# NOTES ON RIEMANN SURFACES 

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## 1. Riemann surfaces

1.1. Riemann surfaces as complex manifolds. Let $M$ be a topological space. A chart on $M$ is a triple $c=(U, \varphi)$ consisting of an open subset $U \subset M$ and a homeomorphism $\varphi$ of $U$ onto an open set in the complex plane $\mathbb{C}$. The open set $U$ is called the domain of the chart $c$.

The charts $c=(U, \varphi)$ and $c^{\prime}=\left(U^{\prime}, \varphi^{\prime}\right)$ on $M$ are compatible if either $U \cap U^{\prime}=\emptyset$ or $U \cap U^{\prime} \neq \emptyset$ and $\varphi^{\prime} \circ \varphi^{-1}: \varphi\left(U \cap U^{\prime}\right) \longrightarrow \varphi^{\prime}\left(U \cap U^{\prime}\right)$ is a bijective holomorphic function (hence the inverse map is also holomorphic).

A family $\mathcal{A}$ of charts on $M$ is an atlas of $M$ if the domains of charts form a covering of $M$ and any two charts in $\mathcal{A}$ are compatible.

Atlases $\mathcal{A}$ and $\mathcal{B}$ of $M$ are compatible if their union is an atlas on $M$. This is obviously an equivalence relation on the set of all atlases on $M$. Each equivalence class of atlases contains the largest element which is equal to the union of all atlases in this class. Such atlas is called saturated.

An one-dimensional complex manifold $M$ is a hausdorff topological space with a saturated atlas. If $M$ is connected we also call it a Riemann surface.

Let $M$ be an Riemann surface. A chart $c=(U, \varphi)$ is a chart around $z \in M$ if $z \in U$. We say that it is centered at $z$ if $\varphi(z)=0$.

Let $M$ and $N$ be two Riemann surfaces. A continuous map $F: M \longrightarrow N$ is a holomorphic map if for any two pairs of charts $c=(U, \varphi)$ on $M$ and $d=(V, \psi)$ on $N$ such that $F(U) \subset V$, the mapping

$$
\psi \circ F \circ \varphi^{-1}: \varphi(U) \longrightarrow \varphi(V)
$$

is a holomorphic function. We denote by $\operatorname{Mor}(M, N)$ the set of all holomorphic maps from $M$ into $N$.

Any open subset $\Omega$ of a Riemann surface $M$ inherits the structure of open submanifold of $M$. If it is connected, it is also a Riemann surface.

The complex line $\mathbb{C}$ has an obvious structure of Riemann surface given by the chart $\left(\mathbb{C}, i d_{\mathbb{C}}\right)$. A holomorphic map $f: M \longrightarrow \mathbb{C}$ from a Riemann surface $M$ into $\mathbb{C}$ is called a holomorphic function on $M$.

If $\Omega$ is a domain in $\mathbb{C}$, a holomorphic function $f$ on $\Omega$ is a holomorphic function in sense of our old definition.

Clearly, Riemann surfaces as objects and holomorphic maps as morphisms form a category. Isomorphisms in this category are called holomorphic isomorphisms.

Let $M$ be a Riemann surface. An holomorphic isomorphism of $M$ with itself is called a holomorphic automorphism of $M$. The set of all holomorphic automorphisms, with composition of maps as an operation, forms a group $\operatorname{Aut}(M)$ of holomorphic automorphisms of $M$.
1.2. Zeros and isolated singularities of holomorphic functions. Let $M$ be a Riemann surface. Let $f$ be a holomorphic function on $M$. Denote by $Z(f)$ the set of all zeros of $f$, i.e. $Z(f)=\{z \in M \mid f(z)=0\}$.
1.2.1. Proposition. Either $f=0$ on $M$ or the set $Z(f)$ of zeros of $f$ has no limit points.

Proof. Let $A$ be the set of all limit points of $Z(f)$. Assume that $A$ is nonempty, and $a$ is a point in $A$. By continuity, $a$ is also a zero of $f$. Let $(U, \phi)$ be a chart of $M$ centered at $a$. Assume that $U$ is connected. Then 0 is a zero of the holomorphic function $f=f \circ \phi^{-1}$ on $\phi(U)$. Since 0 is a limit point of zeros of $f \circ \phi^{-1}$, by [2, Theorem 10.18] it follows that $f$ is zero on $U$. Therefore, any point of $U$ is in $A$. This implies that $A$ is an open set. On the other hand, any limit point of $A$ is a limit point of zeros, i.e., it is in $A$. Hence, $A$ is also closed. Since $M$ is connected, $A$ has to be equal to $M$.

Therefore, a zero $a$ of a nontrivial holomorphic function on a Riemann surface is isolated. Hence, there exists a chart $(U, \varphi)$ centered at $a$ such that $f$ has no other zeros on $U$. Therefore, the holomorphic function $f \circ \varphi^{-1}$ on $\varphi(U)$ has only one zero at 0 . Assume that 0 is a zero of order $m$ of $f \circ \varphi^{-1}$. By [2, Theorem 10.32], there is a (possibly smaller) neighborhood $U^{\prime}$ of $a$ such that $f \circ \varphi^{-1}$ is a m-to-1 map on $\varphi\left(U^{\prime}\right)-\{0\}$. Therefore, $f$ is a $m$-to-1 map on the punctured neighborhood $U^{\prime}-\{a\}$ of $a$. Hence, $m$ is independent of the choice of the chart $(U, \varphi)$. We call it the order of zero $a$ of $f$.

Let $M$ be a Riemann surface and $a$ a point in $M$. If $f$ is a function holomorphic on $M-\{a\}$ we say that $a$ is an isolated singularity of $f$.

The isolated singularity $a$ of $f$ is removable, if we can define $f(a)$ so that the extended function $f$ is holomorphic on $M$. By [2, Theorem 10.20], we have the following result.
1.2.2. Proposition. Let $a$ be an isolated singularity of a holomorphic function $f$ on a Riemann surface M. Assume that there exists a neighborhood $U$ of a such that $f$ is bounded on $U-\{a\}$. Then $a$ is a removable singularity.

Assume that $a$ is an isolated singularity of $f$ and $\lim _{z \rightarrow a}|f(z)|=+\infty$. Then we say that $a$ is a pole of $f$. If $(U, \varphi)$ is a chart of $M$ centered at $a$, the function $f \circ \varphi^{-1}$ has a pole at 0 .

Clearly, if $a$ is a pole of $f$, there is a neighborhood $U$ of $a$ such that $f$ is different from zero on $U-\{a\}$. Therefore, the function $g=\frac{1}{f}$ is holomorphic on $U-\{a\}$, i.e., $a$ is an isolated singularity of $g$. Moreover, $\lim _{z \rightarrow a}|g(z)|=0$ and $g$ is bounded around $a$. Hence, by $1.2 .2, a$ is a removable singularity of $g$. Moreover, if we extend $g$ to a holomorphic function on $U, a$ is a zero of $g$. The order $m$ of the pole of $f \circ \varphi^{-1}$ is equal to the order of zero $g \circ \varphi^{-1}$ at 0 . Therefore, it is equal to the order of zero of $g$ at $a$, and independent of the choice of the chart $(U, \varphi)$. Therefore, we say that $m$ is the order of the pole $a$ of $f$.

If the isolated singularity $a$ is neither a removable singularity nor a pole of $f$, we say that it is an essential singularity of $f$.

By [2, Theorem 10.21], we immediately see that the following result holds.
1.2.3. Proposition. Let $a$ be an isolated singularity of a holomorphic function $f$ on a Riemann surface $M$. Then the following statements are equivalent
(i) $a$ is an essential singularity of $f$;
(ii) For any neighborhood $U$ of a the image $f(U-\{a\}$ of the punctured neighborhood $U-\{a\}$ is dense in $\mathbb{C}$.
1.3. Holomorphic functions on compact Riemann surfaces. Now we want to prove the following simple consequence of the maximum principle.
1.3.1. Theorem. Let $M$ be a compact Riemann surface. Then any holomorphic function on $f$ is constant.

