EQUIVARIANT DERIVED CATEGORIES, ZUCKERMAN FUNCTORS AND LOCALIZATION

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Introduction

In this paper we revisit some now classical constructions of modern representation theory: Zuckerman's cohomological construction and the localization theory of Bernstein and Beilinson. These constructions made an enormous impact on our understanding of representation theory during the last decades (see, for example, [19]). Our present approach and interest is slightly different than usual. We approach these constructions from the point of view of a student in homological algebra and not representation theory. Therefore, we drop certain assumptions natural from the point of view of representation theorists and stress some unifying principles.

Although both constructions have common heuristic origins in various attempts to generalize the classical Borel-Weil-Bott realization of irreducible finite-dimensional representations of compact Lie groups, they are remarkably different in technical details. Still, the duality theorem of Hecht, Miličić, Schmid and Wolf [11] indicated that there must exist a strong common thread between these constructions. This paper is an attempt to explain the unifying homological principles behind these constructions, which lead to the duality theorem as a formally trivial consequence.

In the first section we present an elementary and self-contained introduction to a generalization of the Zuckerman construction. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and K a complex algebraic group that is a finite covering of a closed algebraic subgroup of the complex algebraic group $\operatorname{Int}(\mathfrak{g})$. Let $\mathcal{M}(\mathfrak{g}, K)$ be the category of Harish-Chandra modules for the pair $(\mathfrak{g}, K)$. Let H be a closed algebraic subgroup of K. Zuckerman observed that the forgetful functor $\mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}, H)$ has a right adjoint $\Gamma_{K,H} : \mathcal{M}(\mathfrak{g}, H) \to \mathcal{M}(\mathfrak{g}, K)$. The functor $\Gamma_{K,H}$ is left exact, and its right-derived modules are the core of Zuckerman's approach.

As we mentioned before, Zuckerman's inspiration was in the Borel-Weil-Bott theorem and he wanted to construct a formal analogue of the sheaf cohomology functor. Therefore in his approach it was natural to assume that both groups K and H are reductive. In our exposition we drop this assumption. The main result of this section is a formula for derived Zuckerman functors which is a generalization of a result of Duflo and Vergne [9]. This formula

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allows us to realize the derived category of $\mathcal{M}(\mathfrak{g},K)$ as a full triangulated subcategory of the derived category of $\mathcal{M}(\mathfrak{g},L)$ for a Levi factor L of K.

To relate Zuckerman's construction to the localization theory, we have to reinterpret the construction in terms of the equivariant derived categories. This is done in the second and third sections, where we review the construction of the equivariant derived category $D(\mathfrak{g},K)$ of Harish-Chandra modules due to Beilinson and Ginzburg, construct the equivariant analogues of Zuckerman functors and relate them to the classical Zuckerman construction [17].

In the fourth section we discuss the localization of the previous constructions. The idea of Beilinson and Bernstein was to generalize the Borel-Weil-Bott theorem in the algebro-geometric setting to an equivalence of categories of modules over the enveloping algebra of $\mathfrak{g}$ with categories of sheaves of $\mathcal{D}$-modules on the flag variety X of $\mathfrak{g}$. This approach ties representation theory with the theory of $\mathcal{D}$-modules. To relate this construction with the constructions of the preceding sections, we define the equivariant derived category of Harish-Chandra sheaves on the flag variety X of $\mathfrak{g}$ and discuss the corresponding version of the localization theory. This allows us, on purely formal grounds, to construct a geometric version of the equivariant Zuckerman functor.

The final section contains a sketch of the proof of the duality theorem of [11]. In our approach, this is just a formula for the cohomology of standard Harish-Chandra sheaves on X in terms of derived Zuckerman functors. While the original proof required a tedious and not very illuminating calculation, the argument sketched here is just slightly more than a diagram chase. It is inspired by Bernstein's argument to prove a special case of the duality theorem.

In this paper we freely use the formalism of derived categories. We think that this should be a necessary part of the toolbox of any representation theorist. An interested reader lacking this background should find [10] and [18] invaluable references.

