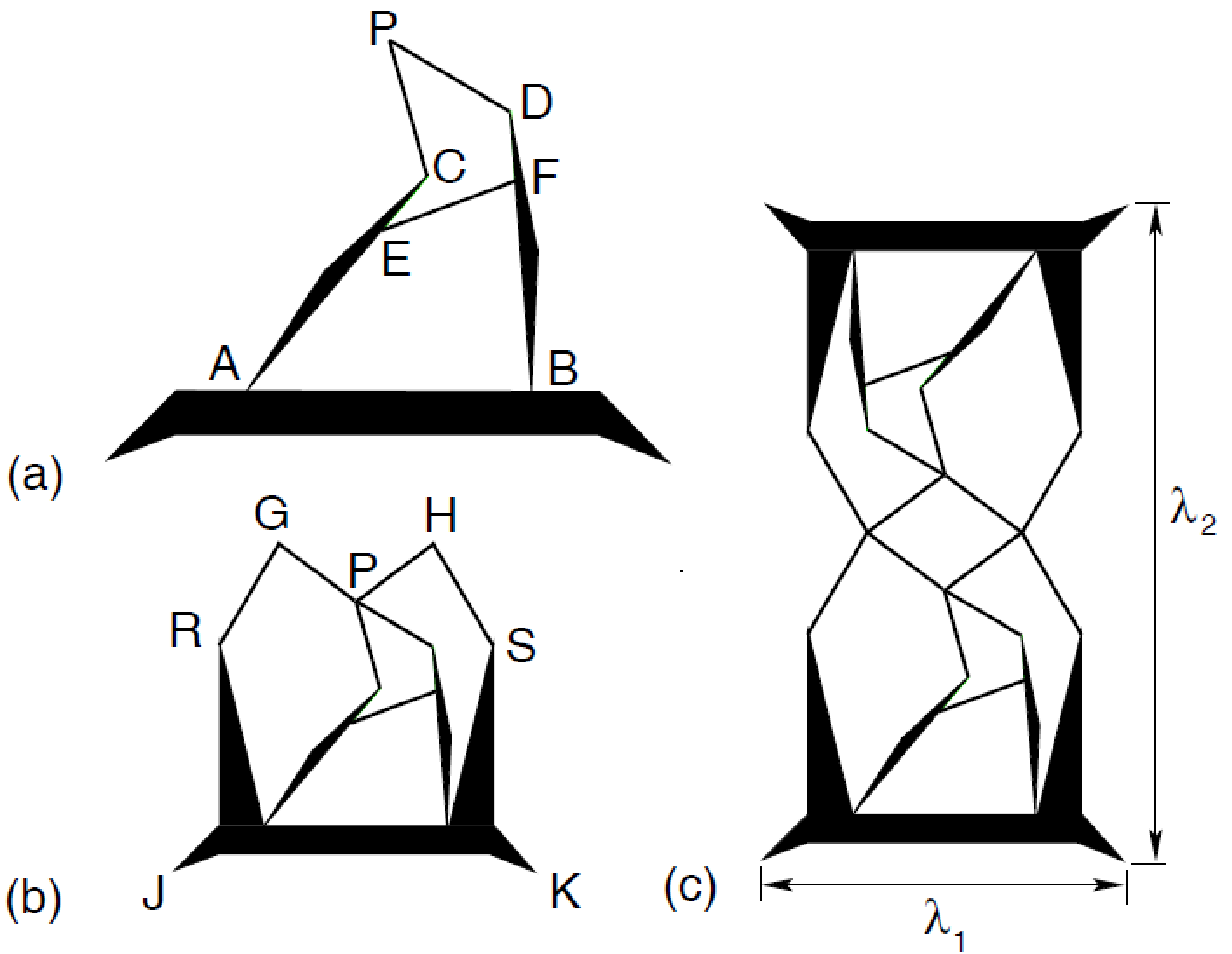
Need 3 angle adjusters

(a) (b)