Proof. Consider the real continuous function $|f|$ on $M$ defined by $|f|(z)=|f(z)|$ for any $z \in M$. Since $M$ is compact this function has a maximum at a point $a \in M$. Let $(U, \phi)$ be a chart centered at $a$ (with $U$ connected). Then the function $f \circ \phi^{-1}$ is a holomorphic function on $\phi(U)$ such that $\left|f \circ \phi^{-1}\right|$ has a maximum at 0 . By the maximum modulus principle [2, Theorem 10.24], $f \circ \phi^{-1}$ is constant on $\phi(U)$. Therefore, $f$ is constant on $U$, i.e., $f(z)=c$ for any $z \in U$ and some $c \in \mathbb{C}$. It follows that $f-c$ is zero on $U$. By 1.2.1, $f-c$ is zero on $M$, i.e., $f$ is constant.
1.4. Riemann sphere. Now we are going to construct a compact Riemann surface.

Let $X=\mathbb{C}^{2}$ and $X^{*}=\mathbb{C}^{2}-\{(0,0)\}$. We define an equivalence relation on $X^{*}$ by $\left(z_{0}, z_{1}\right) \sim\left(\zeta_{0}, \zeta_{1}\right)$ if there exists $t \in \mathbb{C}^{*}$ such that $\zeta_{i}=t z_{i}, i=0,1$. The equivalence class of $\left(z_{0}, z_{1}\right)$ is denoted by $\left[z_{0}, z_{1}\right]$. Let $Y$ be the set of all equivalence classes in $X^{*}$ and denote by $p: X^{*} \longrightarrow Y$ the natural projection given by $p\left(z_{0}, z_{1}\right)=\left[z_{0}, z_{1}\right]$ for any $\left(z_{0}, z_{1}\right) \in X^{*}$. We equip $Y$ with the quotient topology. Then $p: X^{*} \longrightarrow Y$ is a continuous map.

Consider the map $\varphi: X^{*} \longrightarrow \mathbb{R}^{3}$ given by

$$
\varphi\left(z_{0}, z_{1}\right)=\left(\frac{2 \operatorname{Re}\left(z_{0} \bar{z}_{1}\right)}{\left|z_{0}\right|^{2}+\left|z_{1}\right|^{2}}, \frac{2 \operatorname{Im}\left(z_{0} \bar{z}_{1}\right)}{\left|z_{0}\right|^{2}+\left|z_{1}\right|^{2}}, \frac{\left|z_{0}\right|^{2}-\left|z_{1}\right|^{2}}{\left|z_{0}\right|^{2}+\left|z_{1}\right|^{2}}\right) .
$$

This is clearly a continuous map. Let $S^{2}$ be the sphere in $\mathbb{R}^{3}$ of radius 1 centered at the origin. Then, we have

$$
\begin{aligned}
& \left(2 \operatorname{Re}\left(z_{0} \bar{z}_{1}\right)\right)^{2}+\left(2 \operatorname{Im}\left(z_{0} \bar{z}_{1}\right)\right)^{2}+\left(\left|z_{0}\right|^{2}-\left|z_{1}\right|^{2}\right)^{2} \\
& \quad=4\left|z_{0} \bar{z}_{1}\right|^{2}+\left|z_{0}\right|^{4}-2\left|z_{0}\right|^{2}\left|z_{1}\right|^{2}+\left|z_{1}\right|^{4}=\left(\left|z_{0}\right|^{2}+\left|z_{1}\right|^{2}\right)^{2}
\end{aligned}
$$

and $\varphi\left(z_{0}, z_{1}\right)$ is in $S^{2}$ for any $\left(z_{0}, z_{1}\right) \in X^{*}$. Therefore, we can view $\varphi$ as a continuous map of $X^{*}$ into $S^{2}$.

Moreover, $\varphi$ is constant on the equivalence classes in $X^{*}$ and defines a continuous map $\Phi$ of $Y$ into $S^{2}$.
1.4.1. Lemma. The map $\Phi: Y \longrightarrow S^{2}$ is a homeomorphism.

Let $U_{0}=\left\{\left[z_{0}, z_{1}\right] \in Y \mid z_{1} \neq 0\right\}$ and $U_{1}=\left\{\left[z_{0}, z_{1}\right] \in Y \mid z_{0} \neq 0\right\}$. Then $U_{0}$ and $U_{1}$ are open subsets in $Y$ and $Y-U_{0}=\{[1,0]\}$ and $Y-U_{1}=\{[0,1]\}$. Hence, $\left\{U_{0}, U_{1}\right\}$ is an open cover of $Y$. Clearly, $\Phi([1,0])=(0,0,1)$. If $z_{1} \neq 0$, $\left[z_{0}, z_{1}\right]=[z, 1]$ for $z=\frac{z_{0}}{z_{1}}$, and

$$
\Phi([z, 1])=\left(\frac{2 \operatorname{Re} z}{|z|^{2}+1}, \frac{2 \operatorname{Im} z}{|z|^{2}+1}, \frac{|z|^{2}-1}{|z|^{2}+1}\right)
$$

Let $(a, b, c)=\Phi([z, 1])$. Then, $-1 \leq c<1$, and $U_{0}$ maps into $S^{2}-\{(0,0,1)\}$. Moreover, the equality $c=\frac{|z|^{2}-1}{|z|^{2}+1}$ implies that $|z|^{2}=\frac{1+c}{1-c}$. Hence, $|z|^{2}+1=\frac{2}{1-c}$ and

$$
a=\frac{2 \operatorname{Re} z}{|z|^{2}+1}=(1-c) \operatorname{Re} z \text { and } b=\frac{2 \operatorname{Im} z}{|z|^{2}+1}=(1-c) \operatorname{Im} z
$$

Hence,

$$
z=\frac{a+i b}{1-c}
$$

and $\left.\Phi\right|_{U_{0}}: U_{0} \longrightarrow S^{2}-\{(0,0,1)\}$ is a homeomorphism. Analogously, we can prove that $\left.\Phi\right|_{U_{1}}: U_{1} \longrightarrow S^{2}-\{(0,0,-1)\}$ is a homeomorphism. Therefore, $\Phi: Y \longrightarrow S^{2}$ is a homeomorphism. It follows that $Y$ is homeomorphic to a two-dimensional sphere, i.e., $Y$ is connected, compact and hausdorff.

Let $\phi_{0}: U_{0} \longrightarrow \mathbb{C}$ given by $\phi_{0}\left(\left[z_{0}, z_{1}\right]\right)=\frac{z_{0}}{z_{1}}$. Then $\phi_{0}$ is a continuous map. Moreover, $z \longmapsto[z, 1]$ for $z \in \mathbb{C}$, is its inverse map. Hence, $\phi_{0}: U_{0} \longrightarrow \mathbb{C}$ is a homeomorphism and we can view $\left(U_{0}, \phi_{0}\right)$ as a chart on $Y$. Analogously, $\phi_{1}$ : $U_{1} \longrightarrow \mathbb{C}$ given by $\phi_{1}\left(\left[z_{0}, z_{1}\right]\right)=\frac{z_{1}}{z_{0}}$, is a homeomorphism of $U_{1}$ onto $\mathbb{C}$. Hence, $\left(U_{1}, \phi_{1}\right)$ is another chart on $Y$.

Clearly, $U_{0} \cap U_{1}=Y-\{[0,1],[1,0]\}$ and $\phi_{i}\left(U_{0} \cap U_{1}\right)=\mathbb{C}^{*}=\mathbb{C}-\{0\}$ for $i=0,1$. Moreover,

$$
\left(\phi_{0} \circ \phi_{1}^{-1}\right)(z)=\phi_{0}([1, z])=\frac{1}{z}
$$

for any $z \in \mathbb{C}^{*}$, i.e., this is a holomorphic isomorphism.
Therefore, our two charts are compatible and cover $Y$. Therefore, they define a one-dimensional complex manifold structure on $Y$. Clearly, $Y$ is a Riemann surface which is diffeomorphic to a two-dimensional sphere. It called the one-dimensional projective space or Riemann sphere and denoted by $\mathbb{P}^{1}$.

We denote the point $[1,0]$ by $\infty$. The complement of this point is the open submanifold $U_{0}$. The map $\phi_{0}: U_{0} \longrightarrow \mathbb{C}$ is an isomorphism of Riemann surfaces. Therefore, we can identify $U_{0}$ with $\mathbb{C}$ in this way. Therefore, we can view the complex plane $\mathbb{C}$ as this open submanifold of $\mathbb{P}^{1}$.