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1. Zuckerman functors

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and K an algebraic group acting on $\mathfrak{g}$ by a morphism $\phi : K \longrightarrow \mathrm{Int}(\mathfrak{g})$ such that its differential $\mathfrak{k} \longrightarrow \mathfrak{g}$ is an injection. In this situation we can identify $\mathfrak{k}$ with a Lie subalgebra of $\mathfrak{g}$. A Harish-Chandra module (V,π) for the pair $(\mathfrak{g},K)$ is

(HC1) a $\mathcal{U}(\mathfrak{g})$-module;

(HC2) an algebraic K-module, i.e., V is a union of finite-dimensional K-invariant subspaces V_i on which K acts algebraically, that is, via algebraic group morphisms $K \longrightarrow GL(V_i)$;

(HC3) the actions of $\mathfrak{g}$ and K are compatible; i.e.,

  (a) the differential of the K-action agrees with the action of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$;

(b)
$$\pi(k)\pi(\xi)\pi(k^{-1})v = \pi(\phi(k)\xi)v$$
for all $k \in K$, $\xi \in \mathfrak{g}$ and $v \in V$.

A morphism of Harish-Chandra modules is a linear map that intertwines the actions of $\mathfrak{g}$ and K. If V and W are two Harish-Chandra modules for $(\mathfrak{g}, K)$, $\mathrm{Hom}_{(\mathfrak{g},K)}(V, W)$ denotes the space of all morphisms between V and W. Let $\mathcal{M}(\mathfrak{g}, K)$ be the category of Harish-Chandra modules for the pair $(\mathfrak{g}, K)$. This is clearly an abelian $\mathbb{C}$-category.

Let T be a closed algebraic subgroup of K. Then we have a natural forgetful functor $\mathcal{M}(\mathfrak{g}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, T)$. The Zuckerman functor $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is by definition the right adjoint functor to this forgetful functor.

First we describe a construction of this functor. Let $R(K)$ be the ring of regular functions on K. Then for any vector space V, we can view $R(K)\otimes V$ as the vector space of all regular maps from K into V and denote it by $R(K, V)$. We define an algebraic representation ρ of K on $R(K, V)$ as the tensor product of the right regular representation of K on $R(K)$ and trivial action on V.

Now let V be an algebraic K-module. Then we have the natural *matrix coefficient* map $c : V \longrightarrow R(K, V)$ which maps a vector $v \in V$ into the function $k \longmapsto \pi(k)v$. Clearly, c is an injective morphism of K-modules.

If we define the representation λ of K on $R(K, V)$ as the tensor product of the left regular representation of K on $R(K)$ with the natural action on V, it commutes with the action ρ. The image of c is in the space of all λ-invariant functions in $R(K, V)$. Moreover, c is an isomorphism of V onto the space of λ-invariants, and the inverse morphism is the evaluation at $1 \in K$.

If V is a Harish-Chandra module in $\mathcal{M}(\mathfrak{g}, K)$, we define a representation ν of $\mathfrak{g}$ on $R(K, V)$ by
$$(\nu(\xi)F)(k) = \pi(\phi(k)\xi)F(k), \quad k \in K,$$
for $\xi \in \mathfrak{g}$ and $v \in V$. By a direct calculation, we see that $c : V \longrightarrow R(K, V)$ intertwines $\mathfrak{g}$-actions. The representation ν also commutes with the λ-action.

Therefore, the Harish-Chandra module V can be reconstructed from the image of the matrix coefficient map. We use this observation to construct the Zuckerman functor.

