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Algebraic $\mathcal{D}$-modules and Representation Theory of Semisimple Lie Groups

Dragan Miličić

Abstract. This expository paper represents an introduction to some aspects of the current research in representation theory of semisimple Lie groups. In particular, we discuss the theory of "localization" of modules over the enveloping algebra of a semisimple Lie algebra due to Alexander Beilinson and Joseph Bernstein [**1**], [**2**], and the work of Henryk Hecht, Wilfried Schmid, Joseph A. Wolf and the author on the localization of Harish-Chandra modules [**7**], [**8**], [**13**], [**17**], [**18**]. These results can be viewed as a vast generalization of the classical theorem of Armand Borel and André Weil on geometric realization of irreducible finite-dimensional representations of compact semisimple Lie groups [**3**].

1. Introduction

Let G_0 be a connected semisimple Lie group with finite center. Fix a maximal compact subgroup K_0 of G_0. Let $\mathfrak{g}$ be the complexified Lie algebra of G_0 and $\mathfrak{k}$ its subalgebra which is the complexified Lie algebra of K_0. Denote by σ the corresponding Cartan involution, i.e., σ is the involution of $\mathfrak{g}$ such that $\mathfrak{k}$ is the set of its fixed points. Let K be the complexification of K_0. The group K has a natural structure of a complex reductive algebraic group.

Let π be an admissible representation of G_0 of finite length. Then, the submodule V of all K_0-finite vectors in this representation is a finitely generated module over the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$, and also a direct sum of finite-dimensional irreducible representations of K_0. The representation of K_0 extends uniquely to a representation of the complexification K of K_0, and it is also a direct sum of finite-dimensional representations.

We say that a representation of a complex algebraic group K in a linear space V is *algebraic* if V is a union of finite-dimensional K-invariant subspaces V_i, $i \in I$, and for each $i \in I$ the action of K on V_i induces a morphism of algebraic groups $K \to \mathrm{GL}(V_i)$.

This leads us to the definition of a *Harish-Chandra module* V:

(i) V is a finitely generated $\mathcal{U}(\mathfrak{g})$-module;
(ii) V is an algebraic representation of K;

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(iii) the actions of $\mathfrak{g}$ and K are compatible, i.e.,
  (1) (a) the action of $\mathfrak{k}$ as the subalgebra of $\mathfrak{g}$ agrees with the differential of the action of K;
  (2) (b) the action map $\mathcal{U}(\mathfrak{g}) \otimes V \longrightarrow V$ is K-equivariant (here K acts on $\mathcal{U}(\mathfrak{g})$ by the adjoint action).

A morphism of Harish-Chandra modules is a linear map which intertwines the $\mathcal{U}(\mathfrak{g})$- and K-actions. Harish-Chandra modules and their morphisms form an abelian category. We denote it by $\mathcal{M}(\mathfrak{g}, K)$.

Let $\mathcal{Z}(\mathfrak{g})$ be the center of the enveloping algebra of $\mathcal{U}(\mathfrak{g})$. If V is an irreducible Harish-Chandra module, the center $\mathcal{Z}(\mathfrak{g})$ acts on V by multiples of the identity operator, i.e., $\mathcal{Z}(\mathfrak{g}) \ni \xi \longrightarrow \chi_V(\xi)\, 1_V$, where $\chi_V : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ is the *infinitesimal character* of V. In general, if a Harish-Chandra module V is annihilated by an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g})$, it is of finite length.

Since the functor attaching to admissible representations of G_0 their Harish-Chandra modules maps irreducibles into irreducibles, the problem of classification of irreducible admissible representations is equivalent to the problem of classification of irreducible Harish-Chandra modules. This problem was solved in the work of R. Langlands [**11**], Harish-Chandra, A.W. Knapp and G. Zuckerman [**10**], and D. Vogan [**19**]. Their proofs were based on a blend of algebraic and analytic techniques and depended heavily on the work of Harish-Chandra.

In this paper we give an exposition of the classification using entirely the methods of algebraic geometry [**8**], [**14**]. In §2, we recall the Borel-Weil theorem. In §3, we introduce the localization functor of Beilinson and Bernstein, and sketch a proof of the equivalence of the category of $\mathcal{U}(\mathfrak{g})$-modules with an infinitesimal character with a category of $\mathcal{D}$-modules on the flag variety of $\mathfrak{g}$. This equivalence induces an equivalence of the category of Harish-Chandra modules with an infinitesimal character with a category of "Harish-Chandra sheaves" on the flag variety. In §4, we recall the basic notions and constructions of the algebraic theory of $\mathcal{D}$-modules. After discussing the structure of K-orbits in the flag variety of $\mathfrak{g}$ in §5, we classify all irreducible Harish-Chandra sheaves in §6. In §7, we describe a necessary and sufficient condition for vanishing of cohomology of irreducible Harish-Chandra sheaves and complete the geometric classification of irreducible Harish-Chandra modules. The final section 8, contains a discussion of the relationship of this classification with the Langlands classification, and a detailed discussion of the case of the group SU(2, 1).

2. The Borel-Weil theorem

First we discuss the case of a connected compact semisimple Lie group. In this situation $G_0 = K_0$, and we denote by G the complexification of G_0. In this case, the irreducible Harish-Chandra modules are just irreducible finite-dimensional representations of G.

For simplicity, we assume that G_0 (and G) is simply connected. Denote by X the flag variety of $\mathfrak{g}$, i.e., the space of all Borel subalgebras of $\mathfrak{g}$. It has a natural structure of a smooth algebraic variety. Since all Borel subalgebras are mutually conjugate, the group G acts transitively on X. For any $x \in X$, the differential of the orbit map $g \longmapsto g \cdot x$ defines a projection of the Lie algebra $\mathfrak{g}$ onto the tangent space $T_x(X)$ of X at x. Therefore, we have a natural vector bundle morphism from the trivial bundle $X \times \mathfrak{g}$ over X into the tangent bundle $T(X)$ of X. If we consider the adjoint action of G on $\mathfrak{g}$, the trivial bundle $X \times \mathfrak{g}$ is G-homogeneous and the morphism $X \times \mathfrak{g} \longrightarrow T(X)$ is G-equivariant. The kernel of this morphism is a G-homogeneous vector bundle $\mathcal{B}$ over X. The fiber of $\mathcal{B}$ over $x \in X$ is the Borel subalgebra $\mathfrak{b}_x$ which corresponds to the point x. Therefore, we can view $\mathcal{B}$ as the "tautological" vector bundle of Borel subalgebras over X. For any $x \in X$, denote by $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ the nilpotent radical of $\mathfrak{b}_x$. Then $\mathcal{N} = \{(x, \xi) \mid \xi \in \mathfrak{n}_x\} \subset \mathcal{B}$ is a G-homogeneous vector subbundle of $\mathcal{B}$. We denote the quotient vector bundle $\mathcal{B}/\mathcal{N}$ by $\mathcal{H}$. If B_x is the stabilizer of x in G, it acts trivially on the fiber $\mathcal{H}_x$ of $\mathcal{H}$ at x. Therefore, $\mathcal{H}$ is a trivial vector bundle on X. Since X is a projective variety, the only global sections of $\mathcal{H}$ are constants. Let $\mathfrak{h}$ be the space of global sections of $\mathcal{H}$. We can view it as an abelian Lie algebra. The Lie algebra $\mathfrak{h}$ is called *the (abstract) Cartan algebra* of $\mathfrak{g}$. Let $\mathfrak{c}$ be any Cartan subalgebra of $\mathfrak{g}$, R the root system of the pair $(\mathfrak{g}, \mathfrak{c})$ in the dual space $\mathfrak{c}^*$ of $\mathfrak{c}$, and R^+ a set of positive roots in R. Then $\mathfrak{c}$ and the root subspaces of $\mathfrak{g}$ corresponding to the roots in R^+ span a Borel subalgebra $\mathfrak{b}_x$ for some point $x \in X$. We have the sequence $\mathfrak{c} \longrightarrow \mathfrak{b}_x \longrightarrow \mathfrak{b}_x/\mathfrak{n}_x = \mathcal{H}_x$ of linear maps, and their composition is an isomorphism. On the other hand, the evaluation map $\mathfrak{h} \longrightarrow \mathcal{H}_x$ is also an isomorphism, and by composing the previous map with the inverse of the evaluation map, we get the canonical isomorphism $\mathfrak{c} \longrightarrow \mathfrak{h}$. Its dual map is an isomorphism $\mathfrak{h}^* \longrightarrow \mathfrak{c}^*$ which we call *a specialization* at x. It identifies an (abstract) root system Σ in $\mathfrak{h}^*$, and a set of positive roots Σ^+, with R and R^+. One can check that Σ and Σ^+ do not depend on the choice of $\mathfrak{c}$ and x. Therefore, we constructed *the (abstract) Cartan triple* $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ of $\mathfrak{g}$. The dual root system in $\mathfrak{h}$ is denoted by $\Sigma\check{}$.

Let $P(\Sigma)$ be the weight lattice in $\mathfrak{h}^*$. Then to each $\lambda \in P(\Sigma)$ we attach a G-homogeneous invertible $\mathcal{O}_X$-module $\mathcal{O}(\lambda)$ on X. We say that a weight λ is *antidominant* if $\alpha\check{}(\lambda) \leq 0$ for any $\alpha \in \Sigma^+$. The following result is the celebrated Borel-Weil theorem. We include a proof inspired by the localization theory.

2.1. Theorem (Borel-Weil). *Let λ be an antidominant weight. Then*
(i) $H^i(X, \mathcal{O}(\lambda))$ *vanish for* $i > 0$.
(ii) $\Gamma(X, \mathcal{O}(\lambda))$ *is the irreducible finite-dimensional G-module with lowest weight* λ.

Proof. Denote by F_λ the irreducible finite-dimensional G-module with lowest weight λ. Let $\mathcal{F}_\lambda$ be the sheaf of local sections of the trivial vector bundle with fibre F_λ over X. Clearly we have

$$H^i(X, \mathcal{F}_\lambda) = H^i(X, \mathcal{O}_X) \otimes_{\mathbb{C}} F_\lambda \text{ for } i \in \mathbb{Z}_+.$$

Since X is a projective variety, the cohomology groups $H^i(X, \mathcal{O}_X)$ are finite dimensional.

Let Ω be the Casimir element in the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. Then for any local section s of $\mathcal{O}(\mu)$, Ωs is proportional to s. In fact, if we denote by $\langle \cdot, \cdot \rangle$ the natural bilinear form on $\mathfrak{h}^*$ induced by the Killing form of $\mathfrak{g}$, by a simple calculation using Harish-Chandra homomorphism we have $\Omega s = \langle \mu, \mu - 2\rho \rangle s$ for any section s of $\mathcal{O}(\mu)$. In particular, Ω annihilates $\mathcal{O}_X$, hence it also annihilates finite-dimensional $\mathfrak{g}$-modules $H^i(X, \mathcal{O}_X)$. Since finite-dimensional $\mathfrak{g}$-modules are semisimple, and Ω acts trivially only on the trivial irreducible $\mathfrak{g}$-module, we conclude that the action of $\mathfrak{g}$ on $H^i(X, \mathcal{O}_X)$ is trivial. Therefore, $\Omega - \langle \lambda, \lambda - 2\rho \rangle$ annihilates $H^i(X, \mathcal{F}_\lambda)$. On the other hand, the Jordan-Hölder filtration of F_λ, considered as a B-module, induces a filtration of $\mathcal{F}_\lambda$ by G-homogeneous locally free $\mathcal{O}_X$-modules such that $\mathrm{F}_p\,\mathcal{F}_\lambda/\,\mathrm{F}_{p-1}\,\mathcal{F}_\lambda$ is a G-homogeneous invertible $\mathcal{O}_X$-module $\mathcal{O}(\nu_p)$ for a weight ν_p of F_λ. This implies that $\prod_0^{\dim F_\lambda}(\Omega-\langle\nu_p,\nu_p-2\rho\rangle)$ annihilates $\mathcal{F}_\lambda$.

Assume that $\langle\nu_p,\nu_p-2\rho\rangle=\langle\lambda,\lambda-2\rho\rangle$ for some weight ν_p. It leads to $\langle\nu_p-\rho,\nu_p-\rho\rangle=\langle\lambda-\rho,\lambda-\rho\rangle$, and, since λ is the lowest weight, we finally see that $\nu_p=\lambda$. Therefore, $\mathcal{F}_\lambda$ splits into the direct sum of the Ω-eigensheaf $\mathcal{O}(\lambda)$ for eigenvalue $\langle\lambda,\lambda-2\rho\rangle$ and its Ω-invariant complement. Since cohomology commutes with direct sums, we conclude that

$$H^i(X,\mathcal{O}(\lambda))=H^i(X,\mathcal{O}_X)\otimes_{\mathbb{C}}F_\lambda$$

for $i\in\mathbb{Z}_+$. Clearly, $\Gamma(X,\mathcal{O}_X)=\mathbb{C}$ and (ii) follows immediately. This implies that invertible $\mathcal{O}_X$-modules $\mathcal{O}(\lambda)$, for regular antidominant λ, are very ample. By a theorem of Serre, (i) follows for geometrically "very positive" λ (i.e., far from the walls in the negative chamber). Hence $H^i(X,\mathcal{O}_X)=0$ for $i>0$, which in turn implies (i) in general. □

3. Beilinson-Bernstein equivalence of categories

Now we want to describe a generalization of the Borel-Weil theorem established by A. Beilinson and J. Bernstein.

First we have to construct a family of sheaves of algebras on the flag variety X. Let $\mathfrak{g}^\circ=\mathcal{O}_X\otimes_{\mathbb{C}}\mathfrak{g}$ be the sheaf of local sections of the trivial bundle $X\times\mathfrak{g}$. Denote by $\mathfrak{b}^\circ$ and $\mathfrak{n}^\circ$ the corresponding subsheaves of local sections of $\mathcal{B}$ and $\mathcal{N}$, respectively. The differential of the action of G on X defines a natural homomorphism τ of the Lie algebra $\mathfrak{g}$ into the Lie algebra of vector fields on X. We define a structure of a sheaf of complex Lie algebras on $\mathfrak{g}^\circ$ by putting

$$[f\otimes\xi,g\otimes\eta]=f\tau(\xi)g\otimes\eta-g\tau(\eta)f\otimes\xi+fg\otimes[\xi,\eta]$$

for $f,g\in\mathcal{O}_X$ and $\xi,\eta\in\mathfrak{g}$. If we extend τ to the natural homomorphism of $\mathfrak{g}^\circ$ into the sheaf of Lie algebras of local vector fields on X, $\ker\tau$ is exactly $\mathfrak{b}^\circ$. In addition, the sheaves $\mathfrak{b}^\circ$ and $\mathfrak{n}^\circ$ are sheaves of ideals in $\mathfrak{g}^\circ$.

Similarly, we define a multiplication in the sheaf $\mathcal{U}^\circ=\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g})$ by

$$(f\otimes\xi)(g\otimes\eta)=f\tau(\xi)g\otimes\eta+fg\otimes\xi\eta$$

where $f,g\in\mathcal{O}_X$ and $\xi\in\mathfrak{g}$, $\eta\in\mathcal{U}(\mathfrak{g})$. In this way $\mathcal{U}^\circ$ becomes a sheaf of complex associative algebras on X. Evidently, $\mathfrak{g}^\circ$ is a subsheaf of $\mathcal{U}^\circ$, and the natural commutator in $\mathcal{U}^\circ$ induces the bracket operation on $\mathfrak{g}^\circ$. It follows that the sheaf of right ideals $\mathfrak{n}^\circ\mathcal{U}^\circ$ generated by $\mathfrak{n}^\circ$ in $\mathcal{U}^\circ$ is a sheaf of two-sided ideals in $\mathcal{U}^\circ$. Therefore, the quotient $\mathcal{D}_\mathfrak{h}=\mathcal{U}^\circ/\mathfrak{n}^\circ\mathcal{U}^\circ$ is a sheaf of complex associative algebras on X.