Also, we will call $[1,0]$ the north pole, and $[0,1]$ the south pole of the Riemman sphere. These poles correspond to points $(0,0,1)$ and $(0,0,-1)$ in $S^{2}$. The circle $\{(x, y, z) \mid z=0\}$ is the equator of the Riemann sphere. It corresponds to $\{[z, 1] \mid$ $|z|=1\}$. Also we consider open upper hemisphere $\left\{(x, y, z) \in S^{2} \mid z>0\right\}$ and lower hemisphere $\left\{(x, y, z) \in S^{2} \mid z<0\right\}$.

The Riemann surface $\mathbb{P}^{1}$ is compact and simply connected. By 1.3.1, the only holomorphic functions on $\mathbb{P}^{1}$ are constants.

Finally, we remark that this observation is closely related to the Liouville's theorem [2, Theorem 10.23]. If $f$ is a bounded entire function, $f \circ \phi_{0}$ is a bounded holomorphic function on $U_{0}=\mathbb{P}^{1}-\{\infty\}$. Therefore, it has an isolated singularity at $\infty$. By 1.2 .2 , this singularity is removable. Hence, the function extends to a holomorphic function on the Riemann sphere. as we remarked, this function must be constant. Hence, it follows that $f$ is also constant.
1.5. Linear fractional transformations. Let $M_{2}(\mathbb{C})$ be the algebra of all 2-by-2 complex matrices. Let $\operatorname{GL}(2, \mathbb{C})$ be the subset of all regular matrices in $M_{2}(\mathbb{C})$, i.e., the set of all matrices

$$
\gamma=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)
$$

with $a d-b c \neq 0$. This set with matrix multiplication as a binary operation is a group which is called the general linear group. Elements of that group act on $\mathbb{C}^{2}$ by

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{z_{0}}{z_{1}}=\binom{a z_{0}+b z_{1}}{c z_{0}+d z_{1}}
$$

Clearly, under this action the point $(1,0)$ maps into the point $(a, c)$. Therefore, we can find a matrix $\gamma$ in $\operatorname{GL}(2, \mathbb{C})$ which maps $(1,0)$ into an arbitrary point in $\mathbb{C}^{2}-\{(0,0)\}$. It follows that $\mathrm{GL}(2, \mathbb{C})$ acts on $\mathbb{C}^{2}$ with exactly two orbits: the origin $\{(0,0)\}$ and its complement $\mathbb{C}^{2}-\{(0,0)\}$. Clearly, by the above formula, the action of an element $\gamma$ in $\operatorname{GL}(2, \mathbb{C})$ carries an equivalence class with respect to $\sim$ into another equivalence class. Hence, $\gamma$ induces a map $\varphi_{\gamma}: \mathbb{P}^{1} \longrightarrow \mathbb{P}^{1}$ given by the formula

$$
\varphi_{\gamma}\left(\left[z_{0}, z_{1}\right]\right)=\left[a z_{0}+b z_{1}, c z_{0}+d z_{1}\right]
$$

This defines an action of $\operatorname{GL}(2, \mathbb{C})$ on $\mathbb{P}^{1}$ and this action is transitive.
1.5.1. Lemma. The action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{P}^{1}$ is transitive.

We claim that $\varphi_{\gamma}: \mathbb{P}^{1} \longrightarrow \mathbb{P}^{1}$ are holomorphic isomorphisims. Let $V=$ $\left\{\left[z_{0}, z_{1}\right] \in \mathbb{P}^{1} \mid c z_{0}+d z_{1} \neq 0\right\}$. Then $V=\mathbb{P}^{1}-\{[d,-c]\}$ is an open set in $\mathbb{P}^{1}$. Then we have $\varphi_{\gamma}(V) \subset U_{0}$ and

$$
\left(\phi_{0} \circ \varphi_{\gamma}\right)\left(\left[z_{0}, z_{1}\right]\right)=\frac{a z_{0}+b z_{1}}{c z_{0}+d z_{1}}
$$

for any $\left[z_{0}, z_{1}\right]$ in $V$.
If $c=0, V=U_{0}$ and

$$
\left(\phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}\right)(z)=\left(\phi_{0} \circ \varphi_{\gamma}\right)([z, 1])=\frac{a z+b}{d}
$$

for any $z \in \varphi_{0}^{-1}\left(U_{0}\right)=\mathbb{C}$. Therefore, $\varphi_{\gamma}: V \longrightarrow U_{0}$ is a holomorphic map.
If $d=0, V=U_{1}$ and

$$
\left(\phi_{0} \circ \varphi_{\gamma} \circ \phi_{1}^{-1}\right)(z)=\left(\phi_{0} \circ \varphi_{\gamma}\right)([1, z])=\frac{a+b z}{c}
$$

for any $z \in \varphi_{1}^{-1}\left(U_{1}\right)=\mathbb{C}$. Therefore, $\varphi_{\gamma}: V \longrightarrow U_{0}$ is a holomorphic map.
If neither $c=0$ nor $d=0$, we have

$$
\left(\phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}\right)(z)=\left(\phi_{0} \circ \varphi_{\gamma}\right)([z, 1])=\frac{a z+b}{c z+d}
$$

for any $z \in \varphi_{0}^{-1}\left(U_{0} \cap V\right)=\mathbb{C}-\left\{-\frac{d}{c}\right\}$. Therefore, $\varphi_{\gamma}: V \cap U_{0} \longrightarrow U_{0}$ is a holomorphic map.

Analogously, we have

$$
\left(\phi_{0} \circ \varphi_{\gamma} \circ \phi_{1}^{-1}\right)(z)=\left(\phi_{0} \circ \varphi_{\gamma}\right)([1, z])=\frac{a+b z}{c+d z}
$$

for any $z \in \varphi_{1}^{-1}\left(U_{1} \cap V\right)=\mathbb{C}-\left\{-\frac{c}{d}\right\}$. Hence, $\varphi_{\gamma}: V \cap U_{1} \longrightarrow U_{0}$ is a holomorphic map. Since $U_{0}$ and $U_{1}$ cover $\mathbb{P}^{1}$, it follows that $\phi_{\gamma}: V \longrightarrow U_{0}$ is a holomorphic map.

Analogously, let $W=\left\{\left[z_{0}, z_{1}\right] \in \mathbb{P}^{1} \mid a z_{0}+b z_{1} \neq 0\right\}$. Then $W=\mathbb{P}^{1}-\{[b,-a]\}$ is an open set in $\mathbb{P}^{1}$. Clearly, if $[b,-a]=[d,-c]$, we would have $a=t c$ and $b=t d$ for some $t \in \mathbb{C}^{*}$ and $a d-b c=t c d-t d c=0$ contradicting the regularity of $\gamma$. Therefore $V$ and $W$ form an open cover of $\mathbb{P}^{1}$. Moreover, $\phi_{\gamma}(W) \subset U_{1}$ and

$$
\left(\phi_{1} \circ \varphi_{\gamma}\right)\left(\left[z_{0}, z_{1}\right]\right)=\frac{c z_{0}+d z_{1}}{a z_{0}+b z_{1}}
$$

for any $\left[z_{0}, z_{1}\right]$ in $W$.

If $a=0, W=U_{0}$ and

$$
\left(\phi_{1} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}\right)(z)=\left(\phi_{1} \circ \varphi_{\gamma}\right)([z, 1])=\frac{c z+d}{b}
$$

for any $z \in \varphi_{0}^{-1}\left(U_{0}\right)=\mathbb{C}$. Therefore, $\varphi_{\gamma}: W \longrightarrow U_{1}$ is a holomorphic map.
If $b=0, W=U_{1}$ and

$$
\left(\phi_{1} \circ \varphi_{\gamma} \circ \phi_{1}^{-1}\right)(z)=\left(\phi_{0} \circ \varphi_{\gamma}\right)([1, z])=\frac{c+d z}{a}
$$

for any $z \in \varphi_{1}^{-1}\left(U_{1}\right)=\mathbb{C}$. Therefore, $\varphi_{\gamma}: W \longrightarrow U_{1}$ is a holomorphic map.
If neither $a=0$ nor $b=0$, we have

$$
\left(\phi_{1} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}\right)(z)=\left(\phi_{0} \circ \varphi_{\gamma}\right)([z, 1])=\frac{c z+d}{a z+b}
$$

for any $z \in \varphi_{0}^{-1}\left(U_{0} \cap W\right)=\mathbb{C}-\left\{-\frac{b}{a}\right\}$. Therefore, $\varphi_{\gamma}: W \cap U_{0} \longrightarrow U_{1}$ is a holomorphic map.