Let W be a Harish-Chandra module in $\mathcal{M}(\mathfrak{g}, T)$. Then we can define the structure of a $\mathcal{U}(\mathfrak{g})$-module on $R(K, W)$ by the ν-action, and the structure of an algebraic K-module by the ρ-action as above. The action ν is K-equivariant, i.e.,
$$\rho(k)\nu(\xi)\rho(k^{-1}) = \nu(\phi(k)\xi)$$
for $\xi \in \mathcal{U}(\mathfrak{g})$ and $k \in K$. Let λ be the tensor product of the left regular representation of $\mathfrak{k}$ and T on $R(K)$ with the natural action on W. This defines a structure of Harish-Chandra module for $(\mathfrak{k}, T)$ on $R(K, W)$. One can check that these actions of $\mathfrak{k}$ and T commute with the representations ν and ρ.

Therefore, the subspace of $(\mathfrak{k}, T)$-invariants
$$\Gamma_{K,T}(W) = R(K, W)^{(\mathfrak{k},T)}$$

in $R(K,W)$ (with respect to λ) is a $\mathfrak{g}$- and K-submodule.

1.1. Lemma. *Let W be a Harish-Chandra module for $(\mathfrak{g},T)$. Then $\Gamma_{K,T}(W)$ is a Harish-Chandra module for $(\mathfrak{g},K)$.*

Proof. We already mentioned that ν is K-equivariant. Also, for $\xi \in \mathfrak{k}$ and $F \in \Gamma_{K,T}(W)$ we have

$$\begin{aligned}(\rho(\xi)F)(k) = \tfrac{d}{dt}F(k\exp(t\xi))\big|_{t=0} &= \tfrac{d}{dt}F(\exp(t(\phi(k)\xi))k)\big|_{t=0} \\ &= \pi(\phi(k)\xi)F(k) = (\nu(\xi)F)(k), \quad k \in K,\end{aligned}$$

since F is λ-invariant. Therefore, the differential of ρ agrees with the restriction of ν to $\mathfrak{k}$ on $\Gamma_{K,T}(W)$, i.e., the actions ν and ρ define a structure of Harish-Chandra module on $\Gamma_{K,T}(W)$. □

Let V and W be two Harish-Chandra modules for $(\mathfrak{g},T)$ and $\alpha \in \operatorname{Hom}_{(\mathfrak{g},T)}(V,W)$. Then α induces a linear map $1\otimes\alpha : R(K,V) \longrightarrow R(K,W)$. Clearly, $1\otimes\alpha$ intertwines the actions ν, ρ and λ on these modules. Hence, it induces a morphism $\Gamma_{K,T}(\alpha) : \Gamma_{K,T}(V) \longrightarrow \Gamma_{K,T}(W)$. It follows that $\Gamma_{K,T}$ is an additive functor from $\mathcal{M}(\mathfrak{g},T)$ into $\mathcal{M}(\mathfrak{g},K)$.

Let V be a Harish-Chandra module in $\mathcal{M}(\mathfrak{g},K)$. Then, as we saw above, the matrix coefficient map c_V of V is a $(\mathfrak{g},K)$-morphism of V into $\Gamma_{K,T}(V)$. It is easy to check that the maps c_V actually define a natural transformation of the identity functor on $\mathcal{M}(\mathfrak{g},K)$ into the composition of $\Gamma_{K,T}$ with the forgetful functor.

On the other hand, let W be a Harish-Chandra module for $(\mathfrak{g},T)$ and $e_W : \Gamma_{K,T}(W) \longrightarrow W$ the linear map given by $e_W(F) = F(1)$. Then e_W is a $(\mathfrak{g},T)$-morphism from $\Gamma_{K,T}(W)$ into W. Clearly, the maps e_W define a natural transformation of the composition of the forgetful functor with the functor $\Gamma_{K,T}$ into the identity functor on $\mathcal{M}(\mathfrak{g},T)$.

Using these natural transformations, we get the following result.

