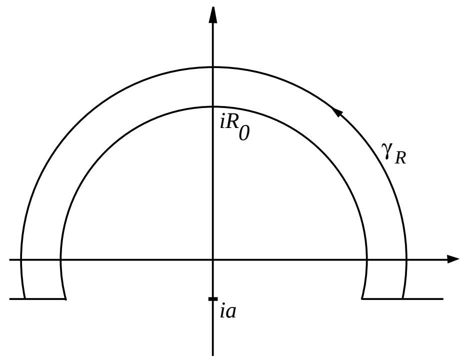
1. Jordan Lemma

Let R_0 be a positive number and a a real number with $|a| < R_0$. Let

$$S = \{z \in \mathbb{C} \mid |z| \geq R_0 \text{ and } \operatorname{Im} z \geq a\}$$

be the region sketched in the following picture:



Let f be a continuous function on S such that $\lim_{z\to\infty} f(z) = 0$, i.e., for any $\epsilon > 0$ there exists $R \geq R_0$ such that $z \in S$ and $|z| > R$ implies that $|f(z)| < \epsilon$.

Let γ_R be the positively oriented arc determined as the intersection of the circle of radius R centered at the origin with S.

1.1. **Lemma** (Jordan Lemma)**.** *For any $m > 0$ we have*

$$\lim_{R\to\infty} \int_{\gamma_R} e^{imz} f(z) dz = 0.$$

Assume first that a is positive. Then γ_R is parameterized as $\gamma_R(\varphi) = Re^{i\varphi}$ for $\varphi \in [\alpha(R), \pi - \alpha(R)]$. Therefore, we have

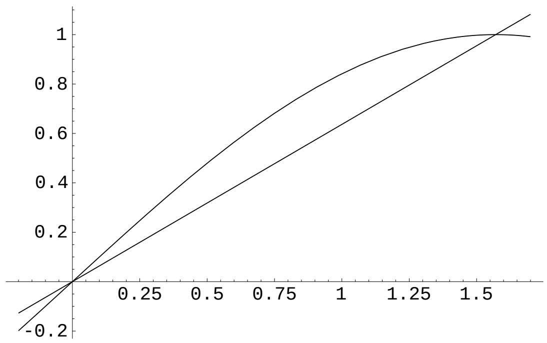
$$\begin{aligned}
\left| \int_{\gamma_R} e^{imz} f(z)\, dz \right| &= \left| iR \int_{\alpha(R)}^{\pi-\alpha(R)} e^{imRe^{i\varphi}} f(Re^{i\varphi}) e^{i\varphi} d\varphi \right| \\
&\leq R \int_{\alpha(R)}^{\pi-\alpha(R)} \left| e^{imR(\cos\varphi + i\sin\varphi)} \right| \left| f(Re^{i\varphi}) \right| d\varphi \\
&= R \int_{\alpha(R)}^{\pi-\alpha(R)} e^{-mR\sin\varphi} \left| f(Re^{i\varphi}) \right| d\varphi \\
&\leq R \max_{z\in\gamma_R^*} |f(z)| \int_{\alpha(R)}^{\pi-\alpha(R)} e^{-mR\sin\varphi}\, d\varphi \\
&\leq R \max_{z\in\gamma_R^*} |f(z)| \int_0^{\pi} e^{-mR\sin\varphi}\, d\varphi = 2R \max_{z\in\gamma_R^*} |f(z)| \int_0^{\frac{\pi}{2}} e^{-mR\sin\varphi}\, d\varphi.
\end{aligned}$$

Now we use the following simple lemma.