Analogously, we have

$$
\left(\phi_{1} \circ \varphi_{\gamma} \circ \phi_{1}^{-1}\right)(z)=\left(\phi_{0} \circ \varphi_{\gamma}\right)([1, z])=\frac{c+d z}{a+b z}
$$

for any $z \in \varphi_{0}^{-1}\left(U_{1} \cap V\right)=\mathbb{C}-\left\{-\frac{a}{b}\right\}$. Hence, $\varphi_{\gamma}: W \cap U_{1} \longrightarrow U_{1}$ is a holomorphic map. Since $U_{0}$ and $U_{1}$ cover $\mathbb{P}^{1}$, it follows that $\phi_{\gamma}: W \longrightarrow U_{1}$ is a holomorphic map.

It follows that $\phi_{\gamma}: \mathbb{P}^{1} \longrightarrow \mathbb{P}^{1}$ is a holomorphic map. Clearly, its inverse is $\phi_{\gamma^{-1}}$. Therefore, $\varphi_{\gamma}$ is a holomorphic automorphism of the Riemann sphere $\mathbb{P}^{1}$.

Holomorphic automorphisms $\varphi_{\gamma}$ for $\gamma \in \mathrm{GL}(2, \mathbb{C})$, are called linear fractional transformations. Clearly, $\gamma \longmapsto \varphi_{\gamma}$ is a group homomorphism of $\operatorname{GL}(2, \mathbb{C})$ into $\operatorname{Aut}\left(\mathbb{P}^{1}\right)$.

Assume that $\phi_{\gamma}$ is identity on $\mathbb{P}^{1}$. Then $[1,0]=\phi_{\gamma}([1,0])=[a, b]$ and $[0,1]=$ $\phi_{\gamma}([0,1])=[c, d]$. Hence, $b=0, c=0$ and $\gamma$ has to be diagonal. On the other hand, for any other point, we have $\left[z_{0}, z_{1}\right]=\phi_{\gamma}\left(\left[z_{0}, z_{1}\right]\right)=\left[a z_{0}, d z_{1}\right]$, i.e., $a=d$. Hence, $\gamma$ is a non-zero multiple of the identity matrix. These matrices form the center $Z$ of the group $\operatorname{GL}(2, \mathbb{C})$.

The subset $\operatorname{SL}(2, \mathbb{C})$ of $\mathrm{GL}(2, \mathbb{C})$ consisting of all matrices with determinant equal to 1 is a subgroup of $G L(2, \mathbb{C})$. This subgroup is called the special linear group. The center of that subgroup is $\pm I$ where $I$ is the identity matrix. The quotient of $\operatorname{SL}(2, \mathbb{C}) /\{ \pm I\}$ is denoted by $\operatorname{PSL}(2, \mathbb{C})$.

Let $\gamma \in \mathrm{GL}(2, \mathbb{C})$. Then $\operatorname{det} \gamma \in \mathbb{C}^{*}$, Let $z$ be a square root of $\operatorname{det} \gamma$. Then the product $\delta$ of $\gamma$ with $z^{-1} I$ is an element of $\operatorname{SL}(2, \mathbb{C})$ and $\varphi_{\delta}=\varphi_{\gamma}$. Therefore, the image of $\mathrm{GL}(2, \mathbb{C})$ under the homomorphism $\gamma \longmapsto \phi_{\gamma}$ is equal to the image of its restriction to $\mathrm{SL}(2, \mathbb{C})$. It follows that the subgroup of all linear fractional transformations in $\operatorname{Aut}\left(\mathbb{P}^{1}\right)$ is isomorphic to $\operatorname{PSL}(2, \mathbb{C})$.

Therefore, we established the monomorphism part of the following result. The epimorphism part will be established in the next section.
1.5.2. Theorem. The homomorphism $\gamma \longmapsto \varphi_{\gamma}$ of $\operatorname{PSL}(2, \mathbb{C})$ into the group of all holomorphic automorphisms $\operatorname{Aut}\left(\mathbb{P}^{1}\right)$ of the Riemann sphere is an isomorphism.
1.6. Automorphisms of the complex plane. Let $\psi$ be a holomorphic automorphism of $\mathbb{C}$. Then we can view $\psi$ as a entire function. Consider the function $f: z \longmapsto \psi\left(\frac{1}{z}\right)$. Then this function has a an isolated singularity at 0 . Assume that this singularity is essential. Then, for any $R>0$, the image under $f$ of the punctured disk $D^{\prime}\left(0, \frac{1}{R}\right)$ would be dense in $\mathbb{C}$. This would imply that the image under $\psi$ of the complement of the disk $D(0, R)$ is dense in $\mathbb{C}$. Since $\psi$ is a homeomorphism of the complex plane, this image would also be closed. Therefore, it would be equal to the whole plane, what is impossible. Therefore, the isolated singularity is either removable or a pole. This implies that there exists $p \in \mathbb{Z}_{+}$such that the function $z \longmapsto z^{p} f(z)$ has a removable singularity at 0 . This in turn implies that the function $z \longmapsto \frac{\psi(z)}{z^{p}}$ is bounded on the complement of $D(0, R)$.

Consider now the Taylor series

$$
\psi(z)=\sum_{n=0}^{\infty} c_{n} z^{n}
$$

of $\psi$. Since $\psi$ is entire it converges in the whole complex plane. Moreover, if $\gamma$ is a positively oriented circle of radius $R$ centered at 0 , we have that

$$
c_{n}=\frac{1}{2 \pi i} \int_{\gamma} \frac{\psi(z)}{z^{n+1}} d z
$$

satisfies

$$
\left|c_{n}\right| \leq \frac{\max _{|z|=R}|\psi(z)|}{R^{n}}=\frac{\max _{|z| \geq R}\left|\frac{\psi(z)}{z^{p}}\right|}{R^{n-p}}
$$

Hence, by taking the limit as $R \rightarrow \infty$, we get that $c_{n}=0$ for $n>p$. It follows that $\psi$ is a polynomial.

This immediately implies that $\psi^{\prime}$ is also a polynomial. Since $\psi$ is a holomorphic automorphism of $\mathbb{C}$, we have, by the chain rule, that $\left(\psi^{-1}\right)^{\prime}(\psi(z)) \psi^{\prime}(z)=1$. Hence, $\psi^{\prime}(z) \neq 0$ for any $z \in \mathbb{C}$. This implies that $\psi^{\prime}$ must be a constant polynomial. Hence $\psi$ is linear function. It follows that $\psi(z)=a z+b$ with $a \neq 0$.

Therefore, we proved the following result.
1.6.1. Lemma. Any holomorphic automorphism of the complex plane $\mathbb{C}$ is given by a nonconstant linear function.

As before, we identify $\mathbb{C}$ with the subset $U_{0}=\mathbb{P}^{1}-\{\infty\}$ of the Riemann sphere via the map $\phi_{0}: U_{0} \longrightarrow \mathbb{C}$.

Consider the fractional linear transformation $\varphi_{\gamma}$ corresponding to the matrix

$$
\gamma=\left(\begin{array}{cc}
a & b \\
0 & a^{-1}
\end{array}\right)
$$

with $a \neq 0$. Then, $\phi_{\gamma}$ fixes the point $\infty$. Therefore, it maps $U_{0}$ into $U_{0}$. Under our identification, we have

$$
\left(\phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}\right)(z)=\phi_{0}\left(\varphi_{\gamma}([z, 1])\right)=\phi_{0}\left(\left[a z+b, a^{-1}\right]\right)=a^{2} z+a b
$$

for all $z \in \mathbb{C}$. It follows that the subgroup of the group $\operatorname{SL}(2, \mathbb{C})$ consisting of all matrices fixing the point $\infty$ is equal to the subgroup $B$ of all upper triangular matrices

$$
\left\{\left.\left(\begin{array}{cc}
a & b \\
0 & a^{-1}
\end{array}\right) \right\rvert\, a \neq 0\right\}
$$

in $\operatorname{SL}(2, \mathbb{C})$. Moreover, every such matrix defines a fractional linear transformation which induces a holomorphic automorphism of $\mathbb{C}$.
1.6.2. Theorem. The homomorphism $\left.\gamma \longmapsto \varphi_{\gamma}\right|_{\mathbb{C}}$ is a surjective homomorphism of the subgroup of all upper triangular matrices $B$ in $\mathrm{SL}(2, \mathbb{C})$ onto the group of all holomorphic automorphisms $\operatorname{Aut}(\mathbb{C})$ of $\mathbb{C}$.

