Math 6220-1 Midterm 1, March 1, 2012

1. Determine if there exists an entire function such that
$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n}$$
for all $n \in \mathbb{N}$.

2. Let f be an entire function. Suppose that in its power series expansions
$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$
for any $a \in \mathbb{C}$, at least one coefficient is equal to zero (where n can depend on a). Show that f is a polynomial.

3. Suppose that f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. What conclusions can you draw?

4. Let
$$f(z) = 1 - \cos z.$$
(i) Find all zeros of this function;
(ii) find the multiplicities of these zeros.

In the next three problems we develop the basic facts about *Laurent series*. A Laurent series
$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$
around $a \in \mathbb{C}$ is the sum of the series
$$\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} \text{ and } \sum_{n=0}^{\infty} c_n (z-a)^n.$$
The first series is called *the principal part* and the second *the regular part* of the Laurent series.

5. Let
$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$
be a Laurent series. Let
$$r = \limsup_{n\to\infty} |c_{-n}|^{\frac{1}{n}} \quad \text{and} \quad R = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}.$$

Then the Laurent series converges absolutely in the open annulus $\{z \in \mathbb{C} \mid r < |z-a| < R\}$ and diverges for $|z-a| < r$ and $|z-a| > R$. If the above annulus is nonempty, the function

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

is holomorphic in the annulus.

6. Let f be a holomorphic function in the open annulus $\{z \in \mathbb{C} \mid r < |z-a| < R\}$. Let γ be a positively oriented circle centered at a of radius ρ such that $r < \rho < R$. Put

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, dz$$

for any $n \in \mathbb{Z}$. Show that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

in the annulus (This series is called *the Laurent series* of f).

7. Let Ω be a domain and $a \in \Omega$. Let f be a function holomorphic in $\Omega - \{a\}$. Let $D(a, R)$ be an open disk in Ω. Then f can be represented by its Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

on the punctured disk $D'(a, R)$. Show:

(a) a is a removable singularity if and only if $c_n = 0$ for all $n < 0$;
(b) a is a pole of order m if and only if $c_{-m} \neq 0$ and $c_n = 0$ for $n < -m$;
(c) a is an essential singularity if and only if infinitely many c_n are different from 0 for $n < 0$.

8. Let

$$f(z) = \sin\left(\frac{z}{z+1}\right).$$

(i) Determine all isolated singularities of f and their type;
(ii) find the Laurent expansions of f at these singularities;
(iii) find the residues of f at these singularities.

9. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 2x + 10} \, dx$$

using the residue theorem.

10. Evaluate the integral

$$\int_0^{2\pi} \frac{\cos^2 3\phi}{1 - 2a\,\cos\phi + a^2}\, d\phi$$

where a is a complex number such that $|a| < 1$, using the residue theorem.