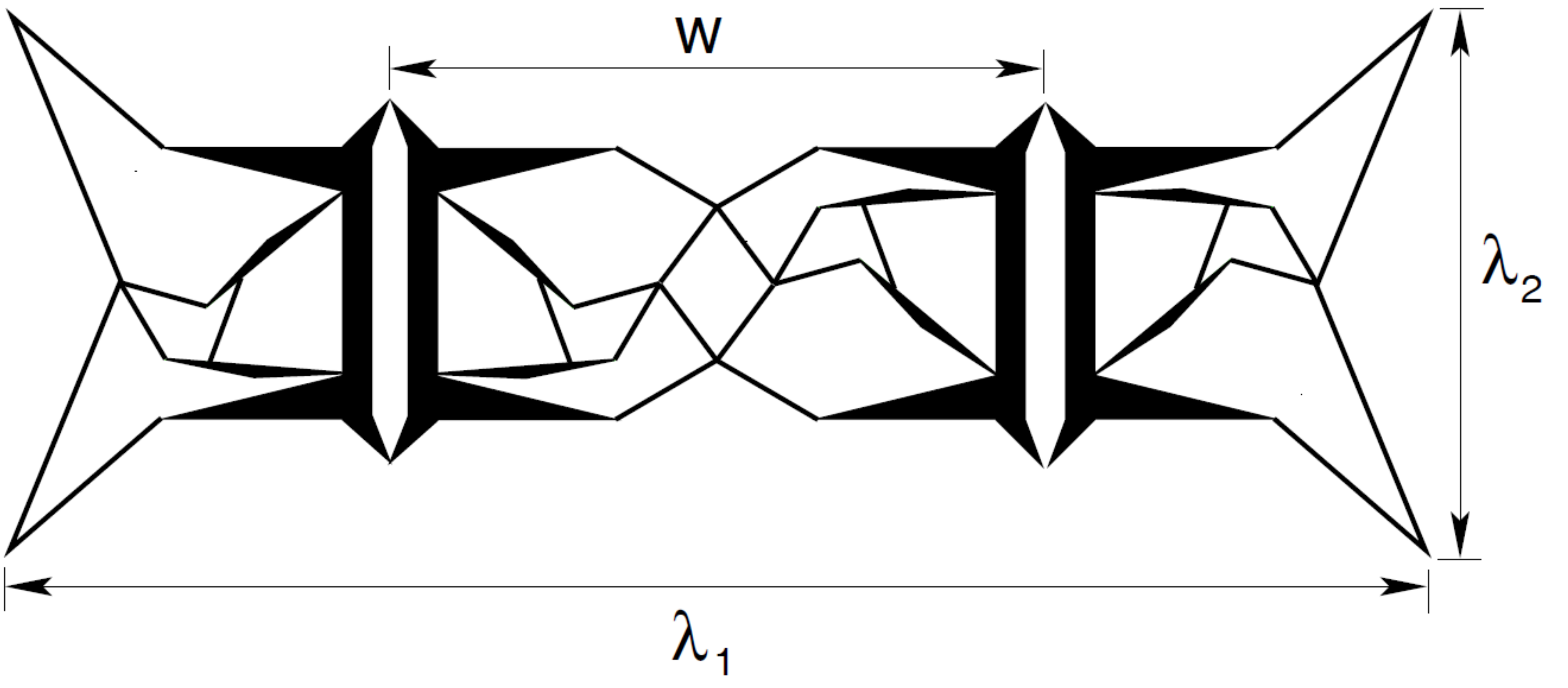
What about non-linear bimode materials?

Do they exist?

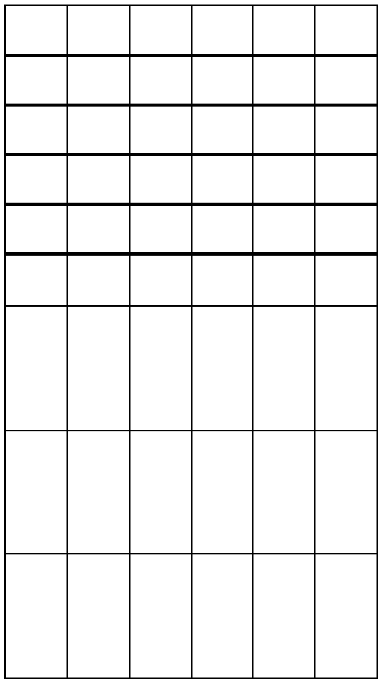
Cell of the perfect expander: a unimode material