The kernel of this homomorphism is $\{ \pm I\}$.
Now we can complete the proof of 1.5.2. Let $\psi$ be an holomorphic automorphism of $\mathbb{P}^{1}$. By 1.5.1, the action of $\operatorname{SL}(2, \mathbb{C})$ on $\mathbb{P}^{1}$ is transitive, and there exists $\gamma \in$ $\mathrm{SL}(2, \mathbb{C})$ such that $\varphi_{\gamma^{-1}} \circ \psi$ fixes $\infty$. By the above discussion, $\varphi_{\gamma^{-1}} \circ \psi$ is $\varphi_{\delta}$ for some upper triangular matrix $\delta \in \operatorname{SL}(2, \mathbb{C})$. Hence, we have $\psi=\varphi_{\gamma \delta}$. This completes the proof of 1.5.2.
1.7. Automorphisms of the unit disk. We consider the unit disk $D(0,1)$ as imbedded into Riemann sphere by the inclusion $\varphi_{0}: \mathbb{C} \longrightarrow U_{0} \subset \mathbb{P}^{1}$. In this way, it corresponds to $\left\{[z, 1]||z|<1\}\right.$. Under the map $\Phi: \mathbb{P}^{1} \longrightarrow S^{2}$, this set is identified with the open lower hemisphere $\left\{(x, y, z) \in S^{2} \mid z<0\right\}$ of $S^{2}$. The equator of the Riemann sphere $\mathbb{P}^{1}$ corresponds under this map to $\left\{(x, y, z) \in S^{2} \mid z=0\right\}$.

We consider the subset

$$
\mathrm{SU}(1,1)=\left\{\left.\left(\begin{array}{cc}
a & b \\
\bar{b} & \bar{a}
\end{array}\right)| | a\right|^{2}-|b|^{2}=1\right\}
$$

of $\operatorname{SL}(2, \mathbb{C})$.
For any matrix

$$
\left(\begin{array}{ll}
a & b \\
\bar{b} & \bar{a}
\end{array}\right)
$$

in $\operatorname{SU}(1,1)$, its inverse is

$$
\left(\begin{array}{cc}
\bar{a} & -b \\
-\bar{b} & a
\end{array}\right)
$$

hence it is again in $\operatorname{SU}(1,1)$. Moreover, if

$$
\left(\begin{array}{ll}
a & b \\
\bar{b} & \bar{a}
\end{array}\right) \text { and }\left(\begin{array}{ll}
c & d \\
\bar{d} & \bar{c}
\end{array}\right)
$$

are in $\mathrm{SU}(1,1)$, then their product

$$
\left(\begin{array}{ll}
a c+b \bar{d} & a d+b \bar{c} \\
\bar{a} \bar{d}+\bar{b} c & \bar{a} \bar{c}+\bar{b} d
\end{array}\right)
$$

is also in $\operatorname{SU}(1,1)$. Hence $\mathrm{SU}(1,1)$ is a subgroup of $\mathrm{SL}(2, \mathbb{C})$.
Clearly, the group $\mathrm{SU}(1,1)$ acts on the Riemann sphere by fractional linear transformations.

Let $\gamma \in \operatorname{SU}(1,1)$. First, if a point $\left[z_{0}, z_{1}\right]$ is on the equator of the Riemann sphere, we have $\left|z_{0}\right|=\left|z_{1}\right|$. Therefore, we have $\frac{z_{0}}{z_{1}}=\frac{z_{1}}{z_{0}}$. Moreover, it follows that

$$
\begin{aligned}
\left|a z_{0}+b z_{1}\right| & =\left|\bar{a} \bar{z}_{0}+\bar{b} \bar{z}_{1}\right|=\left|\frac{\bar{a} z_{1} \bar{z}_{0}+\bar{b} z_{1} \bar{z}_{1}}{z_{1}}\right| \\
= & \frac{\left.\left|\bar{a} z_{1} \bar{z}_{0}+\bar{b}\right| z_{1}\right|^{2} \mid}{\left|z_{1}\right|}=\frac{\left.\left|\bar{a} z_{1} \bar{z}_{0}+\bar{b}\right| z_{0}\right|^{2} \mid}{\left|z_{0}\right|}\left|=\left|\frac{\bar{a} z_{1} \bar{z}_{0}+\bar{b} z_{0} \bar{z}_{0}}{\bar{z}_{0}}\right|=\left|\bar{a} z_{1}+\bar{b} z_{0}\right| .\right.
\end{aligned}
$$

Hence, we have that

$$
\varphi_{\gamma}\left(\left[z_{0}, z_{1}\right]\right)=\left[a z_{0}+b z_{1}, \bar{b} z_{0}+\bar{a} z_{1}\right]
$$

is also on the equator of $\mathbb{P}^{1}$. It follows that the equator is invariant for the action of $\operatorname{SU}(1,1)$. If $b=0$ and $|a|=1$, we see that

$$
\varphi_{\gamma}\left(\left[z_{0}, z_{1}\right]\right)=\left[a z_{0}, \bar{a} z_{1}\right]=\left[a^{2} z_{0}, z_{1}\right] .
$$

Therefore, $\mathrm{SU}(1,1)$ acts transitively on the equator, i.e., the equator is an orbit for the action of $\operatorname{SU}(1,1)$.

This implies that for any $\gamma \in \mathrm{SU}(1,1), \varphi_{\gamma}$ maps the complement of the equator onto itself. Moreover, $\varphi_{\gamma}([1,0])=[a, \bar{b}]$. Since $|a|^{2}-|b|^{2}=1$, we see that this point is in the open upper hemisphere. On the other hand, $\varphi_{\gamma}([0,1])=[b, \bar{a}]$, i.e., this point is in the open lower hemisphere. Since open hemispheres are connected, by continuity we conclude that $\varphi_{\gamma}$ leaves them invariant.

On the other hand, for $|z|<1$, if we put

$$
\gamma=\left(\begin{array}{cc}
\frac{1}{\sqrt{1-|z|^{2}}} & \frac{z}{\sqrt{1-|z|^{2}}} \\
\frac{\bar{z}}{\sqrt{1-|z|^{2}}} & \frac{1}{\sqrt{1-|z|^{2}}}
\end{array}\right)
$$

then $\gamma \in \operatorname{SU}(1,1)$. Moreover, we have $\varphi_{\gamma}([1,0])=[1, \bar{z}]$ and $\varphi_{\gamma}([0,1])=[z, 1]$. Therefore, $\mathrm{SU}(1,1)$ acts transitively on open upper and lower hemisphere. Therefore, they are also orbits of $\operatorname{SU}(1,1)$.
1.7.1. Lemma. The group $\mathrm{SU}(1,1)$ acts on the Riemann sphere with three orbits: the equator, open upper hemisphere and open lower hemisphere.

Since $\phi_{0}$ maps the open lower hemisphere in $\mathbb{P}^{1}$ onto the unit disk, we conclude that for any $\gamma$ in $\operatorname{SU}(1,1), \phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}$ is a holomorphic automorphism of the unit disk $D(0,1)$.

It follows that $\gamma \longmapsto \phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}$ is a group homomorphism of $\operatorname{SU}(1,1)$ into the $\operatorname{group} \operatorname{Aut}(D(0,1))$.

If $\phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}$ is the identity on the unit disk, we have

$$
\left.z=\left(\phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}\right)(z)=\left(\phi_{0} \circ \varphi_{\gamma}\right)(z, 1]\right)=\phi_{0}[a z+b, \bar{b} z+\bar{a}]=\frac{a z+b}{\bar{b} z+\bar{a}}
$$

Evaluating at $z=0$ immediately implies that $b=0$. Hence $|a|=1$ and $z=a^{2} z$ for any $z \in D(0,1)$. Hence, $a= \pm 1$. Hence the kernel of the homomorphism of $\mathrm{SU}(1,1)$ into $\operatorname{Aut}(D(0,1))$ is equal to $\{ \pm I\}$.

Let $\psi$ be an element of $\operatorname{Aut}(D(0,1))$. Then, $\psi(0)=z$ with $|z|<1$. By the above discussion, there exists $\gamma \in \mathrm{SU}(1,1)$ such that $\varphi_{\gamma}([0,1])=[z, 1]$. Hence, the composition of $\phi_{0} \circ \varphi_{\gamma^{-1}} \circ \phi_{0}^{-1}$ with $\psi$ is an automorphism of $D(0,1)$ which fixes the origin. The rest is based on the following observation.
1.7.2. Lemma. Let $\Phi$ be an holomorphic automorphism of $D(0,1)$ which fixes the origin. Then there exists $\lambda \in \mathbb{C},|\lambda|=1$, such that $\Phi(z)=\lambda z$ for all $z \in D(0,1)$.