The natural morphism of $\mathfrak{g}^\circ$ into $\mathcal{D}_\mathfrak{h}$ induces a morphism of the sheaf of Lie subalgebras $\mathfrak{b}^\circ$ into $\mathcal{D}_\mathfrak{h}$ which vanishes on $\mathfrak{n}^\circ$. Hence there is a natural homomorphism ϕ of the enveloping algebra $\mathcal{U}(\mathfrak{h})$ of $\mathfrak{h}$ into the global sections $\Gamma(X,\mathcal{D}_\mathfrak{h})$ of $\mathcal{D}_\mathfrak{h}$. The action of the group G on the structure sheaf $\mathcal{O}_X$ and $\mathcal{U}(\mathfrak{g})$ induces a natural G-action on $\mathcal{U}^\circ$ and $\mathcal{D}_\mathfrak{h}$. On the other hand, triviality of $\mathcal{H}$ and constancy of its global sections imply that the induced G-action on $\mathfrak{h}$ is trivial. It follows that ϕ maps $\mathcal{U}(\mathfrak{h})$ into the G-invariants of $\Gamma(X,\mathcal{D}_\mathfrak{h})$. This implies that the image of ϕ is in the center of $\mathcal{D}_\mathfrak{h}(U)$ for any open set U in X. One can show that ϕ is actually an isomorphism of $\mathcal{U}(\mathfrak{h})$ onto the subalgebra of all G-invariants in $\Gamma(X,\mathcal{D}_\mathfrak{h})$. In addition, the natural homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\Gamma(X,\mathcal{D}_\mathfrak{h})$ induces a homomorphism of the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ into $\Gamma(X,\mathcal{D}_\mathfrak{h})$. Its image is also contained in the subalgebra of G-invariants of $\Gamma(X,\mathcal{D}_\mathfrak{h})$. Hence, it is in $\phi(\mathcal{U}(\mathfrak{h}))$. Finally, we have the canonical Harish-Chandra homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$, defined in the following way. First, for any $x \in X$, the center $\mathcal{Z}(\mathfrak{g})$ is contained in the sum of the subalgebra $\mathcal{U}(\mathfrak{b}_x)$ and the right ideal $\mathfrak{n}_x\mathcal{U}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$. Therefore, we have the natural projection of $\mathcal{Z}(\mathfrak{g})$ into

$$\mathcal{U}(\mathfrak{b}_x)/(\mathfrak{n}_x\mathcal{U}(\mathfrak{g})\cap\mathcal{U}(\mathfrak{b}_x)) = \mathcal{U}(\mathfrak{b}_x)/\mathfrak{n}_x\mathcal{U}(\mathfrak{b}_x) = \mathcal{U}(\mathfrak{b}_x/\mathfrak{n}_x).$$

Its composition with the natural isomorphism of $\mathcal{U}(\mathfrak{b}_x/\mathfrak{n}_x)$ with $\mathcal{U}(\mathfrak{h})$ is independent of x and, by definition, equal to γ. The diagram

$$\begin{tikzcd} \mathcal{Z}(\mathfrak{g}) \arrow[r, "\gamma"] \arrow[d, equal] & \mathcal{U}(\mathfrak{h}) \arrow[d, "\phi"'] \\ \mathcal{Z}(\mathfrak{g}) \arrow[r] & \Gamma(X,\mathcal{D}_\mathfrak{h}) \end{tikzcd}$$

of natural algebra homomorphisms is commutative. We can form $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{Z}(\mathfrak{g})}\mathcal{U}(\mathfrak{h})$, which has a natural structure of an associative algebra. There exists a natural algebra homomorphism

$$\Psi : \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{Z}(\mathfrak{g})}\mathcal{U}(\mathfrak{h}) \to \Gamma(X,\mathcal{D}_\mathfrak{h})$$

given by the tensor product of the natural homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\Gamma(X,\mathcal{D}_\mathfrak{h})$ and ϕ. The next result describes the cohomology of the sheaf of algebras $\mathcal{D}_\mathfrak{h}$. Its proof is an unpublished argument due to Joseph Taylor and the author.

3.1. Lemma.

(i) *The morphism*

$$\Psi : \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{Z}(\mathfrak{g})}\mathcal{U}(\mathfrak{h}) \to \Gamma(X,\mathcal{D}_\mathfrak{h})$$

is an isomorphism of algebras.

(ii) $H^i(X,\mathcal{D}_\mathfrak{h}) = 0$ *for* $i > 0$.

Sketch of the proof. First we construct a left resolution

$$\ldots \to \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \textstyle\bigwedge^p \mathfrak{n}^\circ \to \ldots \to \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \mathfrak{n}^\circ \to \mathcal{U}^\circ \to \mathcal{D}_\mathfrak{h} \to 0$$

of $\mathcal{D}_\mathfrak{h}$ (here $\bigwedge^p \mathfrak{n}^\circ$ is the p^{th} exterior power of $\mathfrak{n}^\circ$). The cohomology of each component in this complex is given by

$$H^q(X,\mathcal{U}^\circ \otimes_{\mathcal{O}_X} \textstyle\bigwedge^p \mathfrak{n}^\circ) = H^q(X,\mathcal{U}(\mathfrak{g})\otimes_\mathbb{C} \bigwedge^p \mathfrak{n}^\circ) = \mathcal{U}(\mathfrak{g})\otimes_\mathbb{C} H^q(X,\bigwedge^p \mathfrak{n}^\circ).$$

Let $\ell : W \to \mathbb{Z}_+$ be the length function on the Weyl group W of Σ with respect to the set of reflections corresponding to simple roots Π in Σ^+. Let $W(p) = \{w \in W \mid \ell(w) = p\}$ and $n(p) = \operatorname{Card} W(p)$. By a lemma of Bott [**5**] (which follows easily from the Borel-Weil-Bott theorem),

$$H^q(X,\textstyle\bigwedge^p \mathfrak{n}^\circ) = 0 \text{ if } p \neq q;$$

and $H^p(X,\bigwedge^p \mathfrak{n}^\circ)$ is a linear space of dimension $n(p)$ with trivial action of G. Now, a standard spectral sequence argument implies that (ii) holds, and that $\Gamma(X,\mathcal{D}_\mathfrak{h})$ has a finite filtration such that the corresponding graded algebra is isomorphic to a direct sum of Card W copies of $\mathcal{U}(\mathfrak{g})$. Taking the G-invariants of this spectral sequence we see that the induced finite filtration of $\Gamma(X, \mathcal{D}_{\mathfrak{h}})^G = \mathcal{U}(\mathfrak{h})$ is such that the corresponding graded algebra is isomorphic to a direct sum of Card W copies of $\mathcal{Z}(\mathfrak{g})$. This implies (i). □

Denote by ρ the half-sum of all positive roots in Σ. The enveloping algebra $\mathcal{U}(\mathfrak{h})$ of $\mathfrak{h}$ is naturally isomorphic to the algebra of polynomials on $\mathfrak{h}^*$, and therefore any $\lambda \in \mathfrak{h}^*$ determines a homomorphism of $\mathcal{U}(\mathfrak{h})$ into $\mathbb{C}$. Let I_λ be the kernel of the homomorphism $\varphi_\lambda : \mathcal{U}(\mathfrak{h}) \to \mathbb{C}$ determined by $\lambda + \rho$. Then $\gamma^{-1}(I_\lambda)$ is a maximal ideal in $\mathcal{Z}(\mathfrak{g})$, and, by a result of Harish-Chandra, for $\lambda, \mu \in \mathfrak{h}^*$,

$$\gamma^{-1}(I_\lambda) = \gamma^{-1}(I_\mu) \text{ if and only if } w\lambda = \mu$$

for some w in the Weyl group W of Σ. For any $\lambda \in \mathfrak{h}^*$, the sheaf $I_\lambda \mathcal{D}_{\mathfrak{h}}$ is a sheaf of two-sided ideals in $\mathcal{D}_{\mathfrak{h}}$; therefore $\mathcal{D}_\lambda = \mathcal{D}_{\mathfrak{h}} / I_\lambda \mathcal{D}_{\mathfrak{h}}$ is a sheaf of complex associative algebras on X. In the case when $\lambda = -\rho$, we have $I_{-\rho} = \mathfrak{h}\,\mathcal{U}(\mathfrak{h})$, hence $\mathcal{D}_{-\rho} = \mathcal{U}^\circ / \mathfrak{b}^\circ \mathcal{U}^\circ$, i.e., it is the sheaf of local differential operators on X. If $\lambda \in P(\Sigma)$, $\mathcal{D}_\lambda$ is the sheaf of differential operators on the invertible $\mathcal{O}_X$-module $\mathcal{O}(\lambda + \rho)$.

Let Y be a smooth complex algebraic variety. Denote by $\mathcal{O}_Y$ its structure sheaf. Let $\mathcal{D}_Y$ be the sheaf of local differential operators on Y. Denote by i_Y the natural homomorphism of the sheaf of rings $\mathcal{O}_Y$ into $\mathcal{D}_Y$. We can consider the category of pairs $(\mathcal{A}, i_{\mathcal{A}})$ where $\mathcal{A}$ is a sheaf of rings on Y and $i_{\mathcal{A}} : \mathcal{O}_Y \to \mathcal{A}$ a homomorphism of sheaves of rings. The morphisms are homomorphisms $\alpha : \mathcal{A} \to \mathcal{B}$ of sheaves of algebras such that $\alpha \circ i_{\mathcal{A}} = i_{\mathcal{B}}$. A pair $(\mathcal{D}, i)$ is called a *twisted sheaf of differential operators* if Y has a cover by open sets U such that $(\mathcal{D}|U, i|U)$ is isomorphic to $(\mathcal{D}_U, i_U)$.

In general, the sheaves of algebras $\mathcal{D}_\lambda$, $\lambda \in \mathfrak{h}^*$, are twisted sheaves of differential operators on X.

Let θ be a Weyl group orbit in $\mathfrak{h}^*$ and $\lambda \in \theta$. Denote by $J_\theta = \gamma^{-1}(I_\lambda)$ the maximal ideal in $\mathcal{Z}(\mathfrak{g})$ determined by θ. We denote by χ_λ the homomorphism of $\mathcal{Z}(\mathfrak{g})$ into $\mathbb{C}$ with $\ker \chi_\lambda = J_\theta$ (as we remarked before, χ_λ depends only on the Weyl group orbit θ of λ). The elements of J_θ map into the zero section of $\mathcal{D}_\lambda$. Therefore, we have a canonical morphism of $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g}) / J_\theta \mathcal{U}(\mathfrak{g})$ into $\Gamma(X, \mathcal{D}_\lambda)$.

3.2. Theorem.

(i) *The morphism*

$$\mathcal{U}_\theta \to \Gamma(X, \mathcal{D}_\lambda)$$

is an isomorphism of algebras.

(ii) $H^i(X, \mathcal{D}_\lambda) = 0$ *for* $i > 0$.

Proof. Let $\mathbb{C}_{\lambda+\rho}$ be a one-dimensional $\mathfrak{h}$-module defined by $\lambda + \rho$. Let

$$\cdots \to F^{-p} \to \cdots \to F^{-1} \to F^0 \to \mathbb{C}_{\lambda+\rho} \to 0$$

be a left free $\mathcal{U}(\mathfrak{h})$-module resolution of $\mathbb{C}_{\lambda+\rho}$. By tensoring with $\mathcal{D}_{\mathfrak{h}}$ over $\mathcal{U}(\mathfrak{h})$ we get

$$\cdots \to \mathcal{D}_{\mathfrak{h}} \otimes_{\mathcal{U}(\mathfrak{h})} F^{-p} \to \cdots \to \mathcal{D}_{\mathfrak{h}} \otimes_{\mathcal{U}(\mathfrak{h})} F^0 \to \mathcal{D}_{\mathfrak{h}} \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}_{\lambda+\rho} \to 0.$$

Since $\mathcal{D}_{\mathfrak{h}}$ is locally $\mathcal{U}(\mathfrak{h})$-free, this is an exact sequence. Therefore, by 1.(ii), it is a left resolution of $\mathcal{D}_{\mathfrak{h}} \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}_{\lambda+\rho} = \mathcal{D}_\lambda$ by $\Gamma(X, -)$-acyclic sheaves. This implies first that all higher cohomologies of $\mathcal{D}_\lambda$ vanish. Also, it gives, using 1.(i), the exact sequence

$$\cdots \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} F^{-p} \longrightarrow \cdots \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} F^0 \longrightarrow \Gamma(X, \mathcal{D}_\lambda) \longrightarrow 0,$$

which yields $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathbb{C}_{\lambda+\rho} = \Gamma(X, \mathcal{D}_\lambda)$. □

Therefore, the twisted sheaves of differential operators $\mathcal{D}_\lambda$ on X can be viewed as "sheafified" versions of the quotients $\mathcal{U}_\theta$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. This allows us to "localize" the modules over $\mathcal{U}_\theta$.

First, denote by $\mathcal{M}(\mathcal{U}_\theta)$ the category of $\mathcal{U}_\theta$-modules. Also, let $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ be the category of quasi-coherent $\mathcal{D}_\lambda$-modules on X. If $\mathcal{V}$ is a quasi-coherent $\mathcal{D}_\lambda$-module, its global sections (and higher cohomology groups) are modules over $\Gamma(X, \mathcal{D}_\lambda) = \mathcal{U}_\theta$. Therefore, we can consider the functors:

$$H^p(X, -) : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$$

for $p \in \mathbb{Z}_+$.

The next two results can be viewed as a vast generalization of the Borel-Weil theorem. In idea, their proof is very similar to our proof of the Borel-Weil theorem. It is also based on the theorems of Serre on cohomology of invertible $\mathcal{O}$-modules on projective varieties, and a splitting argument for the action of $\mathcal{Z}(\mathfrak{g})$ [**1**].

The first result corresponds to 2.1.(i). We say that $\lambda \in \mathfrak{h}^*$ is *antidominant* if $\alpha\check{}(\lambda)$ is not a positive integer for any $\alpha \in \Sigma^+$. This generalizes the notion of antidominance for weights in $P(\Sigma)$ introduced in §2.

3.3. Vanishing Theorem. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Let $\mathcal{V}$ be a quasi-coherent $\mathcal{D}_\lambda$-module on the flag variety X. Then the cohomology groups $H^i(X, \mathcal{V})$ vanish for $i > 0$.*

In particular, the functor

$$\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$$

is exact. The second result corresponds to 2.1.(ii).

3.4. Nonvanishing Theorem. *Let $\lambda \in \mathfrak{h}^*$ be regular and antidominant and $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ such that $\Gamma(X, \mathcal{V}) = 0$. Then $\mathcal{V} = 0$.*

This has the following consequence:

3.5. Corollary. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is generated by its global sections.*

Proof. Denote by $\mathcal{W}$ the $\mathcal{D}_\lambda$-submodule of $\mathcal{V}$ generated by all global sections. Then, we have an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{W}) \longrightarrow \Gamma(X, \mathcal{V}) \longrightarrow \Gamma(X, \mathcal{V}/\mathcal{W}) \longrightarrow 0,$$

of $\mathcal{U}_\theta$-modules, and therefore $\Gamma(X, \mathcal{V}/\mathcal{W}) = 0$. Hence, $\mathcal{V}/\mathcal{W} = 0$, and $\mathcal{V}$ is generated by its global sections. □

Let $\lambda \in \mathfrak{h}^*$ and let θ be the corresponding Weyl group orbit. Then we can define a right exact covariant functor Δ_λ from $\mathcal{M}(\mathcal{U}_\theta)$ into $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ by

$$\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$$

for any $V \in \mathcal{M}(\mathcal{U}_\theta)$. It is called the *localization functor*. Since

$$\Gamma(X, \mathcal{W}) = \operatorname{Hom}_{\mathcal{D}_\lambda}(\mathcal{D}_\lambda, \mathcal{W})$$

for any $\mathcal{W} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$, it follows that Δ_λ is a left adjoint functor to the functor of global sections Γ, i.e.,

$$\operatorname{Hom}_{\mathcal{D}_\lambda}(\Delta_\lambda(V), \mathcal{W}) = \operatorname{Hom}_{\mathcal{U}_\theta}(V, \Gamma(X, \mathcal{W})),$$

for any $V \in \mathcal{M}(\mathcal{U}_\theta)$ and $\mathcal{W} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$. In particular, there exists a functorial morphism φ from the identity functor into $\Gamma \circ \Delta_\lambda$. For any $V \in \mathcal{M}(\mathcal{U}_\theta)$, it is given by the natural morphism $\varphi_V : V \longrightarrow \Gamma(X, \Delta_\lambda(V))$.