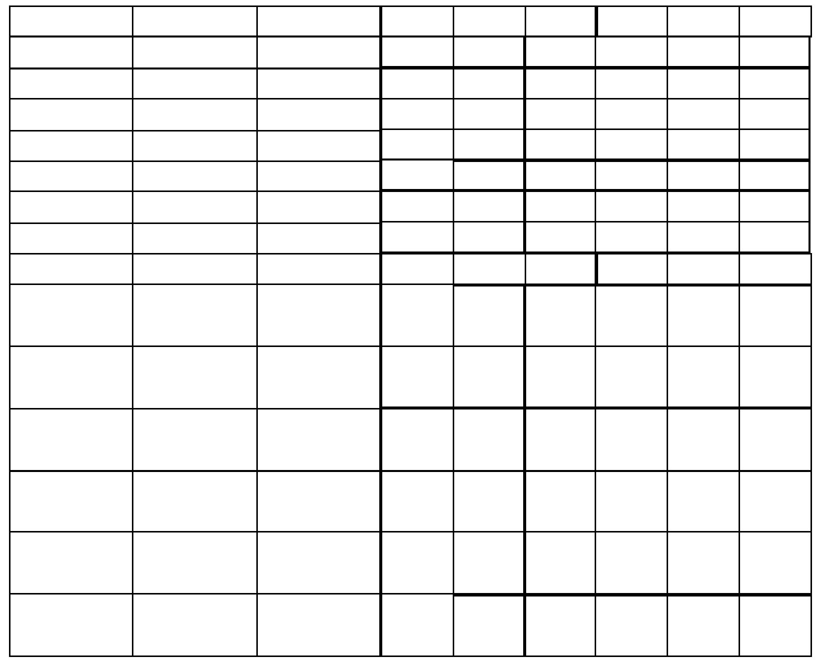
Cell of a bimode material



However neither are affine materials:



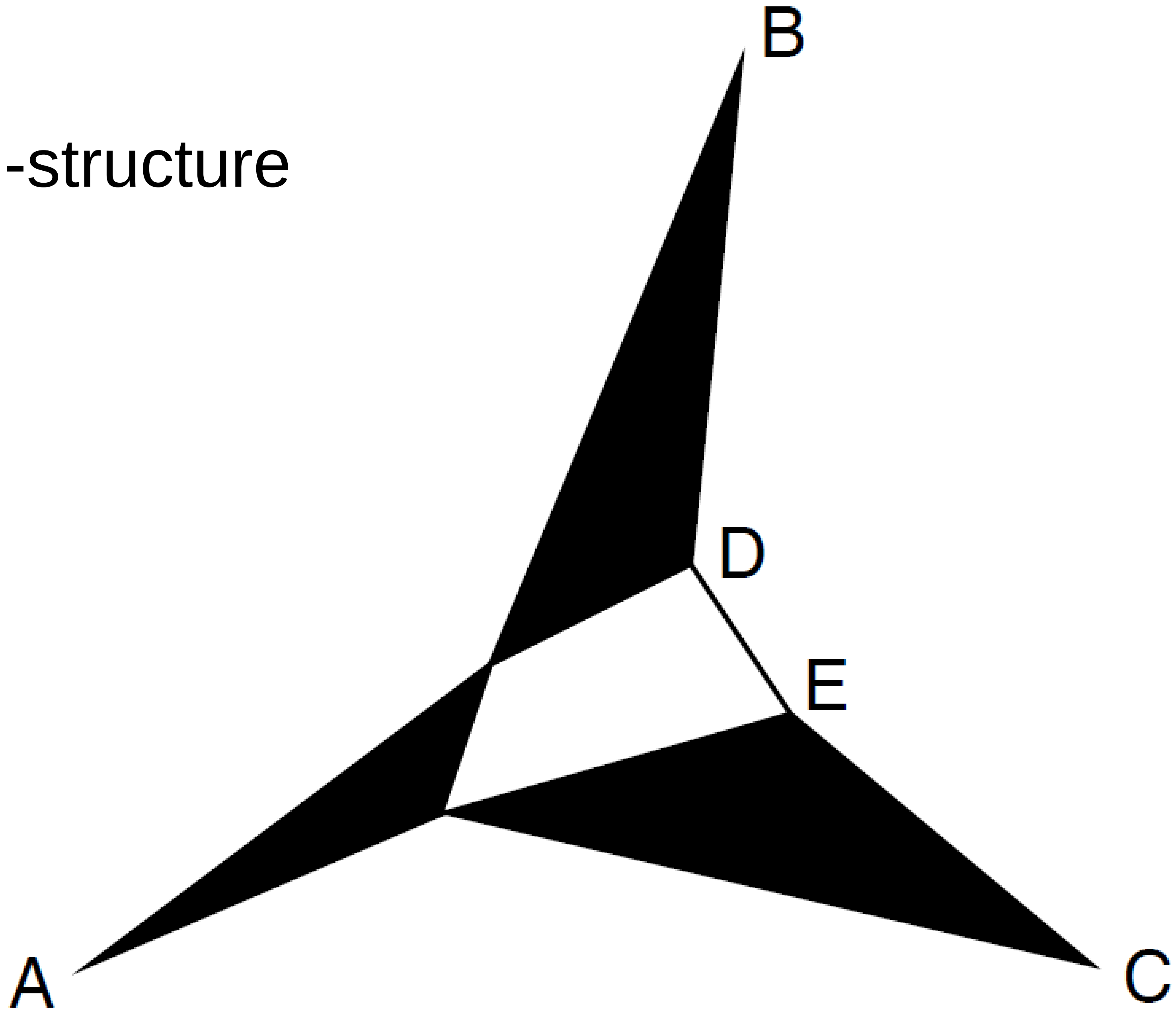
(a)