Therefore, if $a$ is a complex number such that $a^{2}=\lambda, \Phi=\phi_{0} \circ \varphi_{\delta} \circ \phi_{0}^{-1}$ for

$$
\delta=\left(\begin{array}{cc}
a & 0 \\
0 & a^{-1}
\end{array}\right)
$$

in $\mathrm{SU}(1,1)$. Hence, assuming the lemma, we see that $\phi_{0} \circ \varphi_{\delta} \circ \phi_{0}^{-1}=\left(\phi_{0} \circ \varphi_{\gamma^{-1}} \circ\right.$ $\left.\phi_{0}^{-1}\right) \circ \psi$ and $\psi=\phi_{0} \circ \varphi_{\gamma \cdot \delta} \circ \phi_{0}^{-1}$. It follows that $\phi$ is induced by a linear fractional transformation corresponding to an element of $\mathrm{SU}(1,1)$.

This proves the following result.
1.7.3. Theorem. The homomorphism $\gamma \longmapsto \phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}$ is a surjective homomorphism of $\mathrm{SU}(1,1)$ onto the group of all holomorphic automorphisms $\operatorname{Aut}(D(0,1))$ of $D(0,1)$.

The kernel of this homomorphism is $\{ \pm I\}$.
It remains to prove 1.7.2. The proof is based on the Schwartz lemma [2, Theorem 12.2]. Clearly, $\Phi$ and $\Phi^{-1}$ map $D(0,1)$ into itself. Hence, by Schwartz lemma, $\left|\Phi^{\prime}(0)\right| \leq 1$ and $\left|\left(\Phi^{-1}\right)^{\prime}(0)\right| \leq 1$. On the other hand, by the chain rule, $\Phi^{\prime}(0)\left(\Phi^{-1}\right)^{\prime}(0)=1$. Hence, we must have $\left|\Phi^{\prime}(0)\right|=1$. By Schwartz lemma, this implies that $\Phi(z)=\lambda z$ for all $z \in D(0,1)$.
1.8. Fixed points of automorphisms. Let $M$ be one of Riemann surfaces we studied in previous sections, and $\varphi$ a holomorphic automorphism of $M$. We want to check when $\varphi$ has a fixed point in $M$.

The situation is especially simple for the Riemann sphere.
1.8.1. Theorem. Any holomorphic automorphism of the Riemann sphere has a fixed point.

Proof. By 1.5.2, any holomorphic automorphism of the Riemann sphere is represented by linear fractional transformation $\varphi_{\gamma}$ for some matrix $\gamma$ in $\operatorname{SL}(2, \mathbb{C})$. This matrix has an nonzero eigenvalue, and the corresponding eigenvector defines a line in $\mathbb{C}^{2}$ which determines a point in $\mathbb{P}^{1}$. Clearly, this point is fixed by $\varphi_{\gamma}$.

More precisely, any $\gamma \in \mathrm{SL}(2, \mathbb{C})$ has two eigenvalues $\lambda$ and $\frac{1}{\lambda}$ with $\lambda \in \mathbb{C}-\{0\}$. Therefore, if $\lambda$ is also different from $\pm 1$, these eigenvalues are distinct. Hence, in this case $\gamma$ has two linearly independent eigenvectors. Each of them determines a line in $\mathbb{C}^{2}$ which is invariant under $\gamma$. Hence, the corresponding automorphism $\varphi_{\gamma}$ has two fixed points in $\mathbb{P}^{1}$. If the eigenvalues are $\pm 1$, either $\gamma= \pm I$ and $\phi_{\gamma}$ acts trivially on $\mathbb{P}^{1}$, or $\gamma$ is $\pm \delta$ where $\delta$ is a unipotent matrix different from $I$. In this case, $\gamma$ has only one linearly independent eigenvector. The corresponding line determines the unique fixed point of $\gamma$ in $\mathbb{P}^{1}$.

The answer is also simple in the case of complex plane $\mathbb{C}$. By 1.6.1, any holomorphic automorphisms of $\mathbb{C}$ is given by a nonconstant linear function $z \longmapsto a z+b$. If $w$ is a fixed point of that automorphism, we have $a w+b=w$ and $(a-1) w=-b$. This equation has a solution if and only if either $a \neq 1$ or $b=0$. But $a=1$ and $b=0$ implies that the automorphism is the identity map. Therefore, we proved the following result.
1.8.2. Theorem. The only holomorphic automorphisms of the complex plane acting without fixed points are translations $z \longmapsto z+b$ with $b \neq 0$.

It remains to study the action of the elements $\mathrm{SU}(1,1)$ on the unit disk $D(0,1)$.
1.9. Some properties of elements of $\operatorname{SU}(1,1)$. To study $\mathrm{SU}(1,1)$ more carefully it is useful to construct an isomorphism of that group with the group $\operatorname{SL}(2, \mathbb{R})$ of real two-by-two matrices with determinant 1. Consider the matrix

$$
T=\frac{1}{\sqrt{2}}\left(\begin{array}{ll}
1 & i \\
i & 1
\end{array}\right)
$$

Its inverse is

$$
T^{-1}=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}
1 & -i \\
-i & 1
\end{array}\right)
$$

Moreover, $\operatorname{det} T=\operatorname{det} T^{-1}=1$, i.e., $T$ is in $\operatorname{SL}(2, \mathbb{C})$. Clearly, conjugation by $T$, i.e., the map $S \longmapsto T S T^{-1}$ is an automorphism of the group $\operatorname{SL}(2, C)$. Let

$$
S=\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)
$$

be a matrix in $\mathrm{SL}(2, \mathbb{R})$, i.e., $\alpha, \beta, \gamma$ and $\delta$ are real and $\alpha \delta-\beta \gamma=1$. Then

$$
\begin{aligned}
& T S T^{-1}=\frac{1}{2}\left(\begin{array}{ll}
1 & i \\
i & 1
\end{array}\right)\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)\left(\begin{array}{cc}
1 & -i \\
-i & 1
\end{array}\right) \\
&=\frac{1}{2}\left(\begin{array}{cc}
\alpha+i \gamma-i \beta+\delta & -i \alpha+\gamma+\beta+i \delta \\
i \alpha+\gamma+\beta-i \delta & \alpha-i \gamma+i \beta+\delta
\end{array}\right)=\left(\begin{array}{ll}
a & b \\
\bar{b} & \bar{a}
\end{array}\right)
\end{aligned}
$$

where

$$
a=\frac{1}{2}(\alpha+\delta-i(\beta-\gamma)) \text { and } b=\frac{1}{2}(\beta+\gamma-i(\alpha-\delta)) .
$$

Clearly, $T S T^{-1}$ is in $\mathrm{SU}(1,1)$. Hence, we have the group homomorphism $\Psi$ : $\mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{SU}(1,1)$ given by $\Psi(S)=T S T^{-1}$.

Moreover, for any matrix

$$
\left(\begin{array}{ll}
a & b \\
\bar{b} & \bar{a}
\end{array}\right)
$$

in $\operatorname{SU}(1,1)$, we can put $\alpha=\operatorname{Re} a-\operatorname{Im} b, \beta=-\operatorname{Im} a+\operatorname{Re} b, \gamma=\operatorname{Im} a+\operatorname{Re} b$ and $\delta=\operatorname{Re} a+\operatorname{Im} b$. Then

$$
\alpha \delta-\beta \gamma=(\operatorname{Re} a)^{2}-(\operatorname{Im} b)^{2}+(\operatorname{Im} a)^{2}-(\operatorname{Re} b)^{2}=|a|^{2}-|b|^{2}=1
$$

i.e.,

$$
S=\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)
$$

is in $\operatorname{SL}(2, \mathbb{R})$ and $\Phi(S)$ is equal to our matrix.
Therefore, $\Psi: \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{SU}(1,1)$ is a group isomorphism.
Since $\Psi$ is a conjugation, the matrices $S$ and $\Psi(S)$ have the same characteristic polynomial, and the same eigenvalues.