3.6. Lemma. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then the natural map φ_V of V into $\Gamma(X, \Delta_\lambda(V))$ is an isomorphism of $\mathfrak{g}$-modules.*

Proof. If $V = \mathcal{U}_\theta$ this follows from 2. Also, by 3, we know that Γ is exact in this situation. This implies that $\Gamma \circ \Delta_\lambda$ is a right exact functor. Let

$$(\mathcal{U}_\theta)^{(J)} \longrightarrow (\mathcal{U}_\theta)^{(I)} \longrightarrow V \longrightarrow 0$$

be an exact sequence of $\mathfrak{g}$-modules. Then we have the commutative diagram

$$\begin{array}{ccccccc}
(\mathcal{U}_\theta)^{(J)} & \longrightarrow & (\mathcal{U}_\theta)^{(I)} & \longrightarrow & V & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Gamma(X, \Delta_\lambda(\mathcal{U}_\theta))^{(J)} & \longrightarrow & \Gamma(X, \Delta_\lambda(\mathcal{U}_\theta))^{(I)} & \longrightarrow & \Gamma(X, \Delta_\lambda(V)) & \longrightarrow & 0
\end{array}$$

with exact rows, and the first two vertical arrows are isomorphisms. This implies that the third one is also an isomorphism. □

On the other hand, the adjointness gives also a functorial morphism ψ from $\Delta_\lambda \circ \Gamma$ into the identity functor. For any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$, it is given by the natural morphism $\psi_\mathcal{V}$ of $\Delta_\lambda(\Gamma(X, \mathcal{V})) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} \Gamma(X, \mathcal{V})$ into $\mathcal{V}$. Assume that λ is also regular. Then, by 5, $\psi_\mathcal{V}$ is an epimorphism. Let $\mathcal{K}$ be the kernel of $\psi_\mathcal{V}$. Then we have the exact sequence of quasi-coherent $\mathcal{D}_\lambda$-modules

$$0 \rightarrow \mathcal{K} \rightarrow \Delta_\lambda(\Gamma(X, \mathcal{V})) \rightarrow \mathcal{V} \rightarrow 0$$

and by applying Γ and using 3. we get the exact sequence

$$0 \rightarrow \Gamma(X, \mathcal{K}) \rightarrow \Gamma(X, \Delta_\lambda(\Gamma(X, \mathcal{V}))) \rightarrow \Gamma(X, \mathcal{V}) \rightarrow 0.$$

By 6. we see that $\Gamma(X, \mathcal{K}) = 0$. By 4, $\mathcal{K} = 0$ and $\psi_\mathcal{V}$ is an isomorphism. This implies the following result, which is known as the *Beilinson-Bernstein equivalence of categories*.

3.7. Theorem (Beilinson-Bernstein). *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then the functor Δ_λ from $\mathcal{M}(\mathcal{U}_\theta)$ into $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is an equivalence of categories. Its inverse is Γ.*

3.8. Remark. In general, if we assume only that λ is antidominant, we denote by $\mathcal{QM}_{qc}(\mathcal{D}_\lambda)$ the quotient category of $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ with respect to the subcategory of all quasi-coherent $\mathcal{D}_\lambda$-modules with no global sections. Clearly, Γ induces an exact functor from $\mathcal{QM}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}(\mathcal{U}_\theta)$ which we denote also by Γ. Then we have an equivalence of categories

$$\mathcal{QM}_{qc}(\mathcal{D}_\lambda) \xrightarrow{\Gamma} \mathcal{M}(\mathcal{U}_\theta).$$

The equivalence of categories allows one to transfer problems about $\mathcal{U}_\theta$-modules into problems about $\mathcal{D}_\lambda$-modules. The latter problems can be attacked by "local" methods. To make this approach useful we need to introduce a "sheafified" version of Harish-Chandra modules.[1] A *Harish-Chandra sheaf* is

(i) a coherent $\mathcal{D}_\lambda$-module $\mathcal{V}$
(ii) with an algebraic action of K;
(iii) the actions of $\mathcal{D}_\lambda$ and K on $\mathcal{V}$ are compatible, i.e.,
    (1) (a) the action of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g} \subset \mathcal{U}_\theta = \Gamma(X, \mathcal{D}_\lambda)$ agrees with the differential of the action of K;
    (2) (b) the action $\mathcal{D}_\lambda \otimes_{\mathcal{O}_X} \mathcal{V} \to \mathcal{V}$ is K-equivariant.

Morphisms of Harish-Chandra sheaves are K-equivariant $\mathcal{D}_\lambda$-module morphisms. Harish-Chandra sheaves form an abelian category denoted by $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$. Because of completely formal reasons, the equivalence of categories has the following consequence, which is a K-equivariant version of 7.

3.9. Theorem. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then the functor Δ_λ from $\mathcal{M}(\mathcal{U}_\theta, K)$ into $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ is an equivalence of categories. Its inverse is Γ.*

Therefore, by 9. and its analogue in the singular case, the classification of all irreducible Harish-Chandra modules is equivalent to the following two problems:

(a) the classification of all irreducible Harish-Chandra sheaves;
(b) determination of all irreducible Harish-Chandra sheaves $\mathcal{V}$ with $\Gamma(X, \mathcal{V}) \neq 0$ for antidominant $\lambda \in \mathfrak{h}^*$.

In next sections we shall explain how to solve these two problems.

3.10. Remark. Although the setting of 7. is adequate for the formulation of our results, the proofs require a more general setup. The difference between 7. and the general setup is analogous to the difference between the Borel-Weil theorem and its generalization, the Borel-Weil-Bott theorem. To explain this we have to use the language of derived categories.

Let $D^b(\mathcal{D}_\lambda)$ be the bounded derived category of the category of quasi-coherent $\mathcal{D}_\lambda$-modules. Let $D^b(\mathcal{U}_\theta)$ be the bounded derived category of the category of $\mathcal{U}_\theta$-modules. Then, we have the following result:

3.11. Theorem. *For a regular λ, the derived functors $R\Gamma : D^b(\mathcal{D}_\lambda) \to D^b(\mathcal{U}_\theta)$ and $L\Delta_\lambda : D^b(\mathcal{U}_\theta) \to D^b(\mathcal{D}_\lambda)$ are mutually inverse equivalences of categories.*

[1]This requires some technical machinery beyond the scope of this paper, so we shall be rather vague in this definition.

4. Algebraic $\mathcal{D}$-modules

In this section we review some basic notions and results from the algebraic theory of $\mathcal{D}$-modules. They will allow us to study the structure of Harish-Chandra sheaves. Interested readers can find details in [**4**].

Let X be a smooth algebraic variety and $\mathcal{D}$ a twisted sheaf of differential operators on X. Then the opposite sheaf of rings $\mathcal{D}^{\text{opp}}$ is again a twisted sheaf of differential operators on X. We can therefore view left $\mathcal{D}$-modules as right $\mathcal{D}^{\text{opp}}$-modules and vice versa. Formally, the category $\mathcal{M}^L_{qc}(\mathcal{D})$ of quasi-coherent left $\mathcal{D}$-modules on X is isomorphic to the category $\mathcal{M}^R_{qc}(\mathcal{D}^{\text{opp}})$ of quasi-coherent right $\mathcal{D}^{\text{opp}}$-modules on X. Hence one can freely use right and left modules depending on the particular situation.

For a category $\mathcal{M}_{qc}(\mathcal{D})$ of $\mathcal{D}$-modules we denote by $\mathcal{M}_{coh}(\mathcal{D})$ the corresponding subcategory of coherent $\mathcal{D}$-modules.

The sheaf of algebras $\mathcal{D}$ has a natural filtration $(\mathcal{D}_p; p \in \mathbb{Z})$ by the degree. If we take a sufficiently small open set U in X such that $\mathcal{D}|_U \cong \mathcal{D}_U$, this filtration agrees with the standard degree filtration on $\mathcal{D}_U$. If we denote by π the canonical projection of the cotangent bundle $T^*(X)$ onto X, we have $\operatorname{Gr}\mathcal{D} = \pi_*(\mathcal{O}_{T^*(X)})$.

For any coherent $\mathcal{D}$-module $\mathcal{V}$ we can construct a *good filtration* $\mathrm{F}\,\mathcal{V}$ of $\mathcal{V}$ as a $\mathcal{D}$-module:

(a) The filtration $\mathrm{F}\,\mathcal{V}$ is increasing, exhaustive and $\mathrm{F}_p\,\mathcal{V} = 0$ for "very negative" $p \in \mathbb{Z}$;
(b) $\mathrm{F}_p\,\mathcal{V}$ are coherent $\mathcal{O}_X$-modules;
(c) $\mathcal{D}_p\,\mathrm{F}_q\,\mathcal{V} = \mathrm{F}_{p+q}\,\mathcal{V}$ for large $q \in \mathbb{Z}$ and all $p \in \mathbb{Z}_+$.

The annihilator of $\operatorname{Gr}\mathcal{V}$ is a sheaf of ideals in $\pi_*(\mathcal{O}_{T^*(X)})$. Therefore, we can attach to it its zero set in $T^*(X)$. This variety is called the *characteristic variety* $\operatorname{Char}(\mathcal{V})$ of $\mathcal{V}$. One can show that it is independent of the choice of the good filtration of $\mathcal{V}$.

A subvariety Z of $T^*(X)$ is called *conical* if $(x, \omega) \in Z$, with $x \in X$ and $\omega \in T^*_x(X)$, implies $(x, \lambda\omega) \in Z$ for all $\lambda \in \mathbb{C}$.

4.1. Lemma. *Let $\mathcal{V}$ be a coherent $\mathcal{D}$-module on X. Then*
(i) *The characteristic variety* $\operatorname{Char}(\mathcal{V})$ *is conical.*
(ii) $\pi(\operatorname{Char}(\mathcal{V})) = \operatorname{supp}(\mathcal{V})$.

The characteristic variety of a coherent $\mathcal{D}$-module cannot be "too small". More precisely, we have the following result.

4.2. Theorem. *Let $\mathcal{V}$ be a nonzero coherent $\mathcal{D}$-module on X. Then*

$$\dim \operatorname{Char}(\mathcal{V}) \geq \dim X.$$

If $\dim \operatorname{Char}(\mathcal{V}) = \dim X$ or $\mathcal{V} = 0$, we say that $\mathcal{V}$ is a *holonomic* $\mathcal{D}$-module. Holonomic modules form an abelian subcategory of $\mathcal{M}_{coh}(\mathcal{D})$. Any holonomic $\mathcal{D}$-module is of finite length.

Modules in $\mathcal{M}_{coh}(\mathcal{D})$ which are coherent as $\mathcal{O}_X$-modules are called *connections*. Connections are locally free as $\mathcal{O}_X$-modules. Therefore, the support of a connection τ is a union of connected components of X. If $\operatorname{supp}(\tau) = X$, its characteristic variety is the zero section of $T^*(X)$; in particular τ is holonomic. On the other hand, a coherent $\mathcal{D}$-module with characteristic variety equal to the zero section of $T^*(X)$ is a connection supported on X.

Assume that $\mathcal{V}$ is a holonomic module with support equal to X. Since the characteristic variety of a holonomic module $\mathcal{V}$ is conical, and has the same dimension as X, there exists an open and dense subset U in X such that the characteristic variety of $\mathcal{V}|_U$ is the zero section of $T^*(U)$. Therefore, $\mathcal{V}|_U$ is a connection.

Now we define several functors between various categories of $\mathcal{D}$-modules.

Let $\mathcal{V}$ be a quasi-coherent $\mathcal{O}_X$-module. An endomorphism D of the sheaf of linear spaces $\mathcal{V}$ is called a *differential endomorphism* of $\mathcal{V}$ of degree $\leq n$, $n \in \mathbb{Z}_+$, if we have

$$[\dots[[D, f_0], f_1], \dots, f_n] = 0$$

for any $(n+1)$-tuple $(f_0, f_1, \dots, f_n)$ of regular functions on any open set U in X.

First, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ has a natural structure of a right $\mathcal{D}$-module by right multiplication in the second factor. Let $\mathcal{D}^{\mathcal{L}}$ be the sheaf of differential endomorphisms of the $\mathcal{O}_X$-module $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ (for the $\mathcal{O}_X$-module structure given by the left multiplication) which commute with the right $\mathcal{D}$-module structure. Then $\mathcal{D}^{\mathcal{L}}$ is a twisted sheaf of differential operators on X. We can define the *twist* functor from $\mathcal{M}^L_{qc}(\mathcal{D})$ into $\mathcal{M}^L_{qc}(\mathcal{D}^{\mathcal{L}})$ by

$$\mathcal{V} \longmapsto (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{V}$$

for $\mathcal{V}$ in $\mathcal{M}^L_{qc}(\mathcal{D})$. As an $\mathcal{O}_X$-module,

$$(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{V} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{V}.$$

The operation of twist is visibly an equivalence of categories. It preserves coherence of $\mathcal{D}$-modules and their characteristic varieties. Therefore, the twist preserves holonomicity.

Let $f : Y \to X$ be a morphism of smooth algebraic varieties. Put

$$\mathcal{D}_{Y \to X} = f^*(\mathcal{D}) = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}.$$

Then $\mathcal{D}_{Y \to X}$ is a right $f^{-1}\mathcal{D}$-module for the right multiplication in the second factor. Denote by $\mathcal{D}^f$ the sheaf of differential endomorphisms of the $\mathcal{O}_Y$-module $\mathcal{D}_{Y \to X}$ which are also $f^{-1}\mathcal{D}$-module endomorphisms. Then $\mathcal{D}^f$ is a twisted sheaf of differential operators on Y.

Let $\mathcal{V}$ be in $\mathcal{M}^L_{qc}(\mathcal{D})$. Put

$$f^+(\mathcal{V}) = \mathcal{D}_{Y \to X} \otimes_{f^{-1}\mathcal{D}} f^{-1}\mathcal{V}.$$

Then $f^+(\mathcal{V})$ is the *inverse image* of $\mathcal{V}$ (in the category of $\mathcal{D}$-modules), and f^+ is a right exact covariant functor from $\mathcal{M}^L_{qc}(\mathcal{D})$ into $\mathcal{M}^L_{qc}(\mathcal{D}^f)$. Considered as an $\mathcal{O}_Y$-module,

$$f^+(\mathcal{V}) = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{V} = f^*(\mathcal{V}),$$

where $f^*(\mathcal{V})$ is the inverse image in the category of $\mathcal{O}$-modules. The left derived functors $L^p f^+ : \mathcal{M}^L_{qc}(\mathcal{D}) \to M^L_{qc}(\mathcal{D}^f)$ of f^+ have analogous properties. One can show that derived inverse images preserve holonomicity.

Let Y be a smooth subvariety of X and $\mathcal{D}$ a twisted sheaf of differential operators on X. Then $\mathcal{D}^i$ is a twisted sheaf of differential operators on Y and $L^p i^+ : \mathcal{M}^L_{qc}(\mathcal{D}) \to \mathcal{M}^L_{qc}(\mathcal{D}^i)$ vanish for $p < -\operatorname{codim} Y$. Therefore, $i^! = L^{-\operatorname{codim} Y} i^+$ is a left exact functor.

To define the direct image functors for $\mathcal{D}$-modules one has to use derived categories. In addition, it is simpler to define them for right $\mathcal{D}$-modules. Let $D^b(\mathcal{M}^R_{qc}(\mathcal{D}^f))$ be the bounded derived category of quasi-coherent right $\mathcal{D}^f$-modules. Then we define

$$Rf_+(\mathcal{V}^\cdot) = Rf_*(\mathcal{V}^\cdot \overset{L}{\otimes}_{\mathcal{D}^f} \mathcal{D}_{Y \to X})$$

for any complex $\mathcal{V}^\cdot$ in $D^b(\mathcal{M}_{qc}^R(\mathcal{D}^f))$ (here we denote by Rf_* and $\overset{L}{\otimes}$ the derived functors of direct image f_* and tensor product). Let $\mathcal{V}^\cdot$ be the complex in $D^b(\mathcal{M}_{qc}^R(\mathcal{D}^f))$ which is zero in all degrees except 0, where it is equal to a quasi-coherent right $\mathcal{D}^f$-module $\mathcal{V}$. Then we put

$$R^p f_+(\mathcal{V}) = H^p(Rf_+(\mathcal{V}^\cdot)) \text{ for } p \in \mathbb{Z},$$

i.e., we get a family $R^p f_+$, $p \in \mathbb{Z}$, of functors from $\mathcal{M}_{qc}^R(\mathcal{D}^f)$ into $\mathcal{M}_{qc}^R(\mathcal{D})$. We call $R^p f_+$ the p^{th} *direct image* functor. Direct image functors also preserve holonomicity.