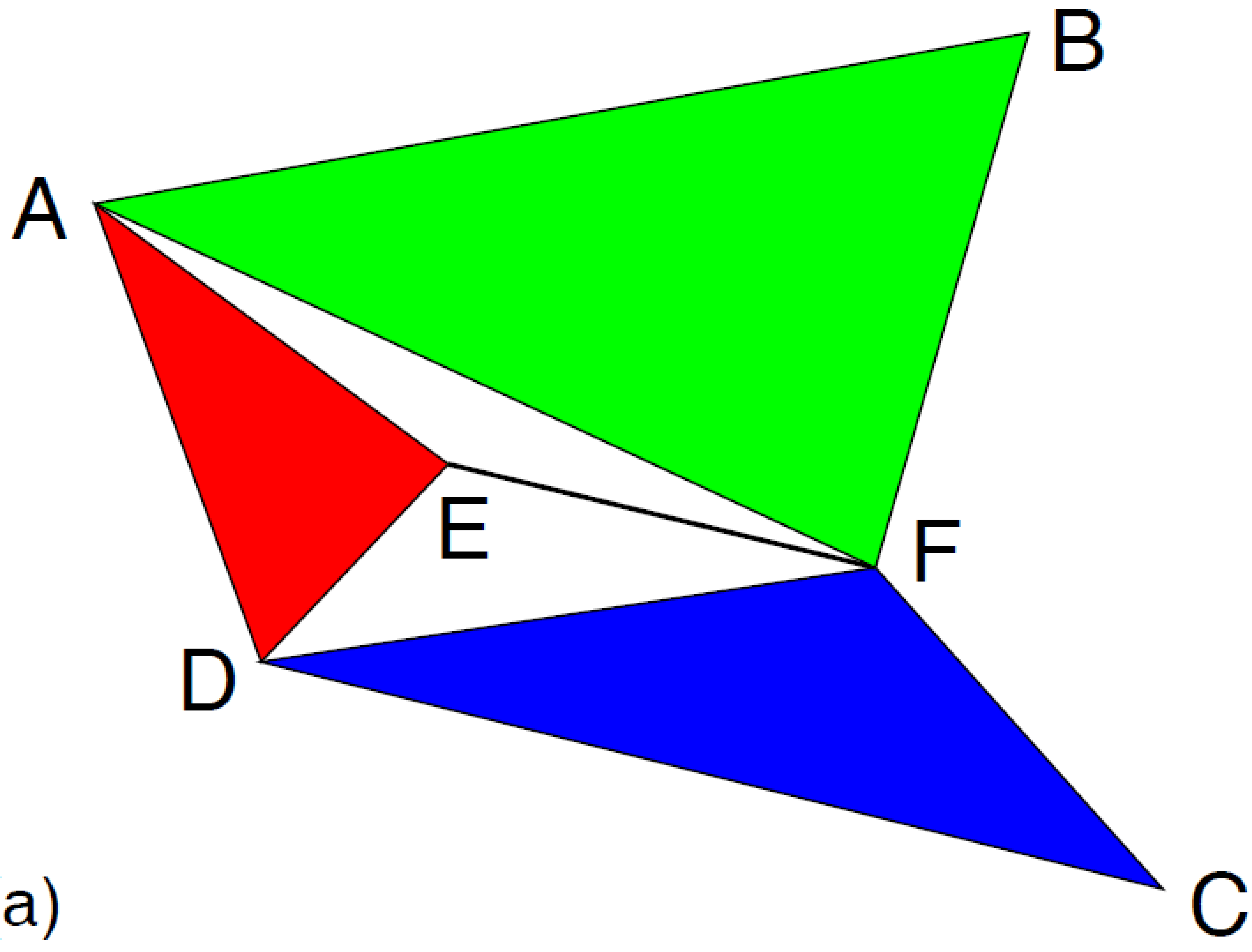
(b)