1.2. Proposition. *The functor $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g},T) \to \mathcal{M}(\mathfrak{g},K)$ is right adjoint to the forgetful functor from $\mathcal{M}(\mathfrak{g},K)$ into $\mathcal{M}(\mathfrak{g},T)$.*

Proof. Let V be a Harish-Chandra module in $\mathcal{M}(\mathfrak{g},K)$ and W a Harish-Chandra module in $\mathcal{M}(\mathfrak{g},T)$. For $\alpha \in \operatorname{Hom}_{(\mathfrak{g},T)}(V,W)$, the composition $\bar\alpha = \Gamma_{K,T}(\alpha)\circ c_V : V \longrightarrow \Gamma_{K,T}(W)$ is in $\operatorname{Hom}_{(\mathfrak{g},K)}(V,\Gamma_{K,T}(W))$. Thus we have a linear map $\alpha \longmapsto \bar\alpha$ of $\operatorname{Hom}_{(\mathfrak{g},T)}(V,W)$ into $\operatorname{Hom}_{(\mathfrak{g},K)}(V,\Gamma_{K,T}(W))$.

Also, if $\beta \in \operatorname{Hom}_{(\mathfrak{g},K)}(V,\Gamma_{K,T}(W))$, $\tilde\beta = e_W \circ \beta \in \operatorname{Hom}_{(\mathfrak{g},T)}(V,W)$. Thus we have a linear map $\beta \longmapsto \tilde\beta$ of $\operatorname{Hom}_{(\mathfrak{g},K)}(V,\Gamma_{K,T}(W))$ into $\operatorname{Hom}_{(\mathfrak{g},T)}(V,W)$.

By a direct calculation, we see that these maps are inverse to each other. □

The functor $\Gamma_{K,T}$ is called the *Zuckerman functor*.

Let $\Gamma_K = \Gamma_{K,\{1\}}$. Since Γ_K is right adjoint to the forgetful functor from $\mathcal{M}(\mathfrak{g},K)$ into $\mathcal{M}(\mathfrak{g})$, it maps injectives into injectives. This has the following consequence.

1.3. Lemma. *The category $\mathcal{M}(\mathfrak{g}, K)$ has enough injectives.*

Proof. Let V be an object in $\mathcal{M}(\mathfrak{g}, K)$. Then there exists an injective object I in $\mathcal{M}(\mathfrak{g})$ and a $\mathfrak{g}$-monomorphism $i : V \to I$. Since Γ_K is left exact, $\Gamma_K(i) : \Gamma_K(V) \to \Gamma_K(I)$ is also a monomorphism. By the arguments in the proof of 1.2, the adjunction morphism $V \to \Gamma_K(V)$ is also a monomorphism. Therefore, the composition of these two morphisms defines a monomorphism $V \to \Gamma_K(I)$ of V into an injective object in $\mathcal{M}(\mathfrak{g}, K)$. □

Let U be an algebraic representation of K. Then $P(U) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} U$, with the K-action given by the tensor product of the action ϕ on $\mathcal{U}(\mathfrak{g})$ and the natural action on U, is an algebraic representation of K. Moreover, it also has a natural structure of a $\mathcal{U}(\mathfrak{g})$-module, given by left multiplication in the first factor. It is straightforward to check that $P(U)$ is a Harish-Chandra module. Therefore, P is an exact functor from the category of algebraic representations of K into $\mathcal{M}(\mathfrak{g}, K)$. In addition,

$$\mathrm{Hom}_{(\mathfrak{g},K)}(P(U), V) = \mathrm{Hom}_K(U, V)$$

for any Harish-Chandra module V, i.e., P is left adjoint to the forgetful functor from $\mathcal{M}(\mathfrak{g}, K)$ into the category of algebraic representations of K.

Assume now that K is reductive. Then the category of algebraic representations of K is semisimple, and every object in it is projective. Therefore, Harish-Chandra modules $P(U)$ are projective in $\mathcal{M}(\mathfrak{g}, K)$ for arbitrary algebraic representation U of K.

This leads to the following result.

1.4. Lemma. *If K is a reductive algebraic group, the category $\mathcal{M}(\mathfrak{g}, K)$ has enough projectives.*

In addition, every finitely generated object in $\mathcal{M}(\mathfrak{g}, K)$ is a quotient of a finitely generated projective object.