If $i : Y \longrightarrow X$ is an immersion, $\mathcal{D}_{Y\longrightarrow X}$ is a locally free $\mathcal{D}^i$-module. This implies that

$$R^p i_+(\mathcal{V}) = R^p i_*(\mathcal{V} \otimes_{\mathcal{D}^i} \mathcal{D}_{Y\longrightarrow X})$$

for $\mathcal{V}$ in $\mathcal{M}_{qc}^R(\mathcal{D}^i)$. Therefore, $i_+ = R^0 i_+$ is left exact and $R^p i_+$ are its right derived functors. In addition, if Y is a closed in X, the functor $i_+ : \mathcal{M}_{qc}^R(\mathcal{D}^i) \longrightarrow \mathcal{M}_{qc}^R(\mathcal{D})$ is exact.

Let $i : Y \longrightarrow X$ be a closed immersion. The support of $i_+(\mathcal{V})$ is equal to the support of $\mathcal{V}$ considered as a subset of $Y \subset X$.

4.3. Theorem (Kashiwara's equivalence of categories). *Let $i : Y \longrightarrow X$ be a closed immersion. Then the direct image functor i_+ is an equivalence of $\mathcal{M}_{qc}^R(\mathcal{D}^i)$ with the full subcategory of $\mathcal{M}_{qc}^R(\mathcal{D})$ consisting of modules with support in Y.*

This equivalence preserves coherence and holonomicity.

The inverse functor is given by $i^!$ (up to a twist caused by our use of right $\mathcal{D}$-modules in the discussion of i_+).

5. K-orbits in the flag variety

In this section we study K-orbits in the flag variety X in more detail. As before, let σ be the Cartan involution of $\mathfrak{g}$ such that $\mathfrak{k}$ is its fixed point set.

We first establish that the number of K-orbits in X is finite.

5.1. Proposition. *The group K acts on X with finitely many orbits.*

To prove this result we can assume that $G = \mathrm{Int}(\mathfrak{g})$. Also, by abuse of notation, denote by σ the involution of G with differential equal to the Cartan involution σ. The key step in the proof is the following lemma. First, define an action of G on $X \times X$ by

$$g(x, y) = (gx, \sigma(g)y)$$

for any $g \in G$, $x, y \in X$.

5.2. Lemma. *The group G acts on $X \times X$ with finitely many orbits.*

Proof. We fix a point $v \in X$. Let B_v be the Borel subgroup of G corresponding to v, and put $B = \sigma(B_v)$. Every G-orbit in $X \times X$ intersects $X \times \{v\}$. Let $u \in X$. Then the intersection of the G-orbit Q through (u, v) with $X \times \{v\}$ is equal to $Bu \times \{v\}$. By the Bruhat decomposition, this implies the finiteness of the number of G-orbits in $X \times X$. □

Now we show that 1. is a consequence of 2. Let Δ be the diagonal in $X \times X$. By 2, the orbit stratification of $X \times X$ induces a stratification of Δ by finitely many irreducible subvarieties which are the irreducible components of the intersections of the G-orbits with Δ. These strata are K-invariant, and therefore unions of K-orbits. Let V be one of these subvarieties, $(x,x) \in V$ and Q the K-orbit of (x,x). If we let $\mathfrak{b}_x$ denote the Borel subalgebra of $\mathfrak{g}$ corresponding to x, the tangent space $T_x(X)$ of X at x can be identified with $\mathfrak{g}/\mathfrak{b}_x$. Let p_x be the projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{b}_x$. The tangent space $T_{(x,x)}(X \times X)$ to $X \times X$ at (x,x) can be identified with $\mathfrak{g}/\mathfrak{b}_x \times \mathfrak{g}/\mathfrak{b}_x$. If the orbit map $f : G \longrightarrow X \times X$ is defined by $f(g) = g(x,x)$, its differential at the identity in G is the linear map $\xi \longrightarrow (p_x(\xi), p_x(\sigma(\xi)))$ of $\mathfrak{g}$ into $\mathfrak{g}/\mathfrak{b}_x \times \mathfrak{g}/\mathfrak{b}_x$. Then the tangent space to V at (x,x) is contained in the intersection of the image of this differential with the diagonal in the tangent space $T_{(x,x)}(X \times X)$, i.e.

$$\begin{aligned} T_{(x,x)}(V) &\subset \{(p_x(\xi), p_x(\xi)) \mid \xi \in \mathfrak{g} \text{ such that } p_x(\xi) = p_x(\sigma(\xi))\} \\ &= \{(p_x(\xi), p_x(\xi)) \mid \xi \in \mathfrak{k}\} = T_{(x,x)}(Q). \end{aligned}$$

Consequently the tangent space to V at (x,x) agrees with the tangent space to Q, and Q is open in V. By the irreducibility of V, this implies that V is a K-orbit, and therefore our stratification of the diagonal Δ is the stratification induced via the diagonal map by the K-orbit stratification of X. Hence, 1. follows.

5.3. Lemma. *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and N the connected subgroup of G determined by $\mathfrak{n}$. Then:*

(i) *$\mathfrak{b}$ contains a σ-stable Cartan subalgebra $\mathfrak{c}$.*
(ii) *any two such Cartan subalgebras are $K \cap N$-conjugate.*

Proof. Clearly, $\sigma(\mathfrak{b})$ is another Borel subalgebra of $\mathfrak{g}$. Therefore, $\mathfrak{b} \cap \sigma(\mathfrak{b})$ contains a Cartan subalgebra $\mathfrak{d}$ of $\mathfrak{g}$. Now, $\sigma(\mathfrak{d})$ is also a Cartan subalgebra of $\mathfrak{g}$ and both $\mathfrak{d}$ and $\sigma(\mathfrak{d})$ are Cartan subalgebras of $\mathfrak{b} \cap \sigma(\mathfrak{b})$. Hence, they are conjugate by $n = \exp(\xi)$ with $\xi \in [\mathfrak{b}\cap\sigma(\mathfrak{b}), \mathfrak{b}\cap\sigma(\mathfrak{b})] \subset \mathfrak{n}\cap\sigma(\mathfrak{n})$. By applying σ to $\sigma(\mathfrak{d}) = \mathrm{Ad}(n)\mathfrak{d}$, we get $\mathfrak{d} = \mathrm{Ad}(\sigma(n))\sigma(\mathfrak{d})$. It follows that

$$\mathfrak{d} = \mathrm{Ad}(\sigma(n))\,\mathrm{Ad}(n)\mathfrak{d} = \mathrm{Ad}(\sigma(n)n)\mathfrak{d}.$$

This implies that the element $\sigma(n)n \in N \cap \sigma(N)$ normalizes $\mathfrak{d}$. Hence, it is equal to 1, i.e. $\sigma(n) = n^{-1}$. Then

$$\exp(\sigma(\xi)) = \sigma(n) = n^{-1} = \exp(-\xi).$$

Since the exponential map on $\mathfrak{n} \cap \sigma(\mathfrak{n})$ is injective, we conclude that $\sigma(\xi) = -\xi$. Hence, the element

$$n^{\frac{1}{2}} = \exp\left(\tfrac{1}{2}\xi\right)$$

satisfies

$$\sigma(n^{\frac{1}{2}}) = \sigma\left(\exp\left(\tfrac{1}{2}\xi\right)\right) = \exp\left(\sigma\left(\tfrac{1}{2}\xi\right)\right) = \exp\left(-\tfrac{1}{2}\xi\right) = (n^{\frac{1}{2}})^{-1}.$$

Put $\mathfrak{c} = \mathrm{Ad}(n^{\frac{1}{2}})\mathfrak{d}$. Then $\mathfrak{c} \subset \mathfrak{b}$ and

$$\sigma(\mathfrak{c}) = \sigma(\mathrm{Ad}(n^{\frac{1}{2}})\mathfrak{d}) = \mathrm{Ad}(\sigma(n^{\frac{1}{2}}))\sigma(\mathfrak{d}) = \mathrm{Ad}((n^{\frac{1}{2}})^{-1})\,\mathrm{Ad}(n)\mathfrak{d} = \mathrm{Ad}(n^{\frac{1}{2}})\mathfrak{d} = \mathfrak{c}$$

and $\mathfrak{c}$ is σ-stable. This proves (i).

(ii) Assume that $\mathfrak{c}$ and $\mathfrak{c}'$ are σ-stable Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{c} \subset \mathfrak{b}$, $\mathfrak{c}' \subset \mathfrak{b}$. Then, as before, there exists $n \in N \cap \sigma(N)$ such that $\mathfrak{c}' = \mathrm{Ad}(n)\mathfrak{c}$. Therefore, by applying σ we get $\mathfrak{c}' = \mathrm{Ad}(\sigma(n))\mathfrak{c}$, and

$$\mathrm{Ad}(n^{-1}\sigma(n))\mathfrak{c} = \mathfrak{c}.$$

As before, we conclude that $n^{-1}\sigma(n) = 1$, i.e. $\sigma(n) = n$. If $n = \exp(\xi)$, $\xi \in \mathfrak{n}$, we get $\sigma(\xi) = \xi$ and $\xi \in \mathfrak{k} \cap \mathfrak{n}$. Hence, $n \in K \cap N$. □

Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra in $\mathfrak{g}$ and $k \in K$. Then $\mathrm{Ad}(k)(\mathfrak{c})$ is also a σ-stable Cartan subalgebra. Therefore, K acts on the set of all σ-stable Cartan subalgebras.

The preceding result implies that to every Borel subalgebra $\mathfrak{b}$ we can attach a K-conjugacy class of σ-stable Cartan subalgebras, i.e., we have a natural map from the flag variety X onto the set of K-conjugacy classes of σ-stable Cartan subalgebras. Clearly, this map is constant on K-orbits, hence to each K-orbit in X we attach a unique K-conjugacy class of σ-stable Cartan subalgebras. Since the set of K-orbits in X is finite by 1, this immediately implies the following classical result.

5.4. Lemma. *The set of K-conjugacy classes of σ-stable Cartan subalgebras in $\mathfrak{g}$ is finite.*

Let Q be a K-orbit in X, x a point of Q, and $\mathfrak{c}$ a σ-stable Cartan subalgebra contained in $\mathfrak{b}_x$. Then σ induces an involution on the root system R in $\mathfrak{c}^*$. Let R^+ be the set of positive roots determined by $\mathfrak{b}_x$. The specialization map from the Cartan triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ into the triple $(\mathfrak{c}^*, R, R^+)$ allows us to pull back σ to an involution of Σ. From the construction, one sees that this involution on Σ depends only on the orbit Q, so we denote it by σ_Q. Let $\mathfrak{h} = \mathfrak{t}_Q \oplus \mathfrak{a}_Q$ be the decomposition of $\mathfrak{h}$ into σ_Q-eigenspaces for the eigenvalue 1 and -1. Under the specialization map this corresponds to the decomposition $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ of $\mathfrak{c}$ into σ-eigenspaces for the eigenvalue 1 and -1. We call $\mathfrak{t}$ the *toroidal part* and $\mathfrak{a}$ the *split part* of $\mathfrak{c}$. The difference $\dim \mathfrak{t} - \dim \mathfrak{a}$ is called the *signature* of $\mathfrak{c}$. Clearly, it is constant on a K-conjugacy class of σ-stable Cartan subalgebras.

We say that a σ-stable Cartan subalgebra is *maximally toroidal* (resp. *maximally split*) if its signature is maximal (resp. minimal) among all σ-stable Cartan subalgebras in $\mathfrak{g}$. It is well-known that all maximally toroidal σ-stable Cartan subalgebras and all maximally split σ-stable Cartan subalgebras are K-conjugate.

A root $\alpha \in \Sigma$ is called *Q-imaginary* if $\sigma_Q\alpha = \alpha$, *Q-real* if $\sigma_Q\alpha = -\alpha$ and *Q-complex* otherwise. This division depends on the orbit Q, hence we have

$$\begin{aligned} \Sigma_{Q,I} &= Q\text{-imaginary roots}, \\ \Sigma_{Q,\mathbb{R}} &= Q\text{-real roots}, \\ \Sigma_{Q,\mathbb{C}} &= Q\text{-complex roots}. \end{aligned}$$

Via specialization, these roots correspond to imaginary, real and complex roots in the root system R in $\mathfrak{c}^*$.

Put

$$D_+(Q) = \{\alpha \in \Sigma^+ \mid \sigma_Q\alpha \in \Sigma^+,\ \sigma_Q\alpha \neq \alpha\};$$

then $D_+(Q)$ is σ_Q-invariant and consists of Q-complex roots. Each σ_Q-orbit in $D_+(Q)$ consists of two roots, hence $d(Q) = \mathrm{Card}\, D_+(Q)$ is even. The complement of the set $D_+(Q)$ in the set of all positive Q-complex roots is

$$D_-(Q) = \{\alpha \in \Sigma^+ | -\sigma_Q\alpha \in \Sigma^+, \sigma_Q\alpha \neq -\alpha\}.$$

In addition, for an imaginary $\alpha \in R$, $\sigma\alpha = \alpha$ and the root subspace $\mathfrak{g}_\alpha$ is σ-invariant. Therefore, σ acts on it either as 1 or as -1. In the first case $\mathfrak{g}_\alpha \subset \mathfrak{k}$ and α is a *compact imaginary* root, in the second case $\mathfrak{g}_\alpha \not\subset \mathfrak{k}$ and α is a *noncompact imaginary* root. We denote by R_{CI} and R_{NI} the sets of compact, resp. noncompact, imaginary roots in R. Also, we denote the corresponding sets of roots in Σ by $\Sigma_{Q,CI}$ and $\Sigma_{Q,NI}$.

5.5. LEMMA.
(i) *The Lie algebra $\mathfrak{k}$ is the direct sum of $\mathfrak{t}$, the root subspaces $\mathfrak{g}_\alpha$ for compact imaginary roots α, and the σ-eigenspaces of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha}$ for the eigenvalue 1 for real and complex roots α.*
(ii) *The Lie algebra $\mathfrak{k} \cap \mathfrak{b}_x$ is spanned by $\mathfrak{t}$, $\mathfrak{g}_\alpha$ for positive compact imaginary roots α, and the σ-eigenspaces of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha}$ for the eigenvalue 1 for complex roots $\alpha \in R^+$ with $\sigma\alpha \in R^+$.*

5.6. LEMMA. *Let Q be a K-orbit in X. Then*

$$\dim Q = \tfrac{1}{2}(\operatorname{Card}\Sigma_{Q,CI} + \operatorname{Card}\Sigma_{Q,\mathbb{R}} + \operatorname{Card}\Sigma_{Q,\mathbb{C}} - d(Q)).$$

PROOF. The tangent space to Q at $\mathfrak{b}_x$ can be identified with $\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{b}_x)$. By 5,

$$\begin{aligned}\dim Q &= \dim\mathfrak{k} - \dim(\mathfrak{k}\cap\mathfrak{b}_x)\\ &= \operatorname{Card}\Sigma_{Q,CI} + \tfrac{1}{2}(\operatorname{Card}\Sigma_{Q,\mathbb{R}} + \operatorname{Card}\Sigma_{Q,\mathbb{C}}) - \tfrac{1}{2}\operatorname{Card}\Sigma_{Q,CI} - \tfrac{1}{2}d(Q).\end{aligned}$$ □

By 6, since $D_+(Q)$ consists of at most half of all Q-complex roots, the dimension of K-orbits attached to $\mathfrak{c}$ lies between

$$\tfrac{1}{2}(\operatorname{Card}\Sigma_{Q,CI} + \operatorname{Card}\Sigma_{Q,\mathbb{R}} + \tfrac{1}{2}\operatorname{Card}\Sigma_{Q,\mathbb{C}})$$

and

$$\tfrac{1}{2}(\operatorname{Card}\Sigma_{Q,CI} + \operatorname{Card}\Sigma_{Q,\mathbb{R}} + \operatorname{Card}\Sigma_{Q,\mathbb{C}}).$$

The first, minimal, value corresponds to the orbits we call *Zuckerman orbits* attached to $\mathfrak{c}$. The second, maximal, value is attained on the K-orbits we call *Langlands orbits* attached to $\mathfrak{c}$. It can be shown that both types of orbits exist for any σ-stable Cartan subalgebra $\mathfrak{c}$. They clearly depend only on the K-conjugacy class of $\mathfrak{c}$.