Let $S$ be a matrix in $\mathrm{SL}(2, \mathbb{R})$. Then its characteristic polynomial is

$$
\operatorname{det}(\lambda I-S)=\left|\begin{array}{cc}
\lambda-a & -b \\
-c & \lambda-d
\end{array}\right|=\lambda^{2}-(a+d) \lambda+a d-b c=\lambda^{2}-(\operatorname{tr} S) \lambda+1
$$

Therefore, the eigenvalues are

$$
\frac{1}{2}\left(\operatorname{tr} S+\sqrt{(\operatorname{tr} S)^{2}-4}\right) \text { and } \frac{1}{2}\left(\operatorname{tr} S-\sqrt{(\operatorname{tr} S)^{2}-4}\right) .
$$

If $|\operatorname{tr} S| \geq 2$, the eigenvalues are real. If $|\operatorname{tr} S|<2$, the eigenvalues are mutually conjugate complex numbers and their absolute value is

$$
\frac{1}{4}\left((\operatorname{tr} S)^{2}+4-(\operatorname{tr} S)^{2}\right)=1
$$

Therefore, we have the following result.
1.9.1. Lemma. Any matrix from $\mathrm{SL}(2, \mathbb{R})$ (or $\mathrm{SU}(1,1)$ ) has eigenvalues which are either
(1) $\lambda$ and $\frac{1}{\lambda}$ for a real $\lambda$;
(2) $\lambda$ and $\bar{\lambda}$ for complex number $\lambda$ such that $|\lambda|=1$.

Now we want to discuss fixed points of automorphisms of $D(0,1)$.
First, we study the fixed points of $\varphi_{\gamma}$ for $\gamma \in \operatorname{SU}(1,1)$.
Let $\gamma \in \mathrm{SU}(1,1)$ be represented by a matrix

$$
\left(\begin{array}{ll}
a & b \\
\bar{b} & \bar{a}
\end{array}\right)
$$

Assume that $\left[z_{0}, z_{1}\right]$ is a fixed point of $\varphi_{\gamma}$ in $\mathbb{P}^{1}$. Then we have

$$
\left[a z_{0}+b z_{1}, \bar{b} z_{0}+\bar{a} z_{1}\right]=\left[z_{0}, z_{1}\right]
$$

and

$$
\left(\begin{array}{ll}
a & b \\
\bar{b} & \bar{a}
\end{array}\right)\binom{z_{0}}{z_{1}}=t\binom{z_{0}}{z_{1}}
$$

for some $t \in \mathbb{C}^{*}$. This implies that

$$
a z_{0}+b z_{1}=t z_{0} \text { and } \bar{b} z_{0}+\bar{a} z_{1}=t z_{1}
$$

By complex conjugation we get

$$
\bar{a} \bar{z}_{0}+\bar{b} \bar{z}_{1}=\bar{t} \bar{z}_{0} \text { and } b \bar{z}_{0}+a \bar{z}_{1}=\bar{t} \bar{z}_{1}
$$

This implies

$$
\left(\begin{array}{ll}
a & b \\
\bar{b} & \bar{a}
\end{array}\right)\binom{\bar{z}_{1}}{\bar{z}_{0}}=\bar{t}\binom{\bar{z}_{1}}{\bar{z}_{0}}
$$

Assume that the eigenvalue $t$ is real and different from $\pm 1$. From the previous discussion, we know that $\varphi_{\gamma}$ has two fixed points in $\mathbb{P}^{1}$. Therefore, the eigenspace for the eigenvalue $t$ is one-dimensional. Hence we have

$$
\binom{z_{0}}{z_{1}}=s\binom{\bar{z}_{1}}{\bar{z}_{0}}
$$

for some $s \in \mathbb{C}^{*}$. It follows that $z_{0}=s \bar{z}_{1}$ and $z_{1}=s \bar{z}_{0}$. This implies that $z_{0} \neq 0$ and $z_{1} \neq 0$. Moreover, $z_{0}=|s|^{2} z_{0}$, i.e., $|s|=1$. Hence, the fixed point $\left[z_{0}, z_{1}\right]=[s, 1]$ is on the equator of the Riemann sphere. Applying the same argument to the other eigenvalue $\frac{1}{t}$, we see that in this case the automorphism $\varphi_{\gamma}$ has two different fixed points in $\mathbb{P}^{1}$ and both of them are on the equator.

If the eigenvalue $t$ is equal to either 1 or $-1, \gamma$ is either $I$, or $-I$, or all of its eigenvectors are proportional. In the first two cases the action on $\mathbb{P}^{1}$ is trivial. In the third case, we know that we have only one fixed point. By the same argument as above, we see that it is on the equator of $\mathbb{P}^{1}$. Therefore, in all of these cases, nontrivial $\varphi_{\gamma}$ has no fixed points in the open lower hemisphere.

Assume now that $t$ is on the unit circle, but different form $\pm 1$. Then we know that $\varphi_{\gamma}$ has two fixed points in $\mathbb{P}^{1}$. Moreover, $t$ and $\bar{t}$ are two different eigenvalues and

$$
\binom{z_{0}}{z_{1}} \text { and }\binom{\bar{z}_{1}}{\bar{z}_{0}}
$$

are the corresponding linearly independent eigenvectors. If $z_{0}=0$, then the first eigenvector corresponds to the point $[0,1]$, and the second one to $[1,0]$ in $\mathbb{P}^{1}$. If $z_{1}=0$, then the first eigenvector corresponds to the point $[1,0]$, and the second one to $[0,1]$ in $\mathbb{P}^{1}$. Therefore, each open hemisphere of the Riemann sphere contains exactly one fixed point.

If $z_{0} \neq 0$ and $z_{1} \neq 0$, then the first eigenvector corresponds to $\left[1, \frac{z_{1}}{z_{0}}\right]=\left[\frac{z_{0}}{z_{1}}, 1\right]$, and the second one to $\left[\frac{\bar{z}_{1}}{\bar{z}_{0}}, 1\right]$. Since these two eigenvectors are linearly independent, $\frac{z_{0}}{z_{1}} \neq \frac{\bar{z}_{1}}{z_{0}}$ and $\left|z_{0}\right|^{2} \neq\left|z_{1}\right|^{2}$. Therefore, either $\left|\frac{z_{0}}{z_{1}}\right|<1$ or $\left|\frac{z_{0}}{z_{1}}\right|>1$. In the latter case,
we have $\left|\frac{\bar{z}_{1}}{z_{0}}\right|<1$. Hence, we again see that each open hemisphere of the Riemann sphere contains exactly one fixed point.

Hence, if $t$ is on the unit circle, but different form $\pm 1$, the automorphism $\varphi_{\gamma}$ has exactly one fixed point in open power hemisphere.

Therefore, we proved the following result.
1.9.2. Lemma. Let $\gamma \in \operatorname{SU}(1,1), \gamma \neq \pm I$. Then $\phi_{0} \circ \varphi_{\gamma} \circ \phi_{0}^{-1}$ has a fixed point in the unit disk $D(0,1)$ if and only if the eigenvalues of $\gamma$ are $\lambda$ and $\frac{1}{\lambda}$ with $|\lambda|=1$ and $\lambda \neq \pm 1$.
1.10. Some discrete subgroups of $\mathrm{SU}(1,1)$. Let $R$ be a commutative ring with identity 1. Then all two-by-two matrices with entries in $R$ form a ring. Let

$$
\mathrm{SL}(2, R)=\left\{\left.\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right) \right\rvert\, \alpha \delta-\beta \gamma=1\right\}
$$