Proof. This is analogous to the proof of 1.3, using the fact that for any Harish-Chandra module V, the adjointness morphism $P(V) \to V$ is surjective. The last remark is obvious from the previous discussion. □

Let $D^+(\mathcal{M}(\mathfrak{g}, K))$ and $D^+(\mathcal{M}(\mathfrak{g}, T))$ be the derived categories of complexes bounded from below corresponding to $\mathcal{M}(\mathfrak{g}, K)$ and $\mathcal{M}(\mathfrak{g}, T)$. Since the category $\mathcal{M}(\mathfrak{g}, T)$ has enough injectives, there exists the derived functor

$$R\Gamma_{K,T} : D^+(\mathcal{M}(\mathfrak{g}, T)) \to D^+(\mathcal{M}(\mathfrak{g}, K))$$

of $\Gamma_{K,T}$. Also, $R\Gamma_{K,T}$ is the right adjoint of the natural "forgetful" functor from $D^+(\mathcal{M}(\mathfrak{g}, K))$ into $D^+(\mathcal{M}(\mathfrak{g}, T))$.

This immediately implies the following remark. Let H be a closed algebraic subgroup of K such that $T \subset H \subset K$. Then the functors $R\Gamma_{K,H} \circ R\Gamma_{H,T}$ and $R\Gamma_{K,T}$ are right adjoint to the natural functor $D^+(\mathcal{M}(\mathfrak{g}, K)) \to D^+(\mathcal{M}(\mathfrak{g}, T))$, hence they are isomorphic, i.e., we have the following theorem.

1.5. Theorem. $$R\Gamma_{K,T} = R\Gamma_{K,H} \circ R\Gamma_{H,T}.$$

The next theorem is our version of the main result of [9]. In the following we assume that T is in addition reductive. For any Harish-Chandra module W in $\mathcal{M}(\mathfrak{k}, T)$ we denote by

$$H^p(\mathfrak{k}, T; W) = \operatorname{Ext}^p_{(\mathfrak{k},T)}(\mathbb{C}, W)$$

the p^{th} relative Lie algebra cohomology group of W.

1.6. Theorem. *Assume that T is reductive. Let V be a Harish-Chandra module in $\mathcal{M}(\mathfrak{g}, T)$. Then*

$$R^p\Gamma_{K,T}(V) = H^p(\mathfrak{k}, T; R(K, V))$$

for $p \in \mathbb{Z}_+$, where the relative Lie algebra cohomology is calculated with respect to the λ-action.

To prove this result we need some preparation.

As we mentioned before, the category $\mathcal{M}(\mathfrak{g}, T)$ has enough injectives. Let V be a Harish-Chandra module for $(\mathfrak{g}, T)$ and $V \to I^\cdot$ a right resolution of V by injective modules in $\mathcal{M}(\mathfrak{g}, T)$. Then

$$R^p\Gamma_{K,T}(V) = H^p(\operatorname{Hom}_{(\mathfrak{k},T)}(\mathbb{C}, R(K, I^\cdot))).$$

Here $R(K, I^\cdot)$ are viewed as $(\mathfrak{k}, T)$-modules with respect to the actions λ of $\mathfrak{k}$ and T. To prove the theorem it is enough to prove that for any injective object I in $\mathcal{M}(\mathfrak{g}, T)$, the module $R(K, I)$ viewed as a Harish-Chandra module for $(\mathfrak{k}, T)$ with respect to the action λ, is acyclic for the functor $\operatorname{Hom}_{(\mathfrak{k},T)}(\mathbb{C}, -)$. This is proved in the next lemma.

1.7. Lemma. *Let I be an injective object in $\mathcal{M}(\mathfrak{g}, T)$. Then $R(K, I)$, viewed as a Harish-Chandra module for $(\mathfrak{k}, T)$ with respect to the action λ, satisfies*

$$\operatorname{Ext}^p_{(\mathfrak{k},T)}(\mathbb{C}, R(K, I)) = 0$$

for $p > 0$.