Since X is connected, it has a unique open K-orbit. Its dimension is obviously $\frac{1}{2}\operatorname{Card}\Sigma$, hence by the preceding formulas, it corresponds to the Langlands orbit attached to the conjugacy class of σ-stable Cartan subalgebras with no noncompact imaginary roots. This immediately implies the following remark.

5.7. COROLLARY. *The open K-orbit in X is the Langlands orbit attached to the conjugacy class of maximally split σ-stable Cartan subalgebras in $\mathfrak{g}$.*

On the other hand, we have the following characterization of closed K-orbits in X.

5.8. LEMMA. *A K-orbit in the flag variety X is closed if and only if it consists of σ-stable Borel subalgebras.*

Proof. Consider the action of G on $X \times X$ from 2. Let $(x, x) \in \Delta$. If B_x is the Borel subgroup which stabilizes $x \in X$, the stabilizer of (x, x) equals $B_x \cap \sigma(B_x)$. Therefore, if the Lie algebra $\mathfrak{b}_x$ of B_x is σ-stable, the stabilizer of (x, x) is B_x, and the G-orbit of (x, x) is closed. Let C be the connected component containing (x, x) of the intersection of this orbit with the diagonal Δ. Then C is closed. Via the correspondence set up in the proof of 1, C corresponds to the K-orbit of x under the diagonal imbedding of X in $X \times X$.

Let Q be a closed K-orbit, and $x \in Q$. Then the stabilizer of x in K is a solvable parabolic subgroup, i.e., it is a Borel subgroup of K. Therefore, by 5,

$$\dim Q = \tfrac{1}{2}(\dim \mathfrak{k} - \dim \mathfrak{t}) = \tfrac{1}{2}(\operatorname{Card} \Sigma_{Q,CI} + \tfrac{1}{2}(\operatorname{Card} \Sigma_{Q,\mathbb{C}} + \operatorname{Card} \Sigma_{Q,\mathbb{R}})).$$

Comparing this with 6, we get

$$\operatorname{Card} \Sigma_{Q,\mathbb{R}} + \operatorname{Card} \Sigma_{Q,\mathbb{C}} = 2d(Q).$$

Since $D_+(Q)$ consists of at most half of all Q-complex roots, we see that there are no Q-real roots, and all positive Q-complex root lie in $D_+(Q)$. This implies that all Borel subalgebras $\mathfrak{b}_x$, $x \in Q$, are σ-stable. □

5.9. Corollary. *The closed K-orbits in X are the Zuckerman orbits attached to the conjugacy class of maximally toroidal Cartan subalgebras in $\mathfrak{g}$.*

5.10. The K-orbits for $\mathrm{SL}(2, \mathbb{R})$. The simplest example corresponds to the group $\mathrm{SL}(2, \mathbb{R})$. For simplicity of the notation, we shall discuss the group $\mathrm{SU}(1, 1)$ isomorphic to it. In this case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. We can identify the flag variety X of $\mathfrak{g}$ with the one-dimensional projective space $\mathbb{P}^1$. If we denote by $[x_0, x_1]$ the projective coordinates of $x \in \mathbb{P}^1$, the corresponding Borel subalgebra $\mathfrak{b}_x$ is the Lie subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ which leaves the line x invariant. The Cartan involution σ is given by $\sigma(T) = J\,T\,J$, $T \in \mathfrak{g}$, where

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\mathfrak{k}$ is the subalgebra of diagonal matrices in $\mathfrak{g}$, and K is the torus of diagonal matrices in $\mathrm{SL}(2, \mathbb{C})$ which stabilizes $0 = [1, 0]$ and $\infty = [0, 1]$. Hence, the K-orbits in $X = \mathbb{P}^1$ are $\{0\}$, $\{\infty\}$ and $\mathbb{C}^*$. There are two K-conjugacy classes of σ-stable Cartan subalgebras in $\mathfrak{g}$, the class of toroidal Cartan subalgebras and the class of split Cartan subalgebras. The K-orbits $\{0\}$, $\{\infty\}$ correspond to the toroidal class, the open K-orbit $\mathbb{C}^*$ corresponds to the split class.

5.11. The K-orbits for $G_0 = \mathrm{SU}(2, 1)$. This is a more interesting example. In this case, $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Let

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Cartan involution σ on $\mathfrak{g}$ is given $\sigma(T) = J\,T\,J$, $T \in \mathfrak{g}$. The subalgebra $\mathfrak{k}$ consists of matrices

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ 0 & 0 & -\operatorname{tr} A \end{pmatrix},$$

where A is an arbitrary 2×2 matrix. In addition, $K = \{A \in \mathrm{SL}(3, \mathbb{C}) \mid \sigma(A) = A\}$ consists of matrices

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{B}} & 0 \\ & & 0 \\ 0 & 0 & (\det B)^{-1} \end{pmatrix},$$

where B is an arbitrary regular 2×2 matrix. There exist two K-conjugacy classes of σ-stable Cartan subalgebras. The conjugacy class of toroidal Cartan subalgebras is represented by the Cartan subalgebra of the diagonal matrices in $\mathfrak{g}$. The conjugacy class of maximally split Cartan subalgebras is represented by the Cartan subalgebra of all matrices of the form

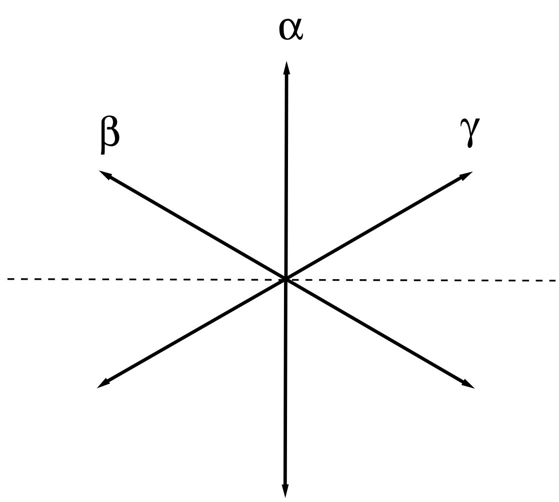
$$\begin{pmatrix} a & 0 & b \\ 0 & -2a & 0 \\ b & 0 & a \end{pmatrix}$$

where $a, b \in \mathbb{C}$ are arbitrary. The Cartan involution acts on this Cartan subalgebra by

$$\sigma \begin{pmatrix} a & 0 & b \\ 0 & -2a & 0 \\ b & 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 & -b \\ 0 & -2a & 0 \\ -b & 0 & a \end{pmatrix}.$$

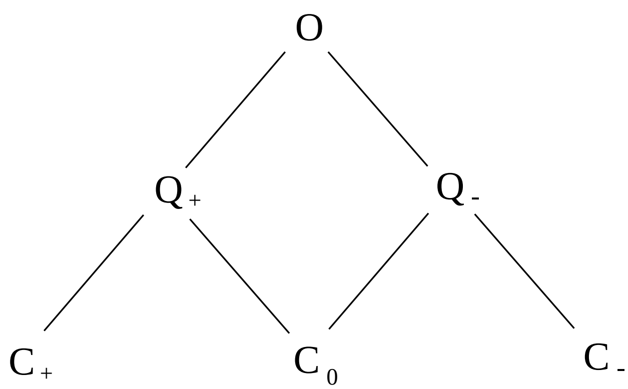
All roots attached to a toroidal Cartan subalgebra are imaginary. A pair of roots is compact imaginary and the remaining ones are noncompact imaginary. Hence, by 6. and 8, all K-orbits are one-dimensional and closed. Since the normalizer of such Cartan subalgebra in K induces the reflection with respect to the compact imaginary roots, the number of these K-orbits is equal to three. One of these, which we denote by C_0, corresponds to a set of simple roots consisting of two noncompact imaginary roots. The other two, C_+ and C_-, correspond to sets of simple roots containing one compact imaginary root and one noncompact imaginary root. The latter two are the "holomorphic" and "antiholomorphic" K-orbits.

If we consider a maximally split Cartan subalgebra, one pair of roots is real and the other roots are complex.



In the above figure, σ is the reflection with respect to the dotted line, the roots $\alpha, -\alpha$ are real, and the other roots are complex. By 6, we see that the K-orbits attached to the class of this Cartan subalgebra can have dimension equal to either 3 or 2. Since $J \in K$, the action of the Cartan involution on this Cartan subalgebra is given by an element of K, i.e., the sets of positive roots conjugate by σ determine the same orbit. Since the flag variety is three-dimensional, the open K-orbit O corresponds to the set of positive roots consisting of α, β and γ. The remaining two two-dimensional K-orbits, Q_+ and Q_-, correspond to the sets of positive roots α, β and $-\gamma$ and α, $-\beta$ and γ respectively.

Therefore, we have the following picture of the K-orbit structure in X.



The top three K-orbits are attached to the K-conjugacy class of maximally split Cartan subalgebras, the bottom three are the closed K-orbits attached to the K-conjugacy class of toroidal Cartan subalgebras. The boundary of one K-orbit is equal to the union of all K-orbits below it connected to it by lines.

6. Standard Harish-Chandra sheaves

Now we shall apply the results from the algebraic theory of $\mathcal{D}$-modules we discussed in §4. to the study of Harish-Chandra sheaves. First we prove the following basic result.

6.1. Theorem. *Harish-Chandra sheaves are holonomic $\mathcal{D}_\lambda$-modules. In particular, they are of finite length.*

This result is based on an analysis of characteristic varieties of Harish-Chandra modules. We start with the following observation.

6.2. Lemma. *Any Harish-Chandra sheaf $\mathcal{V}$ has a good filtration $\mathrm{F}\,\mathcal{V}$ consisting of K-homogeneous coherent $\mathcal{O}_X$-modules.*

Proof. By shifting with $\mathcal{O}(\mu)$ for sufficiently negative $\mu \in P(\Sigma)$ we can assume that λ is antidominant and regular. In this case, by the equivalence of categories, $\mathcal{V} = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$, where $V = \Gamma(X, \mathcal{V})$. Since V is an algebraic K-module and a finitely generated $\mathcal{U}_\theta$-module, there is a finite-dimensional K-invariant subspace U which generates V as a $\mathcal{U}_\theta$-module. Then $F_p\mathcal{D}_\lambda \otimes_{\mathbb{C}} U$, $p \in \mathbb{Z}_+$, are K-homogeneous coherent $\mathcal{O}_X$-modules. Since the natural map of $F_p\mathcal{D}_\lambda \otimes_{\mathbb{C}} U$ into $\mathcal{V}$ is K-equivariant, the image $\mathrm{F}_p\,\mathcal{V}$ is a K-homogeneous coherent $\mathcal{O}_X$-submodule of $\mathcal{V}$ for arbitrary $p \in \mathbb{Z}_+$.

We claim that $\mathrm{F}\,\mathcal{V}$ is a good filtration of the $\mathcal{D}_\lambda$-module $\mathcal{V}$. Clearly, this is a $\mathcal{D}_\lambda$-module filtration of $\mathcal{V}$ by K-homogeneous coherent $\mathcal{O}_X$-modules. Since $\mathcal{V}$

is generated by its global sections, to show that it is exhaustive it is enough to show that any global section v of $\mathcal{V}$ lies in $\mathrm{F}_p \mathcal{V}$ for sufficiently large p. Since V is generated by U as a $\mathcal{U}_\theta$-module, there are $T_i \in \mathcal{U}_\theta$, $u_i \in U$, $1 \leq i \leq m$, such that $v = \sum_{i=1}^m T_i u_i$. On the other hand, there exists $p \in \mathbb{Z}_+$ such that T_i, $1 \leq i \leq m$, are global sections of $\mathrm{F}_p \mathcal{D}_\lambda$. This implies that $v \in \mathrm{F}_p \mathcal{V}$. Finally, by the construction of $\mathrm{F}\,\mathcal{V}$, it is evident that $\mathrm{F}_p \mathcal{D}_\lambda \mathrm{F}_q \mathcal{V} = \mathrm{F}_{p+q} \mathcal{V}$ for all $p, q \in \mathbb{Z}_+$, i.e., $\mathrm{F}\,\mathcal{V}$ is a good filtration. $\square$

We also need some notation. Let Y be a smooth algebraic variety and Z a smooth subvariety of Y. Then we define a smooth subvariety $N_Z(Y)$ of $T^*(Y)$ as the variety of all points $(z, \omega) \in T^*(Y)$ where $z \in Z$ and $\omega \in T^*_z(Y)$ is a linear form vanishing on $T_z(Z) \subset T_z(Y)$. We call $N_Z(Y)$ the *conormal variety* of Z in Y. The dimension of the conormal variety $N_Z(Y)$ of Z in Y is equal to $\dim Y$. To see this, we remark that the dimension of the space of all linear forms in $T^*_z(Y)$ vanishing on $T_z(Z)$ is equal to $\dim T_z(Y) - \dim T_z(Z) = \dim Y - \dim_z Z$. Hence, $\dim_z N_Z(Y) = \dim Y$.

Let $\lambda \in \mathfrak{h}^*$. Then, as we remarked before, $\mathrm{Gr}\,\mathcal{D}_\lambda = \pi_*(\mathcal{O}_{T^*(X)})$, where $\pi : T^*(X) \longrightarrow X$ is the natural projection. Let $\xi \in \mathfrak{g}$. Then ξ determines a global section of $\mathcal{D}_\lambda$ of order ≤ 1, i.e. a global section of $\mathrm{F}_1 \mathcal{D}_\lambda$. The symbol of this section is a global section of $\pi_*(\mathcal{O}_{T^*(X)})$ independent of λ. Let $x \in X$. Then the differential at $1 \in G$ of the orbit map $f_x : G \longrightarrow X$, given by $f_x(g) = gx$, $g \in G$, maps the Lie algebra $\mathfrak{g}$ onto the tangent space $T_x(X)$ at x. The kernel of this map is $\mathfrak{b}_x$, i.e. the differential $T_1(f_x)$ of f_x at 1 identifies $\mathfrak{g}/\mathfrak{b}_x$ with $T_x(X)$. The symbol of the section determined by ξ is given by the function $(x, \omega) \longmapsto \omega(T_1(f_x)(\xi))$ for $x \in X$ and $\omega \in T^*_x(X)$.

Denote by $\mathcal{I}_K$ the ideal in the $\mathcal{O}_X$-algebra $\pi_*(\mathcal{O}_{T^*(X)})$ generated by the symbols of sections attached to elements of $\mathfrak{k}$. Let $\mathcal{N}_K$ be the set of zeros of this ideal in $T^*(X)$.

6.3. Lemma. *The variety $\mathcal{N}_K$ is the union of the conormal varieties $N_Q(X)$ for all K-orbits Q in X. Its dimension is equal to* $\dim X$.

Proof. Let $x \in X$ and denote by Q the K-orbit through x. Then,

$$\begin{aligned}\mathcal{N}_K \cap T^*_x(X) &= \{\omega \in T^*_x(X) \mid \omega \text{ vanishes on } T_1(f_x)(\mathfrak{k})\,\} \\ &= \{\omega \in T^*_x(X) \mid \omega \text{ vanishes on } T_x(Q)\,\} = N_Q(X) \cap T^*_x(X),\end{aligned}$$

i.e. $\mathcal{N}_K$ is the union of all $N_Q(X)$.

For any K-orbit Q in X, its conormal variety $N_Q(X)$ has dimension equal to $\dim X$. Since the number of K-orbits in X is finite, $\mathcal{N}_K$ is a finite union of subvarieties of dimension $\dim X$. $\square$

Therefore, 1. is an immediate consequence of the following result.

6.4. Proposition. *Let $\mathcal{V}$ be a Harish-Chandra sheaf. Then the characteristic variety* $\mathrm{Char}(\mathcal{V})$ *of $\mathcal{V}$ is a closed subvariety of $\mathcal{N}_K$.*

Proof. By 2, $\mathcal{V}$ has a good filtration $\mathrm{F}\,\mathcal{V}$ consisting of K-homogeneous coherent $\mathcal{O}_X$-modules. Therefore, the global sections of $\mathcal{D}_\lambda$ corresponding to $\mathfrak{k}$ map $F_p\mathcal{V}$ into itself for $p \in \mathbb{Z}$. Hence, their symbols annihilate $\mathrm{Gr}\,\mathcal{V}$ and $\mathcal{I}_K$ is contained in the annihilator of $\mathrm{Gr}\,\mathcal{V}$ in $\pi_*(\mathcal{O}_{T^*(X)})$. This implies that the characteristic variety $\mathrm{Char}(\mathcal{V})$ is a closed subvariety of $\mathcal{N}_K$. $\square$

Now we want to describe all irreducible Harish-Chandra sheaves. We start with the following remark.