So can one get affine bimode materials?

A u-structure

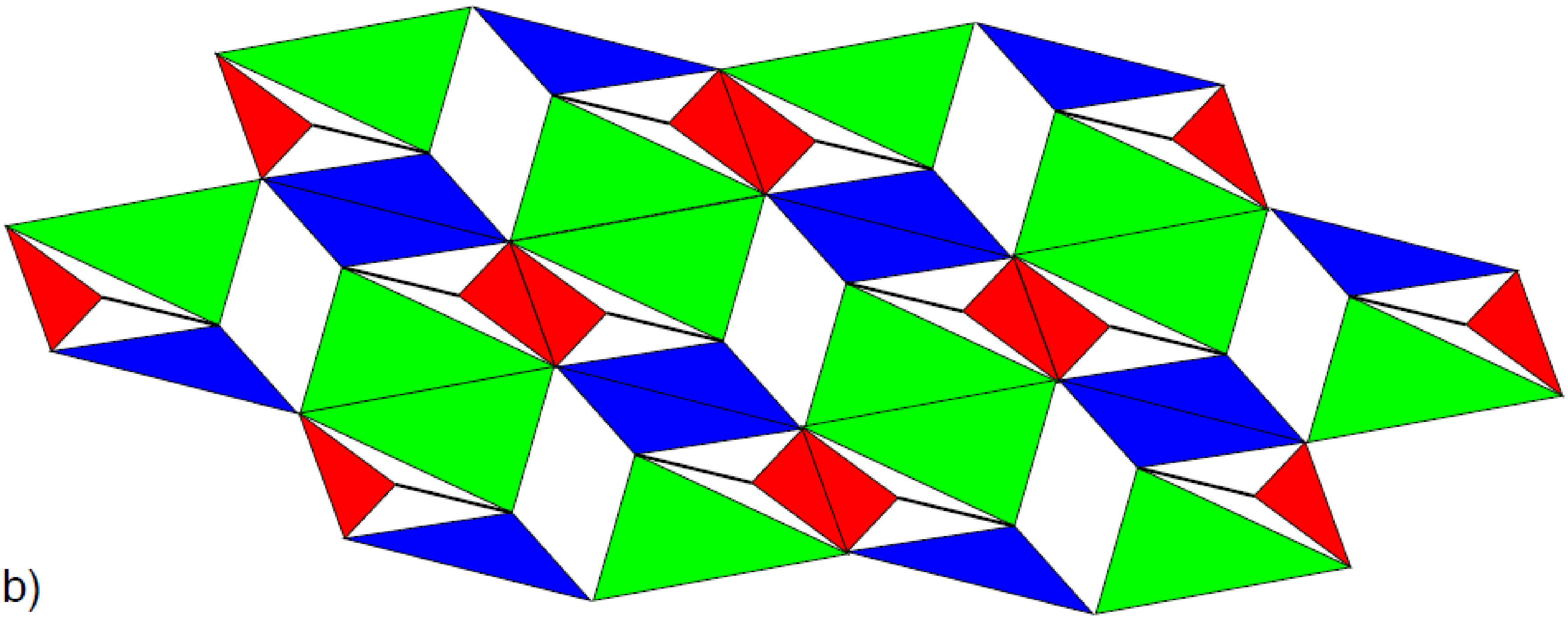


A b-structure



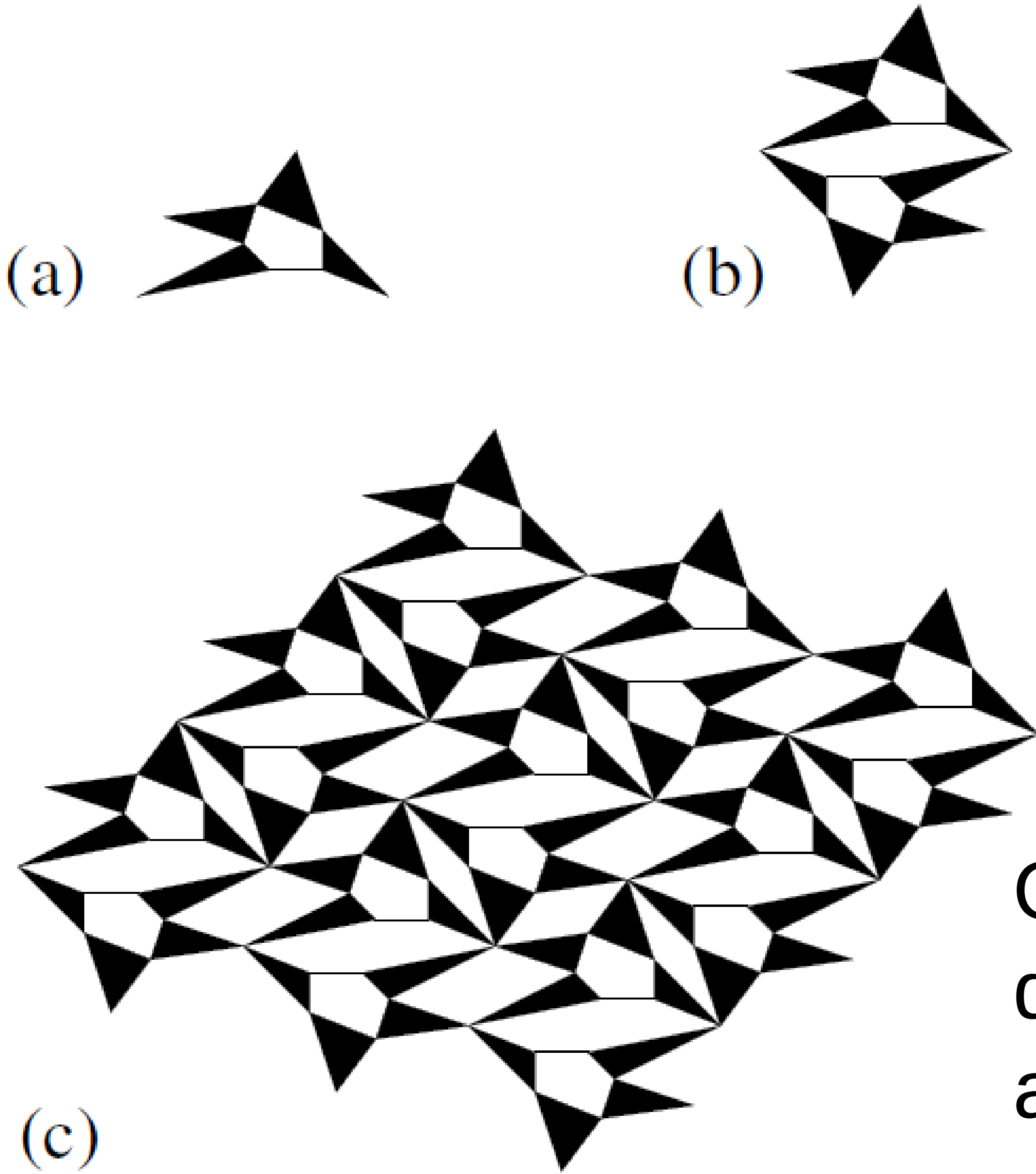
(a)

Bimode material formed from a tiling of a b-structure.



(b)

A non-linear bimode material: awaits construction



Black=Rigid
White=Void

Only macroscopic deformations are affine ones

OPEN PROBLEM:

In two-dimensional materials, can one get non-linear affine trimode materials?