If

$$
\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right) \text { and }\left(\begin{array}{ll}
\alpha^{\prime} & \beta^{\prime} \\
\gamma^{\prime} & \delta^{\prime}
\end{array}\right)
$$

are two elements in $\operatorname{SL}(2, R)$, their product is

$$
\left(\begin{array}{ll}
\alpha \alpha^{\prime}+\beta \gamma^{\prime} & \alpha \beta^{\prime}+\beta \delta^{\prime} \\
\gamma \alpha^{\prime}+\delta \gamma^{\prime} & \gamma \beta^{\prime}+\delta \delta^{\prime}
\end{array}\right)
$$

and its determinant is

$$
\begin{aligned}
& \left(\alpha \alpha^{\prime}+\beta \gamma^{\prime}\right)\left(\gamma \beta^{\prime}+\delta \delta^{\prime}\right)-\left(\alpha \beta^{\prime}+\beta \delta^{\prime}\right)\left(\gamma \alpha^{\prime}+\delta \gamma^{\prime}\right) \\
& =\alpha \gamma \alpha^{\prime} \beta^{\prime}+\beta \gamma \beta^{\prime} \gamma^{\prime}+\alpha \delta \alpha^{\prime} \delta^{\prime}+\beta \delta \gamma^{\prime} \delta^{\prime}-\alpha \gamma \alpha^{\prime} \beta^{\prime}-\beta \gamma \alpha^{\prime} \delta^{\prime}-\alpha \delta \beta^{\prime} \gamma^{\prime}-\beta \delta \gamma^{\prime} \delta^{\prime} \\
& =(\alpha \delta-\beta \gamma)\left(\alpha^{\prime} \delta^{\prime}-\beta^{\prime} \gamma^{\prime}\right)=1
\end{aligned}
$$

i.e., it is also in $\mathrm{SL}(2, R)$. Hence, $\mathrm{SL}(2, R)$ is closed under multiplication, and for any $S$ in $\mathrm{SL}(2, R)$,

$$
S^{-1}=\left(\begin{array}{cc}
\delta & -\beta \\
-\gamma & \alpha
\end{array}\right)
$$

is the inverse of $S$. Hence $\operatorname{SL}(2, R)$ is a group.
Clearly, all integral matrices in $\mathrm{SL}(2, \mathbb{R})$ form a subgroup $\mathrm{SL}(2, \mathbb{Z})$. Since the integral matrices form a discrete subset in the space of all real two-by-two matrices, $\mathrm{SL}(2, \mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$.
1.10.1. Lemma. The group $\mathrm{SL}(2, \mathbb{Z})$ is infinite.

Proof. Let $\alpha$ and $\beta$ be two relatively prime integers. Then the ideal in $\mathbb{Z}$ generated by $\alpha$ and $\beta$ is equal to $\mathbb{Z}$. In particular, there exist $\delta$ and $\gamma$ in $\mathbb{Z}$ such that $\alpha \delta-\beta \gamma=$ 1, i.e., the matrix

$$
\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)
$$

is in $\operatorname{SL}(2, \mathbb{Z})$. Since there are infinitely many primes in $\mathbb{Z}$, the number of relatively prime integers is also infinite.

Let $\mathbb{Z}_{p}=\mathbb{Z} / p \mathbb{Z}$ for any positive integer $p$. Then, $\mathbb{Z}_{p}$ is a finite ring with identity. Therefore, $\operatorname{SL}\left(2, \mathbb{Z}_{p}\right)$ is a finite group. Let $\rho_{p}: \mathbb{Z} \longrightarrow \mathbb{Z}_{p}$ be the quotient homomorphism. Then it induces a group homomorphism $R_{p}: \operatorname{SL}(2, \mathbb{Z}) \longrightarrow \mathrm{SL}\left(2, \mathbb{Z}_{p}\right)$. The kernel of this homomorphism is a normal subgroup $\Gamma_{p}$ of $\operatorname{SL}(2, \mathbb{Z})$ which is called
the $p^{\text {th }}$ congruence subgroup of $\operatorname{SL}(2, \mathbb{Z})$. Since $\operatorname{SL}\left(2, \mathbb{Z}_{p}\right)$ is finite, $\Gamma_{p}$ is an infinite subgroup.

Let $\gamma$ be in $\Gamma_{p}$. Then $\gamma=I+p S$ where $S$ is a two-by-two integral matrix. Hence, $\operatorname{tr} \gamma=2+p m$ for some integer $m \in \mathbb{Z}$. In particular, if $p>3, \operatorname{tr} \gamma$ cannot be 1,0 and -1 . It follows that $|\operatorname{tr} \gamma| \geq 2$ in this case. Combining this with the discussion preceding the proof of 1.9 .1 we obtain the following result.
1.10.2. Lemma. Let $p>3$. Then any element in $\Gamma_{p}$ has real eigenvalues.

Combining this with 1.9.2, we have the following consequence.
Let $p$ be a positive integer. Under the isomorphism $\Psi: \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{SU}(1,1)$ the congruence subgroup $\Gamma_{p}$ maps into an infinite discrete subgroup $\Gamma_{p}$ of $\mathrm{SU}(1,1)$.
1.10.3. Theorem. Let $p>3$. Then any element of $\Gamma_{p}$ different from the identity acts on the unit disk $D(0,1)$ without fixed points.
1.11. Uniformization theorem. In the preceding sections we considered three Riemann surfaces: complex plane $\mathbb{C}$, Riemann sphere $\mathbb{P}^{1}$ and the unit disk $D(0,1)$ in the complex plane. All of these spaces are connected and simply connected. Clearly, the Riemann sphere is compact and the complex plane and the unit disk are not. Moreover, the complex plane and the unit disk are diffeomorphic.

On the other hand, if we consider holomorphic functions on these Riemann surfaces we observe the following simple facts. First, since $\mathbb{P}^{1}$ is compact, the only holomorphic functions on $\mathbb{P}^{1}$ are constants.

On the other hand, there exist nonconstant entire functions, i.e., holomorphic functions on $\mathbb{C}$. Moreover, by Liouville's theorem [2, Theorem 10.23], the only bounded holomorphic functions on $\mathbb{C}$ are constants. On the other hand, any entire function restricted to the unit disk is holomorphic and bounded. This clearly implies that $\mathbb{P}^{1}, \mathbb{C}$ and $D(0,1)$ are not isomorphic as Riemann surfaces.

Therefore, $\mathbb{P}^{1}, \mathbb{C}$ and $D(0,1)$ are nonisomorphic simply connected Riemann surfaces. The following result, called the uniformization theorem states that this list is exhaustive. Its proof can be found in [1, Theorem 10-3].
1.11.1. Theorem. Any simply connected Riemann surface is isomorphic to either the Riemann sphere, the complex plane or the unit disk.

The uniformization theorem allows to describe more precisely general Riemann surfaces using the theory of covering spaces. In the next few paragraphs we give a rough sketch this classification.

Let $X$ be a Riemann surface, and $\tilde{X}$ its universal cover. Denote by $\pi: \tilde{X} \longrightarrow X$ the covering projection. Then $\tilde{X}$ is a simply connected Riemann surface. Therefore, it is isomorphic to either the Riemann sphere $\mathbb{P}^{1}$, complex plane $\mathbb{C}$ or the unit disk $U=D(0,1)$. The fundamental group of $X$ act on deck transformations on $\tilde{X}$. The nontrivial deck transformations are automorphisms of $\tilde{X}$ which act without fixed points.

Since any automorphism of $\mathbb{P}^{1}$ has a fixed point, the only Riemann surface covered by $\mathbb{P}^{1}$ is $\mathbb{P}^{1}$ itself.

The only automorphisms of $\mathbb{C}$ without fixed points are translations $z \longrightarrow z+b$. The possible groups of deck transformations are discrete subgroups of $\mathbb{C}$. They are isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$.

In the first case, by applying the automorphism $z \longmapsto a z$ with $a \neq 0$, we can assume that the group of deck transformations is generated by $z \longmapsto z+2 \pi i$. In
this case, $X$ is isomorphic to $\mathbb{C}^{*}$ and the covering projection corresponds to the exponential function $\exp : \mathbb{C} \longrightarrow \mathbb{C}^{*}$.

In the second case, the group of deck transformations is a rank two lattice $L$ in $\mathbb{C}=\mathbb{R}^{2}$. The deck transformations are translations $z \longmapsto z+l$ for $l \in L$. By applying the automorphism $z \longmapsto a z$ with $a \neq 0$, we can assume that one of the generators of $L$ is 1 . The other is a complex number $c$ such that $\operatorname{im} c \neq 0$. Since these two elements are linearly independent over $\mathbb{R}$, as a real manifold $X$ is isomorphic to the two dimensional torus $T^{2}$. These Riemann surfaces are called elliptic curves. The theory of functions on them is equivalent to the theory of elliptic functions.

Finally, if $\tilde{X}$ is isomorphic to $U$, the Riemann surface $X$ is the quotient of $U$ with respect to a torsion free discrete subgroup $\Gamma$ of $\mathrm{SU}(1,1)$. A series of nontrivial examples of such subgroups was constructed in the preceding section. The theory of functions on these Riemann surfaces is equivalent to the theory of automorphic functions.

## References

[1] Lars Ahlfors, Conformal Invariants: Topics in Geometric Function Theory, McGraw-Hill, New York, 1973.
[2] Walter Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1987.