6.5. Lemma. *Let $\mathcal{V}$ be an irreducible Harish-Chandra sheaf. Then its support* $\operatorname{supp}(\mathcal{V})$ *is the closure of a K-orbit Q in X.*

Proof. Since K is connected, the Harish-Chandra sheaf $\mathcal{V}$ is irreducible if and only if it is irreducible as a $\mathcal{D}_\lambda$-module. To see this we may assume, by twisting with $\mathcal{O}(\mu)$ for sufficiently negative μ, that λ is antidominant and regular. In this case the statement follows from the equivalence of categories and the analogous statement for Harish-Chandra modules (which is evident).

Therefore, we know that $\operatorname{supp}(\mathcal{V})$ is an irreducible closed subvariety of X. Since it must also be K-invariant, it is a union of K-orbits. The finiteness of K-orbits implies that there exists an orbit Q in $\operatorname{supp}(\mathcal{V})$ such that $\dim Q = \dim \operatorname{supp}(\mathcal{V})$. Therefore, $\bar{Q}$ is a closed irreducible subset of $\operatorname{supp}(\mathcal{V})$ and $\dim \bar{Q} = \dim \operatorname{supp}(\mathcal{V})$. This implies that $\bar{Q} = \operatorname{supp}(\mathcal{V})$. □

Let $\mathcal{V}$ be an irreducible Harish-Chandra sheaf and Q the K-orbit in X such that $\operatorname{supp}(\mathcal{V}) = \bar{Q}$. Let $X' = X - \partial Q$. Then X' is an open subvariety of X and Q is a closed subvariety of X'. The restriction $\mathcal{V}|_{X'}$ of $\mathcal{V}$ to X' is again irreducible. Let $i : Q \longrightarrow X$, $i' : Q \longrightarrow X'$ and $j : X' \longrightarrow X$ be the natural immersions. Hence, $i = j \circ i'$. Then $\mathcal{V}|_{X'}$ is an irreducible module supported in Q. Since Q is a smooth closed subvariety of X', by Kashiwara's equivalence of categories, $i'_+(\tau) = \mathcal{V}|_{X'}$ for $\tau = i^!(\mathcal{V})$. Also, τ is an irreducible $(\mathcal{D}_\lambda^i, K)$-module. Since $\mathcal{V}$ is holonomic by 1, τ is a holonomic $\mathcal{D}_\lambda^i$-module with the support equal to Q. This implies that there exists an open dense subset U in Q such that $\tau|_U$ is a connection. Since K acts transitively on Q, τ must be a K-homogeneous connection on Q.

Therefore, to each irreducible Harish-Chandra sheaf we attach a pair (Q, τ) consisting of a K-orbit Q and an irreducible K-homogeneous connection τ on Q such that:

(i) $\operatorname{supp}(\mathcal{V}) = \bar{Q}$;

(ii) $i^!(\mathcal{V}) = \tau$.

We call the pair (Q, τ) *the standard data* attached to $\mathcal{V}$.

Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q in $\mathcal{M}_{coh}(\mathcal{D}_\lambda^i, K)$. Then, $\mathcal{I}(Q, \tau) = i_+(\tau)$ is a $(\mathcal{D}_\lambda, K)$-module. Moreover, it is holonomic and therefore coherent. Hence, $\mathcal{I}(Q, \tau)$ is a Harish-Chandra sheaf. We call it *the standard Harish-Chandra sheaf* attached to (Q, τ).

6.6. Lemma. *Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q. Then the standard Harish-Chandra sheaf $\mathcal{I}(Q, \tau)$ contains a unique irreducible Harish-Chandra subsheaf.*

Proof. Clearly,

$$\mathcal{I}(Q, \tau) = i_+(\tau) = j_+(i'_+(\tau)) = j_\cdot(i'_+(\tau)),$$

where $j_\cdot$ is the sheaf direct image functor. Therefore, $\mathcal{I}(Q, \tau)$ contains no sections supported in ∂Q. Hence, any nonzero $\mathcal{D}_\lambda$-submodule $\mathcal{U}$ of $\mathcal{I}(Q, \tau)$ has a nonzero restriction to X'. By Kashiwara's equivalence of categories, $i'_+(\tau)$ is an irreducible $\mathcal{D}_\lambda|_{X'}$-module. Hence, $\mathcal{U}|_{X'} = \mathcal{I}(Q, \tau)|_{X'}$. Therefore, for any two nonzero $\mathcal{D}_\lambda$-submodules $\mathcal{U}$ and $\mathcal{U}'$ of $\mathcal{I}(Q, \tau)$, $\mathcal{U} \cap \mathcal{U}' \neq 0$. Since $\mathcal{I}(Q, \tau)$ is of finite length, it has a minimal $\mathcal{D}_\lambda$-submodule and by the preceding remark this module is unique. By its uniqueness it must be K-equivariant, therefore it is a Harish-Chandra subsheaf. □

We denote by $\mathcal{L}(Q,\tau)$ the unique irreducible Harish-Chandra subsheaf of $\mathcal{I}(Q,\tau)$. The following result gives a classification of irreducible Harish-Chandra sheaves.

6.7. Theorem (Beilinson-Bernstein).

(i) *An irreducible Harish-Chandra sheaf $\mathcal{V}$ with the standard data (Q,τ) is isomorphic to $\mathcal{L}(Q,\tau)$.*

(ii) *Let Q and Q' be K-orbits in X, and τ and τ' irreducible K-homogeneous connections on Q and Q' respectively. Then $\mathcal{L}(Q,\tau)\cong\mathcal{L}(Q',\tau')$ if and only if $Q=Q'$ and $\tau\cong\tau'$.*

Proof. (i) Let $\mathcal{V}$ be an irreducible Harish-Chandra sheaf and (Q,τ) the corresponding standard data. Then, as we remarked before, $\mathcal{V}|X'=(i')_+(\tau)$. By the universal property of $j_\cdot$, there exists a nontrivial morphism of $\mathcal{V}$ into $\mathcal{I}(Q,\tau)=j_\cdot(i'_+(\tau))$ which extends this isomorphism. Since $\mathcal{V}$ is irreducible, the kernel of this morphism must be zero. Clearly, by 6, its image is equal to $\mathcal{L}(Q,\tau)$.

(ii) Since $\bar{Q}=\operatorname{supp}\mathcal{L}(Q,\tau)$, it is evident that $\mathcal{L}(Q,\tau)\cong\mathcal{L}(Q',\tau')$ implies $Q=Q'$. The rest follows from the formula $\tau=i^!(\mathcal{L}(Q,\tau))$. □

From the construction it is evident that the quotient of the standard module $\mathcal{I}(Q,\tau)$ by the irreducible submodule $\mathcal{L}(Q,\tau)$ is supported in the boundary ∂Q of Q. In particular, if Q is closed, $\mathcal{I}(Q,\tau)$ is irreducible.

Let Q be a K-orbit and τ an irreducible K-homogeneous connection on Q in $\mathcal{M}_{coh}(\mathcal{D}^i_\lambda,K)$. Let $x\in Q$ and $T_x(\tau)$ be the geometric fibre of τ at x. Then $T_x(\tau)$ is finite dimensional, and the stabilizer S_x of x in K acts irreducibly in $T_x(\tau)$. The connection τ is completely determined by the representation ω of S_x in $T_x(\tau)$. Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra in $\mathfrak{b}_x$. The Lie algebra $\mathfrak{s}_x=\mathfrak{k}\cap\mathfrak{b}_x$ of S_x is the semidirect product of the toroidal part $\mathfrak{t}$ of $\mathfrak{c}$ with the nilpotent radical $\mathfrak{u}_x=\mathfrak{k}\cap\mathfrak{n}_x$ of $\mathfrak{s}_x$. Let U_x be the unipotent subgroup of K corresponding to $\mathfrak{u}_x$; it is the unipotent radical of S_x. Let T be the Levi factor of S_x with Lie algebra $\mathfrak{t}$. Then S_x is the semidirect product of T with U_x. The representation ω is trivial on U_x, hence it can be viewed as a representation of the group T. The differential of the representation ω, considered as a representation of $\mathfrak{t}$, is a direct sum of a finite number of copies of the one dimensional representation defined by the restriction of the specialization of $\lambda+\rho$ to $\mathfrak{t}$. Therefore, we say that τ is *compatible with* $\lambda+\rho$.

If the group G_0 is linear, T is contained in a complex torus in the complexification of G_0, hence it is abelian. Therefore, in this case, ω is one-dimensional. Hence, if S_x is connected, it is completely determined by $\lambda+\rho$. Otherwise, Q can admit several K-homogeneous connections compatible with the same $\lambda+\rho$, as we can see from the following basic example.

6.8. Standard Harish-Chandra sheaves for $\mathrm{SL}(2,\mathbb{R})$. Now we discuss the structure of standard Harish-Chandra sheaves for $\mathrm{SL}(2,\mathbb{R})$ (the more general situation of finite covers of $\mathrm{SL}(2,\mathbb{R})$ is discussed in [**13**]). In this case, as we discussed in 5.10, K has three orbits in $X=\mathbb{P}^1$, namely $\{0\}$, $\{\infty\}$ and $\mathbb{C}^*$.

The standard $\mathcal{D}_\lambda$-modules corresponding to the orbits $\{0\}$ and $\{\infty\}$ exist if and only if λ is a weight in $P(\Sigma)$. Since these orbits are closed, these standard modules are irreducible.

Therefore, it remains to study the standard modules attached to the open orbit $\mathbb{C}^*$. First we want to construct suitable trivializations of $\mathcal{D}_\lambda$ on the open cover of $\mathbb{P}^1$ consisting of $\mathbb{P}^1 - \{0\}$ and $\mathbb{P}^1 - \{\infty\}$. We denote by $\alpha \in \mathfrak{h}^*$ the positive root of $\mathfrak{g}$ and put $\rho = \frac{1}{2}\alpha$ and $t = \alpha\check{}(\lambda)$, where $\alpha\check{}$ is the dual root of α.

Let $\{E, F, H\}$ denote the standard basis of $\mathfrak{sl}(2, \mathbb{C})$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the commutation relations

$$[H, E] = 2E \quad [H, F] = -2F \quad [E, F] = H.$$

Also, H spans the Lie algebra $\mathfrak{k}$. Moreover, if we specialize at 0, H corresponds to the dual root $\alpha\check{}$, but if we specialize at ∞, H corresponds to the negative of $\alpha\check{}$.

First we discuss $\mathbb{P}^1 - \{\infty\}$. On this set we define the coordinate z by $z([1, x_1]) = x_1$. In this way one identifies $\mathbb{P}^1 - \{\infty\}$ with the complex plane $\mathbb{C}$. After a short calculation we get

$$E = -z^2\partial - (t+1)z, \quad F = \partial, \quad H = 2z\partial + (t+1)$$

in this coordinate system. Analogously, on $\mathbb{P}^1 - \{0\}$ with the natural coordinate $\zeta([x_0, 1]) = x_0$, we have

$$E = \partial, \quad F = -\zeta^2\partial - (t+1)\zeta, \quad H = -2\zeta\partial - (t+1).$$

On $\mathbb{C}^*$ these two coordinate systems are related by the inversion $\zeta = \frac{1}{z}$. This implies that $\partial_\zeta = -z^2\partial_z$, i. e., on $\mathbb{C}^*$ the second trivialization gives

$$E = -z^2\partial, \quad F = \partial - \frac{1+t}{z} \quad H = 2z\partial - (t+1).$$

It follows that the first and the second trivialization on $\mathbb{C}^*$ are related by the automorphism of $\mathcal{D}_{\mathbb{C}^*}$ induced by

$$\partial \longrightarrow \partial - \frac{1+t}{z} = z^{1+t}\,\partial\, z^{-(1+t)}.$$

Now we want to analyze the standard Harish-Chandra sheaves attached to the open K-orbit $\mathbb{C}^*$. If we identify K with another copy of $\mathbb{C}^*$, the stabilizer in K of any point in the orbit $\mathbb{C}^*$ is the group $M = \{\pm 1\}$. Let η_0 be the trivial representation of M and η_1 the identity representation of M. Denote by τ_k the irreducible K-equivariant connection on $\mathbb{C}^*$ corresponding to the representation η_k of M, and by $\mathcal{I}(\mathbb{C}^*, \tau_k)$ the corresponding standard Harish-Chandra sheaf in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$. To analyze these $\mathcal{D}_\lambda$-modules it is convenient to introduce a trivialization of $\mathcal{D}_\lambda$ on $\mathbb{C}^* = \mathbb{P}^1 - \{0, \infty\}$ such that H corresponds to the differential operator $2z\partial$ on the orbit $\mathbb{C}^*$ and $t \in K \cong \mathbb{C}^*$ acts on it by multiplication by t^2. We obtain this trivialization by restricting the original z-trivialization to $\mathbb{C}^*$ and twisting it by the automorphism

$$\partial \longrightarrow \partial - \frac{1+t}{2z} = z^{\frac{1+t}{2}}\,\partial\, z^{-\frac{1+t}{2}}.$$

This gives a trivialization of $\mathcal{D}_\lambda|_{\mathbb{C}^*}$ which satisfies

$$E = -z^2\partial - \frac{1+t}{2}z, \quad F = \partial - \frac{1+t}{2z}, \quad H = 2z\partial.$$

The global sections of τ_k on $\mathbb{C}^*$ form the linear space spanned by functions $z^{p+\frac{k}{2}}$, $p \in \mathbb{Z}$. To analyze irreducibility of the standard $\mathcal{D}_\lambda$-module $\mathcal{I}(\mathbb{C}^*, \tau_k)$ we have to study its behavior at 0 and ∞. By the preceding discussion, if we use the z-trivialization of $\mathcal{D}_\lambda$ on $\mathbb{C}^*$, $\mathcal{I}(\mathbb{C}^*, \tau_k)|\mathbb{P}^1 - \{\infty\}$ looks like the $\mathcal{D}_\mathbb{C}$-module which is the direct image of the $\mathcal{D}_{\mathbb{C}^*}$-module generated by $z^{\frac{k-t-1}{2}}$. This module is clearly reducible if and only if it contains functions regular at the origin, i.e., if and only if $\frac{k-t-1}{2}$ is an integer. Analogously, $\mathcal{I}(\mathbb{C}^*, \tau_k)|\mathbb{P}^1 - \{0\}$ is reducible if and only if $\frac{k+t+1}{2}$ is an integer. Therefore, $\mathcal{I}(\mathbb{C}^*, \tau_k)$ is irreducible if and only if $t + k$ is an odd integer.

We can summarize this as the *parity condition*: The following conditions are equivalent:

(i) $\alpha\check{}(\lambda) + k \notin 2\mathbb{Z} + 1$;

(ii) the standard module $\mathcal{I}(\mathbb{C}^*, \tau_k)$ is irreducible.

Therefore, if λ is not a weight, the standard Harish-Chandra sheaves $\mathcal{I}(\mathbb{C}^*, \tau_k)$, $k = 0, 1$, are irreducible. If λ is a weight, $\alpha\check{}(\lambda)$ is an integer, and depending on its parity, one of the standard Harish-Chandra sheaves $\mathcal{I}(\mathbb{C}^*, \tau_0)$ and $\mathcal{I}(\mathbb{C}^*, \tau_1)$ is reducible while the other one is irreducible. Assume that $\mathcal{I}(\mathbb{C}^*, \tau_k)$ is reducible. Then it contains the module $\mathcal{O}(\lambda + \rho)$ as the unique irreducible submodule and the quotient by this submodule is the direct sum of standard Harish-Chandra sheaves at $\{0\}$ and $\{\infty\}$.

Under the equivalence of categories, this describes basic results on classification of irreducible Harish-Chandra modules for $\mathrm{SL}(2, \mathbb{R})$. If $\mathrm{Re}\, \alpha\check{}(\lambda) \leq 0$ and $\lambda \neq 0$, the global sections of the standard Harish-Chandra sheaves at $\{0\}$ and $\{\infty\}$ represent the discrete series representations (holomorphic and antiholomorphic series correspond to the opposite orbits). The global sections of the standard Harish-Chandra sheaves attached to the open orbit are the principal series representations. They are reducible if $\alpha\check{}(\lambda)$ is an integer and k is of the appropriate parity. In this case, they have irreducible finite-dimensional submodules, and their quotients by these submodules are direct sums of holomorphic and antiholomorphic discrete series. If $\lambda = 0$, the global sections of the irreducible standard Harish-Chandra sheaves attached to $\{0\}$ and $\{\infty\}$ are the limits of discrete series, the space of global sections of the irreducible standard Harish-Chandra sheaf attached to the open orbit is the irreducible principal series representation and the space of global sections of the reducible standard Harish-Chandra sheaf attached to the open orbit is the reducible principal series representation which splits into the sum of two limits of discrete series. The latter phenomenon is caused by the vanishing of global sections of $\mathcal{O}(\rho)$.