1.2. **Lemma.** *Let $\varphi \in [0, \frac{\pi}{2}]$. Then*

$$\sin\varphi \geq \frac{2}{\pi}\varphi.$$

Proof. This is clear from the following picture representing the graphs of the functions $y = \sin x$ and $y = \frac{2}{\pi}x$ which intersect at the origin and the point $(\frac{\pi}{2}, 1)$:



To get an analytic proof, consider the function

$$F(x) = \frac{\sin x}{x}$$

on $(0, \frac{\pi}{2}]$. Clearly, the function is differentiable on $(0, \infty)$ and

$$F'(x) = \frac{x\cos x - \sin x}{x^2}.$$

Moreover, if $G(x) = x\cos x - \sin x$, we have

$$G'(x) = \cos x - x\sin x - \cos x = -x\sin x.$$

Hence, $G'(x) \leq 0$ for $x \in [0, \pi]$, and the function is decreasing there, i.e., $G(x) \leq G(0) = 0$ for $x \in [0, \pi]$. This implies that $F'(x) \leq 0$ for $x \in (0, \pi]$ and the function is decreasing on this interval. In particular, for $x \in (0, \frac{\pi}{2}]$, we have $F(x) \geq F(\frac{\pi}{2}) = \frac{2}{\pi}$. $\square$

Using this result, we see that

$$\int_0^{\frac{\pi}{2}} e^{-mR\sin\varphi}\, d\varphi \leq \int_0^{\frac{\pi}{2}} e^{-\frac{2mR}{\pi}\varphi}\, d\varphi$$
$$= -\frac{\pi}{2mR}\left[e^{-\frac{2mR}{\pi}\varphi}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2mR}\left(1 - e^{-mR}\right) \leq \frac{\pi}{2mR}.$$

Combining this with the above we get

$$\left|\int_{\gamma_R} e^{imz} f(z)\, dz\right| \leq \frac{\pi}{m} \max_{z\in\gamma_R^*} |f(z)|.$$

Therefore, the left side tends to 0 as $R \to \infty$. This proves the lemma in this case.

If $a < 0$, the path γ_R is parameterized as $\gamma_R(\varphi) = Re^{i\varphi}$ where $\varphi \in [-\alpha(R), \pi + \alpha(R)]$. Therefore, γ_R consists of three positively oriented arcs: the arc γ_R'' for $\varphi \in [-\alpha(R), 0]$, followed by γ_R' for $\varphi \in [0, \pi]$ and γ_R''' for $\varphi \in [\pi, \pi + \alpha(R)]$. It follows that

$$\int_{\gamma_R} e^{imz} f(z) dz = \int_{\gamma_R''} e^{imz} f(z) dz + \int_{\gamma_R'} e^{imz} f(z) dz + \int_{\gamma_R'''} e^{imz} f(z) dz.$$

The second integral tends to zero as $R \to \infty$ by the first part of the proof. As before, the first integral satisfies

$$\begin{aligned}\left|\int_{\gamma_R''} e^{imz} f(z)dz\right| &= \left|iR\int_{-\alpha(R)}^{0} e^{imRe^{i\varphi}} f(Re^{i\varphi})e^{i\varphi}d\varphi\right| \\ &\leq R\int_{-\alpha(R)}^{0} e^{-mR\sin\varphi}|f(Re^{i\varphi})|d\varphi.\end{aligned}$$

From the geometry of our situation, we see that $\sin\alpha(R) = \frac{|a|}{R}$. Since $\alpha(R) < \frac{\pi}{2}$, we see that

$$-\sin\varphi \leq \sin\alpha(R) = \frac{|a|}{R}$$

for $\varphi \in [-\alpha(R), 0]$. It follows that

$$\begin{aligned}\left|\int_{\gamma_R''} e^{imz} f(z)dz\right| &\leq e^{m|a|}\max_{z\in\gamma_R^*}|f(z)|\ \ell(\gamma_R'') \\ &= e^{m|a|}\max_{z\in\gamma_R^*}|f(z)|\ R\,\alpha(R) = e^{m|a|}\max_{z\in\gamma_R^*}|f(z)|\,R\arcsin\left(\frac{|a|}{R}\right).\end{aligned}$$

Now, we have

$$\lim_{R\to\infty} R\arcsin\left(\frac{|a|}{R}\right) = \lim_{s\to 0}\frac{\arcsin(|a|s)}{s} = |a|\lim_{s\to 0}\frac{\arcsin s}{s} = |a|\lim_{t\to 0}\frac{t}{\sin t} = |a|,$$

and this implies that the above integral tends to zero as $R \to \infty$. The argument for γ_R''' is analogous.