To handle the analogous phenomena in general, we have to formulate an analogous parity condition. We restrict ourselves to the case of linear group G_0 (the general case is discussed in [**13**]). In this case we can assume that K is a subgroup of the complexification G of G_0. Let α be a Q-real root. Denote by $\mathfrak{s}_\alpha$ the three-dimensional simple algebra generated by the root subspaces corresponding to α and $-\alpha$. Let S_α be the connected subgroup of G with Lie algebra $\mathfrak{s}_\alpha$; it is isomorphic either to $\mathrm{SL}(2, \mathbb{C})$ or to $\mathrm{PSL}(2, \mathbb{C})$. Denote by H_α the element of $\mathfrak{s}_\alpha \cap \mathfrak{c}$ such that $\alpha(H_\alpha) = 2$. Then $m_\alpha = \exp(\pi i H_\alpha) \in G$ satisfies $m_\alpha^2 = 1$. Moreover, $\sigma(m_\alpha) = \exp(-\pi i H_\alpha) = m_\alpha^{-1} = m_\alpha$. Clearly, $m_\alpha = 1$ if $S_\alpha \cong \mathrm{PSL}(2, \mathbb{C})$, and $m_\alpha \neq 1$ if $S_\alpha \cong \mathrm{SL}(2, \mathbb{C})$. In the latter case m_α corresponds to the negative of the identity matrix in $\mathrm{SL}(2, \mathbb{C})$. In both cases, m_α lies in T.

The set $D_-(Q)$ is the union of $-\sigma_Q$-orbits consisting of pairs $\{\beta, -\sigma_Q\beta\}$. Let A be a set of representatives of $-\sigma_Q$-orbits in $D_-(Q)$. Then, for an arbitrary Q-real root α, the number

$$\delta_Q(m_\alpha) = \prod_{\beta \in A} e^\beta(m_\alpha)$$

is independent of the choice of A and equal to ± 1.

Following B. Speh and D. Vogan [**16**][2], we say that τ satisfies the SL_2*-parity condition with respect to the Q-real root* α if the number $e^{i\pi\alpha\check{}(\lambda)}$ is not equal to $-\delta_Q(m_\alpha)\omega(m_\alpha)$. Clearly, this condition specializes to the condition (i) in 8.

The relation of the SL_2-parity condition with irreducibility of the standard modules can be seen from the following result. First, let

$$\Sigma_\lambda = \{\alpha \in \Sigma \mid \alpha\check{}(\lambda) \in \mathbb{Z}\}$$

be the root subsystem of Σ consisting of all roots integral with respect to λ. The following result is established in [**8**]. We formulate it in the case of linear group G_0, where it corresponds to the result of Speh and Vogan [**16**]. The discussion of the general situation can be found in [**13**].

6.9. Theorem. *Let Q be a K-orbit in X, $\lambda \in \mathfrak{h}^*$, and τ an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$. Then the following conditions are equivalent:*

(i) *$D_-(Q) \cap \Sigma_\lambda = \emptyset$, and τ satisfies the* SL_2*-parity condition with respect to every Q-real root in Σ; and*

(ii) *the standard $\mathcal{D}_\lambda$-module $\mathcal{I}(Q,\tau)$ is irreducible.*

6.10. Standard Harish-Chandra sheaves for $\mathrm{SU}(2,1)$. Consider again the case of $G_0 = \mathrm{SU}(2,1)$. In this case, the stabilizers in K of any point $x \in X$ are connected, so each K-orbit admits at most one irreducible K-homogeneous connection compatible with $\lambda + \rho$ for a given $\lambda \in \mathfrak{h}^*$. Therefore, we can denote the corresponding standard Harish-Chandra sheaf by $\mathcal{I}(Q,\lambda)$. If Q is any of the closed K-orbits, these standard Harish-Chandra sheaves exist if and only if $\lambda \in P(\Sigma)$. If Q is a nonclosed K-orbit, these standard Harish-Chandra sheaves exist if and only if $\lambda + \sigma_Q\lambda \in P(\Sigma)$.

Clearly, the standard Harish-Chandra sheaves attached to the closed orbits are always irreducible. By analyzing 9, we see that the standard Harish-Chandra sheaves for the other orbits are reducible if and only if λ is a weight. If Q is the open orbit O, the standard Harish-Chandra sheaf $\mathcal{I}(Q,\lambda)$ attached to $\lambda \in P(\Sigma)$ contains the homogeneous invertible $\mathcal{O}_X$-module $\mathcal{O}(\lambda+\rho)$ as its unique irreducible submodule, the standard Harish-Chandra sheaf $\mathcal{I}(C_0,\lambda)$ is its unique irreducible quotient, and the direct sum $\mathcal{L}(Q_+,\lambda) \oplus \mathcal{L}(Q_-,\lambda)$ is in the "middle" of the composition series. The standard Harish-Chandra sheaves $\mathcal{I}(Q_+,\lambda)$ and $\mathcal{I}(Q_-,\lambda)$ have unique irreducible submodules $\mathcal{L}(Q_+,\lambda)$ and $\mathcal{L}(Q_-,\lambda)$ respectively, and the quotients are

$$\mathcal{I}(Q_+,\lambda)/\mathcal{L}(Q_+,\lambda) = \mathcal{I}(C_+,\lambda) \oplus \mathcal{I}(C_0,\lambda)$$
$$\text{and } \mathcal{I}(Q_-,\lambda)/\mathcal{L}(Q_-,\lambda) = \mathcal{I}(C_-,\lambda) \oplus \mathcal{I}(C_0,\lambda).$$

[2] In fact, they consider the reducibility condition, while ours is the irreducibility condition.

7. Geometric classification of irreducible Harish-Chandra modules

In the preceding section we described the classification of all irreducible Harish-Chandra sheaves. Now, we use this classification to classify irreducible Harish-Chandra modules.

First, it is useful to use a more restrictive condition than antidominance. We say that $\lambda \in \mathfrak{h}^*$ is *strongly antidominant* if $\operatorname{Re} \alpha\check{}(\lambda) \leq 0$ for any $\alpha \in \Sigma^+$. Clearly, a strongly antidominant λ is antidominant.

Let V be an irreducible Harish-Chandra module. We can view V as an irreducible object in the category $\mathcal{M}(\mathcal{U}_\theta, K)$. We fix a strongly antidominant $\lambda \in \theta$. Then, as we remarked in §3, there exists a unique irreducible $\mathcal{D}_\lambda$-module $\mathcal{V}$ such that $\Gamma(X, \mathcal{V}) = V$. Since this $\mathcal{D}_\lambda$-module must be a Harish-Chandra sheaf, it is of the form $\mathcal{L}(Q, \tau)$ for some K-orbit Q in X and an irreducible K-homogeneous connection τ on Q compatible with $\lambda + \rho$. Hence, there is a unique pair (Q, τ) such that $\Gamma(X, \mathcal{L}(Q, \tau)) = V$. If λ is regular in addition, this correspondence gives a parametrization of equivalence classes of irreducible Harish-Chandra modules by all pairs (Q, τ). On the other hand, if λ is not regular, some of the pairs (Q, τ) correspond to irreducible Harish-Chandra sheaves $\mathcal{L}(Q, \tau)$ with $\Gamma(X, \mathcal{L}(Q, \tau)) = 0$. Therefore, to give a precise formulation of this classification of irreducible Harish-Chandra modules, we have to determine a necessary and sufficient condition for nonvanishing of global sections of irreducible Harish-Chandra sheaves $\mathcal{L}(Q, \tau)$.

For any root $\alpha \in \Sigma$ we have $\alpha\check{}(\lambda + \sigma_Q \lambda) \in \mathbb{R}$. In particular, if α is Q-imaginary, $\alpha\check{}(\lambda)$ is real.

Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Let

$$\Sigma_0 = \{\alpha \in \Sigma \mid \operatorname{Re} \alpha\check{}(\lambda) = 0\}.$$

Let Π be the basis in Σ corresponding to Σ^+. Put $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$ and $\Pi_0 = \Pi \cap \Sigma_0$. Since λ is strongly antidominant, Π_0 is the basis of the root system Σ_0 determined by the set of positive roots Σ_0^+.

Let $\Sigma_1 = \Sigma_0 \cap \sigma_Q(\Sigma_0)$; equivalently, Σ_1 is the largest root subsystem of Σ_0 invariant under σ_Q. Let

$$\Sigma_2 = \{\alpha \in \Sigma_1 \mid \alpha\check{}(\lambda) = 0\}.$$

This set is also σ_Q-invariant. Let $\Sigma_2^+ = \Sigma_2 \cap \Sigma^+$, and denote by Π_2 the corresponding basis of the root system Σ_2. Clearly, $\Pi_0 \cap \Sigma_2 \subset \Pi_2$, but this inclusion is strict in general.

The next theorem gives the simple necessary and sufficient condition for $\Gamma(X, \mathcal{L}(Q, \tau)) \neq 0$, that was alluded to before. In effect, this completes the classification of irreducible Harish-Chandra modules. The proof can be found in [**8**].

7.1. THEOREM. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$. Then the following conditions are equivalent:*

(i) $\Gamma(X, \mathcal{L}(Q, \tau)) \neq 0$;

(ii) *the following conditions hold:*

  (1) (a) *the set Π_2 contains no compact Q-imaginary roots;*

  (2) (b) *for any positive Q-complex root α with $\alpha\check{}(\lambda) = 0$, the root $\sigma_Q \alpha$ is also positive;*

  (3) (c) *for any Q-real α with $\alpha\check{}(\lambda) = 0$, τ must satisfy the* SL_2*-parity condition with respect to α.*

The proof of this result is based on the use of the intertwining functors I_w for w in the subgroup W_0 of the Weyl group W generated by reflections with respect to roots in Σ_0 [**2**], [**13**]. The vanishing of $\Gamma(X, \mathcal{L}(Q,\tau))$ is equivalent with $I_w(\mathcal{L}(Q,\tau)) = 0$ for some $w \in W_0$. Let $\alpha \in \Pi_0$ and s_α the corresponding reflection. Then, essentially by an $\mathrm{SL}(2,\mathbb{C})$-calculation, $I_{s_\alpha}(\mathcal{L}(Q,\tau)) = 0$ if and only if a condition in (ii) fails for α, i.e., $\alpha\check{}(\lambda) = 0$ and α is either a compact Q-imaginary root, or a Q-complex root with $-\sigma_Q\alpha \in \Sigma^+$, or a Q-real root and the SL_2-parity condition for τ fails for α. Otherwise, either $\alpha\check{}(\lambda) = 0$ and $\mathcal{L}(Q,\tau)$ is a quotient of $I_{s_\alpha}(\mathcal{L}(Q,\tau))$, or $\alpha\check{}(\lambda) \neq 0$ and $I_{s_\alpha}(\mathcal{L}(Q,\tau)) = \mathcal{L}(Q',\tau')$ for some K-orbit Q' and irreducible K-homogeneous connection τ' on Q' compatible with $s_\alpha\lambda + \rho$ and $\Gamma(X, \mathcal{L}(Q,\tau)) = \Gamma(X, \mathcal{L}(Q',\tau'))$. Since intertwining functors satisfy the product formula

$$I_{w'w''} = I_{w'}I_{w''} \text{ for } w', w'' \in W \text{ such that } \ell(w'w'') = \ell(w') + \ell(w''),$$

by induction in the length of $w \in W_0$, one checks that (i) holds if and only if (ii) holds for all roots in Σ_0.

In general, there are several strongly antidominant λ in θ, and an irreducible Harish-Chandra module V correspond to different standard data (Q,τ). Still, all such K-orbits Q correspond to the same K-conjugacy class of σ-stable Cartan subalgebras [**8**].

8. Geometric classification versus Langlands classification

At the first glance it is not clear how the "geometric" classification in §7 relates to the other classification schemes. To see its relation to the Langlands classification, it is critical to understand the asymptotic behavior of the matrix coefficients of the irreducible Harish-Chandra modules $\Gamma(X, \mathcal{L}(Q,\tau))$. Although the asymptotic behavior of the matrix coefficients is an "analytic" invariant, its connection with the $\mathfrak{n}$-homology of Harish-Chandra modules studied by Casselman and the author in [**6**], [**12**], shows that it also has a simple, completely algebraic, interpretation. Together with the connection of the $\mathfrak{n}$-homology of $\Gamma(X, \mathcal{L}(Q,\tau))$ with the derived geometric fibres of $\mathcal{L}(Q,\tau)$ (see, for example, [**9**]), this establishes a precise relationship between the standard data and the asymptotics of $\Gamma(X, \mathcal{L}(Q,\tau))$ [**8**].

To formulate some important consequences of this relationship, for $\lambda \in \mathfrak{h}^*$ and a K-orbit Q, we introduce the following invariant:

$$\lambda_Q = \frac{1}{2}(\lambda - \sigma_Q\lambda).$$

8.1. Theorem. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant, Q a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$ such that $V = \Gamma(X, \mathcal{L}(Q,\tau)) \neq 0$. Then:*

(i) *V is tempered if and only if $\operatorname{Re}\lambda_Q = 0$;*
(ii) *V is square-integrable if and only if $\sigma_Q = 1$ and λ is regular.*

If $\operatorname{Re}\lambda_Q = 0$, then $\operatorname{Re}\alpha\check{}(\lambda) = \operatorname{Re}(\sigma_Q\alpha)\check{}(\lambda)$. Hence, if α is Q-real, $\operatorname{Re}\alpha\check{}(\lambda) = 0$ and α is in the subset Σ_1 introduced in the preceding section. If α is in $D_-(Q)$, $\alpha, -\sigma_Q\alpha \in \Sigma^+$ and, since λ is strongly dominant, we conclude that $\operatorname{Re}\alpha\check{}(\lambda) = \operatorname{Re}(\sigma_Q\alpha)\check{}(\lambda) = 0$, i.e., α is also in Σ_1. It follows that all roots in $D_-(Q)$ and all Q-real roots are in Σ_1.

Hence, 1, 7.1. and 6.9. have the following consequence which was first proved by Ivan Mirković [**15**].

8.2. THEOREM. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q. Assume that* $\operatorname{Re}\lambda_Q = 0$. *Then* $\Gamma(X, \mathcal{L}(Q,\tau)) \neq 0$ *implies that* $\mathcal{I}(Q,\tau)$ *is irreducible, i.e.,* $\mathcal{L}(Q,\tau) = \mathcal{I}(Q,\tau)$.

Thus 2. explains the simplicity of the classification of tempered irreducible Harish-Chandra modules: every tempered irreducible Harish-Chandra module is the space of global sections of an irreducible standard Harish-Chandra sheaf.

The analysis becomes especially simple in the case of square-integrable irreducible Harish-Chandra modules. By 1.(ii) they exist if and only if $\operatorname{rank}\mathfrak{g} = \operatorname{rank} K$ – this is a classical result of Harish-Chandra. If this condition is satisfied, the Weyl group orbit θ must in addition be regular and real. Since it is real, θ contains a unique strongly antidominant λ. This λ is regular and $\Gamma(X, \mathcal{L}(Q,\tau))$ is square-integrable if and only if $\sigma_Q = 1$. Therefore, all Borel subalgebras in Q are σ-stable. By 5.8, the K-orbit Q is necessarily closed. The stabilizer in K of a point in Q is a Borel subgroup of K. Hence, an irreducible K-homogeneous connection τ compatible with $\lambda + \rho$ exists on the K-orbit Q if and only if $\lambda + \rho$ specializes to a character of this Borel subgroup. If G_0 is linear, this means that λ is a weight in $P(\Sigma)$. The connection $\tau = \tau_{Q,\lambda}$ is completely determined by $\lambda+\rho$. Hence, the map $Q \to \Gamma(X, \mathcal{I}(Q, \tau_{Q,\lambda}))$ is a bijection between closed K-orbits in X and equivalence classes of irreducible square-integrable Harish-Chandra modules with infinitesimal character determined by θ.

By definition, the *discrete series* is the set of equivalence classes of irreducible square-integrable Harish-Chandra modules.

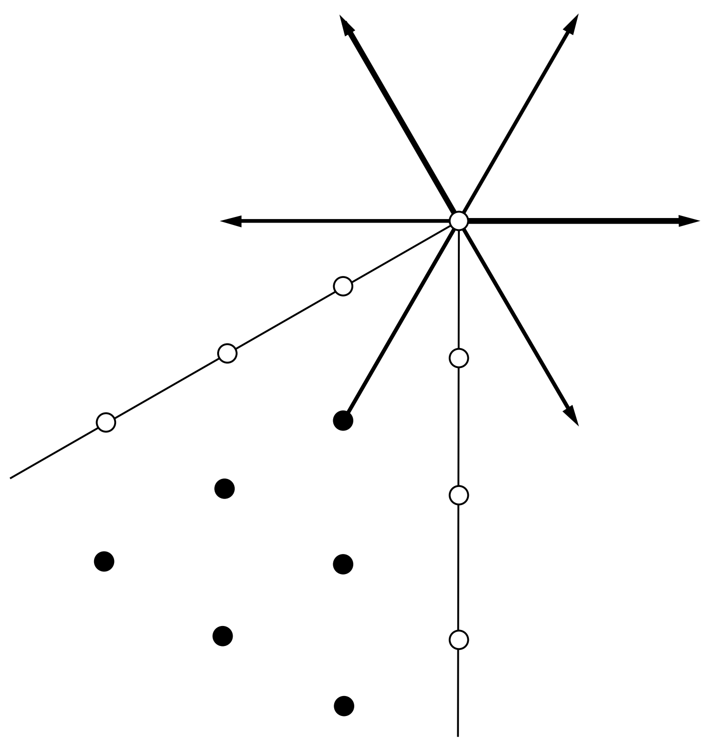
If we drop the regularity assumption on λ, for a closed K-orbit Q in X and an irreducible K-homogeneous connection τ compatible with $\lambda+\rho$, $\Gamma(X, \mathcal{I}(Q,\tau)) \neq 0$ if and only if there exists no compact Q-imaginary root $\alpha \in \Pi$ such that $\alpha\check{}(\lambda) = 0$. These representations are tempered irreducible Harish-Chandra modules. They constitute the *limits of discrete series*.

Using the duality theorem of [**7**], one shows that the space of global sections of a standard Harish-Chandra sheaf is a standard Harish-Chandra module, as is explained in [**18**]. In particular, irreducible tempered representations are irreducible unitary principal series representations induced from limits of discrete series [**10**]. More precisely, if $\Gamma(X, \mathcal{I}(Q,\tau))$ is not a limit of discrete series, we have $\mathfrak{a}_Q \neq \{0\}$. Then $\mathfrak{a}_Q$ determines a parabolic subgroup in G_0. The standard data (Q,τ) determine, by "restriction", the standard data of a limit of discrete series representation of its Levi factor. The module $\Gamma(X, \mathcal{I}(Q,\tau))$ is the irreducible unitary principal series representation induced from the limits of discrete series representation attached to these "restricted" data. If the standard Harish-Chandra sheaf $\mathcal{I}(Q,\tau)$ with $\operatorname{Re}\lambda_Q = 0$ is reducible, its space of global sections represents a reducible unitary principal series representation induced from a limits of discrete series representation. These reducible standard Harish-Chandra sheaves can be analyzed in more detail. This leads to a $\mathcal{D}$-module theoretic explanation of the results of Knapp and Zuckerman on the reducibility of unitary principal series representations [**10**]. This analysis has been done by Ivan Mirković in [**15**].

It remains to discuss nontempered irreducible Harish-Chandra modules, i.e., the Langlands representations. In this case $\operatorname{Re}\lambda_Q \neq 0$ and it defines a nonzero linear form on $\mathfrak{a}_Q$. This form determines a parabolic subgroup of G_0 such that the roots of its Levi factor are orthogonal to the specialization of $\operatorname{Re}\lambda_Q$. The "restriction" of the standard data (Q,τ) to this Levi factor determines tempered standard data. The module $\Gamma(X, \mathcal{L}(Q, \tau))$ is equal to the unique irreducible submodule of the principal series representation $\Gamma(X, \mathcal{I}(Q, \tau))$ corresponding to this parabolic subgroup, and induced from the tempered representation of the Levi factor attached to the "restricted" standard data. By definition, this unique irreducible submodule is a Langlands representation. A detailed analysis of this construction leads to a completely algebraic proof of the Langlands classification [**8**].

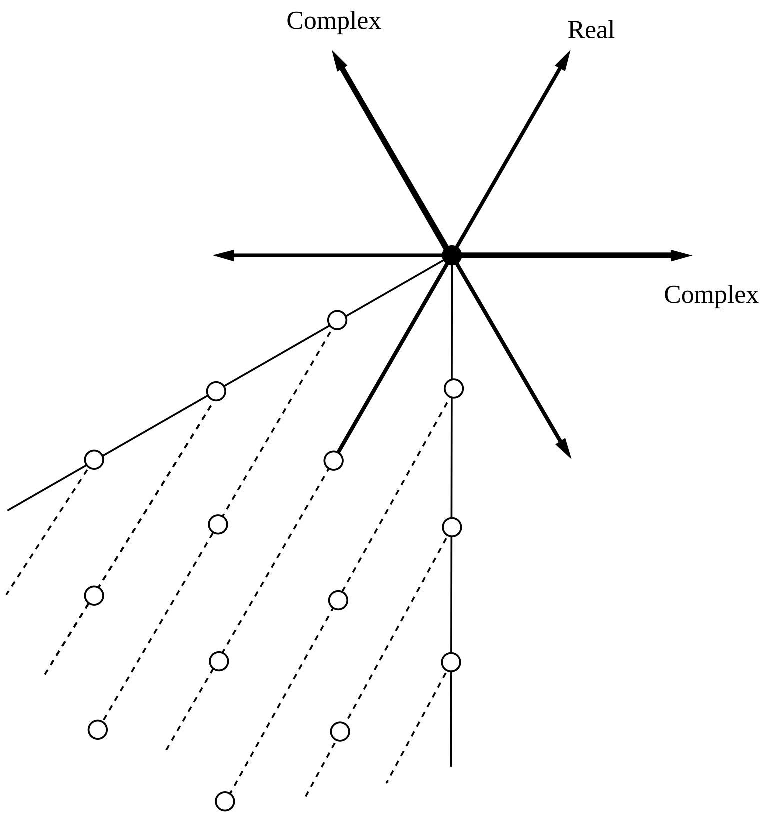
In the following we analyze in detail the case of SU(2, 1). In this case the K-orbit structure and the structure of standard Harish-Chandra sheaves are rather simple. Still, all situations from 7.1.(ii) appear there.

8.3. Discrete series of SU(2, 1). If G_0 is SU(2, 1), we see that the discrete series are attached to all regular weights λ in the negative chamber. Therefore, we have the following picture:



The black dots correspond to weights λ to which a discrete series representation is attached for a particular orbit. If the orbit in question is C_0, these are the "non-holomorphic" discrete series and the white dots in the walls correspond to the limits of discrete series. If the orbit is either C_+ or C_-, these are either "holomorphic" or "anti-holomorphic" discrete series. Since one of the simple roots is compact imaginary in these cases, the standard Harish-Chandra sheaves corresponding to the white dots in the wall orthogonal to this root have no global sections. The white dots in the other wall are again the limits of discrete series.

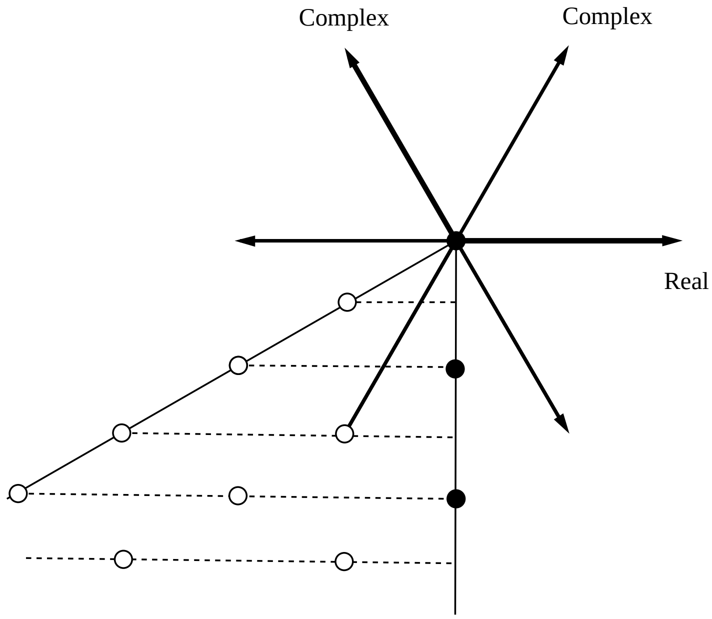
8.4. Tempered representations of SU(2, 1). Except the discrete series and the limits of discrete series we already discussed, the other irreducible Harish-Chandra modules are attached to the open orbit O and the two-dimensional orbits Q_+ and Q_-. The picture for the open orbit is:



As we discussed in 6.10, the standard Harish-Chandra sheaves $\mathcal{I}(O, \lambda)$ on the open orbit O exist (in the negative chamber) only for $\operatorname{Re}\lambda$ on the dotted lines. As we remarked, $\mathcal{I}(O, \lambda)$ are reducible if and only if λ is a weight (i.e. one of the dots in the picture). At these points, $\mathcal{I}(O, \lambda)$ have the invertible $\mathcal{O}_X$-modules $\mathcal{O}(\lambda + \rho)$ as the unique irreducible submodules, i.e., $\mathcal{L}(O, \lambda) = \mathcal{O}(\lambda + \rho)$. The length of these standard Harish-Chandra sheaves is equal to 4. Their composition series consist of the irreducible Harish-Chandra sheaves attached to K-orbits O, Q_+, Q_- and C_0. The standard Harish-Chandra sheaf corresponding to C_0 is the unique irreducible quotient of $\mathcal{I}(O, \lambda)$ and $\mathcal{I}(O, \lambda)/\mathcal{O}(\lambda + \rho)$ contains the direct sum of $\mathcal{L}(Q_+, \lambda)$ and $\mathcal{L}(Q_-, \lambda)$ as a submodule.

The only tempered modules can be obtained for $\operatorname{Re}\lambda_O = 0$, which in this situation corresponds to $\operatorname{Re}\lambda = 0$. Since $\lambda = 0$ corresponds to the invertible $\mathcal{O}_X$-module $\mathcal{O}(\rho)$ with no cohomology, we see that the only irreducible tempered Harish-Chandra modules in this case correspond to $\operatorname{Re}\lambda = 0$, $\lambda \neq 0$. These representations are irreducible unitary spherical principal series.

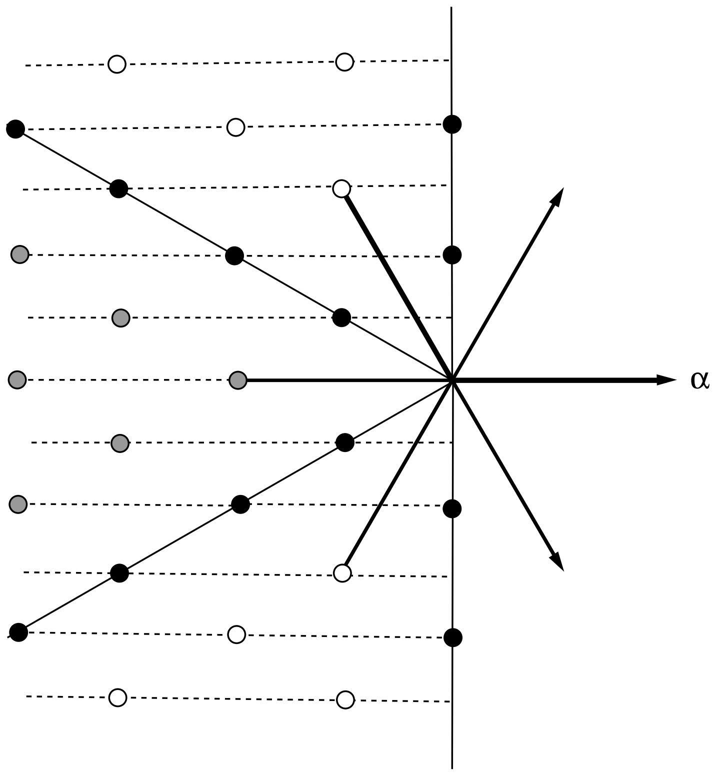
It remains to study the case of tempered Harish-Chandra modules attached to the orbits Q_+ and Q_-. The picture in these cases is:



Assume that we are looking at the picture for Q_+. Again, the standard Harish-Chandra sheaves on this orbit (in the negative chamber) exist only for $\operatorname{Re}\lambda$ on the dotted lines. The standard Harish-Chandra sheaves $\mathcal{I}(Q_+,\lambda)$ are reducible if and only if λ is a weight (i.e., one of the dots in the picture). In this case $\mathcal{I}(Q_+,\lambda)$ has length 3, and the quotient $\mathcal{I}(Q_+,\lambda)/\mathcal{L}(Q_+,\lambda)$ is the direct sum of the standard modules on C_0 and C_+. The temperedness condition is satisfied for $\operatorname{Re}\lambda$ in the wall corresponding to the real root. The corresponding standard Harish-Chandra sheaves are irreducible, except in the case of λ being one of the black dots. Their global sections are various irreducible unitary principal series. The standard modules at the black dots correspond to the reducible unitary principal series. Since in these cases the global sections of $\mathcal{L}(Q_+,\lambda)$ vanish, the global sections of $\mathcal{I}(Q_+,\lambda)$, for $\lambda \neq 0$, are direct sums of "non-holomorphic" and "holomorphic" limits of discrete series representations. If $\lambda = 0$, the "holomorphic" limit of discrete series also "disappears," hence the spherical unitary principal series representation is irreducible and equal to the "non-holomorphic" limit of discrete series.

8.5. Langlands representations of $\mathrm{SU}(2,1)$. As we already remarked, Langlands representations are attached only to non-closed K-orbits. They are either irreducible non-unitary principal series representations, or unique irreducible submodules of reducible principal series. In the latter case, for the open K-orbit, the Langlands representations are irreducible finite-dimensional representations by the Borel-Weil theorem.

Putting all of this information together we can describe the structure of principal series representations for $\mathrm{SU}(2,1)$. Let P be the minimal parabolic subgroup of $\mathrm{SU}(2,1)$ and $P = MAN$ its Langlands decomposition. Then the group MA is a connected maximally split Cartan subgroup in G_0. Therefore the principal series representations are parametrized by pairs (δ,μ) where δ is a representation of the circle group M and μ is a linear form on the complexified Lie algebra $\mathfrak{a}$ of A. The Lie algebra $\mathfrak{a}$ is spanned by the dual root $\alpha\check{}$ of a real root α. Because of the duality between principal series, it is enough to describe their structure for $\operatorname{Re}\alpha\check{}(\mu) \leq 0$. These parameters correspond to the dotted lines in our next figure.



This picture is a "union" of the pictures for the orbits O, Q_+ and Q_-. A detailed explanation of this phenomenon in general can be found in [**18**]. The principal series are generically irreducible. The length of their composition series is two at the black dots, three at the white dots and four at the gray dots. The representations on the intersection of the dotted lines with the vertical wall are unitary principal series. They are either irreducible or sums of two limits of the discrete series, one "holomorphic" and one "non-holomorphic". The spherical unitary principal series at the origin is actually equal to a limit of "non-holomorphic" discrete series. The white dots correspond to the representations which contain infinite dimensional Langlands representations as unique irreducible submodules and direct sums of two discrete series, one "holomorphic" and one "non-holomorphic", as quotients. At two non-vertical walls the composition series consists of an infinite dimensional Langlands representation as a submodule and a limit of "non-holomorphic" discrete series as a quotient (since the limit of "holomorphic" discrete series "vanishes" in these walls). The gray dots correspond to representations which contain finite-dimensional representations as unique irreducible submodules, the "non-holomorphic" discrete series as unique irreducible quotients, and the direct sums of the infinite dimensional Langlands representations attached to Q_+ and Q_- in the "middle" of the composition series.

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Department of Mathematics, University of Utah, Salt Lake City, Utah 84112
E-mail address: milicic@math.utah.edu