Proof. First, let S be a module in $\mathcal{M}(\mathfrak{k}, T)$. Then, we can define the action of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} S$ as left multiplication in the first variable and the action of T as the tensor product of the adjoint action on the first factor with the natural action on the second factor. It is easy to check that for any Harish-Chandra module W in $\mathcal{M}(\mathfrak{g}, T)$ we have

$$\operatorname{Hom}_{(\mathfrak{g},T)}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} S, W) = \operatorname{Hom}_{(\mathfrak{k},T)}(S, W).$$

Therefore, the forgetful functor from $\mathcal{M}(\mathfrak{g}, T)$ into $\mathcal{M}(\mathfrak{k}, T)$ is the right adjoint of the exact functor $S \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} S$. Hence it preserves injectives; in particular, I is an injective object in $\mathcal{M}(\mathfrak{k}, T)$.

Second, if F is a finite-dimensional algebraic representation of K and F^* its contragredient, we have

$$\operatorname{Hom}_{(\mathfrak{k},T)}(R, F\otimes W) = \operatorname{Hom}_{(\mathfrak{k},T)}(R\otimes F^*, W)$$

for any two Harish-Chandra modules R and W in $\mathcal{M}(\mathfrak{k},T)$. Therefore, $W \longmapsto F\otimes W$ is the right adjoint of the exact functor $R\longmapsto R\otimes F^*$. It follows that $F\otimes I$ is an injective object in $\mathcal{M}(\mathfrak{k},T)$.

Let U be an algebraic representation of K considered as a Harish-Chandra module for $(\mathfrak{k},T)$. Then on $U\otimes I$ we can define the tensor product structure of Harish-Chandra module for $(\mathfrak{k},T)$. Since U is an algebraic representation, U is a union of finite-dimensional K-invariant subspaces U_j, $j\in J$; i.e., $U = \varinjlim_{j\in J} U_j$. Therefore, for any R in $\mathcal{M}(\mathfrak{k},T)$, finitely generated over $\mathcal{U}(\mathfrak{k})$,

$$\operatorname{Hom}_{(\mathfrak{k},T)}(R, U\otimes I) = \varinjlim_{j\in J} \operatorname{Hom}_{(\mathfrak{k},T)}(R, U_j\otimes I).$$

By 1.4, since T is reductive, $\mathbb{C}$ has a left resolution $P^{\cdot}$ by projective finitely generated Harish-Chandra modules in $\mathcal{M}(\mathfrak{k},T)$. Therefore, since the direct limit functor is exact,

$$\begin{aligned}\operatorname{Ext}^p_{(\mathfrak{k},T)}(\mathbb{C}, U\otimes I) &= H^p(\operatorname{Hom}_{(\mathfrak{k},T)}(P^{\cdot}, U\otimes I))\\ &= H^p(\varinjlim_{j\in J}\operatorname{Hom}_{(\mathfrak{k},T)}(P^{\cdot}, U_j\otimes I)) = \varinjlim_{j\in J} H^p(\operatorname{Hom}_{(\mathfrak{k},T)}(P^{\cdot}, U_j\otimes I))\\ &= \varinjlim_{j\in J}\operatorname{Ext}^p_{(\mathfrak{k},T)}(\mathbb{C}, U_j\otimes I).\end{aligned}$$

But the last expression is zero, since $U_j\otimes I$ is an injective $(\mathfrak{k},T)$-module by the above discussion. In particular,

$$\operatorname{Ext}^p_{(\mathfrak{k},T)}(\mathbb{C}, U\otimes I) = 0$$

for $p>0$. Applying this for $U = R(K)$, we get 1.7, and thus also 1.6. □

Now we want to study one of the adjointness morphisms attached to the adjoint pair consisting of the forgetful functor and the derived Zuckerman functor $R\Gamma_{K,T}$.

Let V be a Harish-Chandra module in $\mathcal{M}(\mathfrak{g},K)$, with action π. We can view it as an object in $\mathcal{M}(\mathfrak{g},T)$. We want to calculate the derived Zuckerman functors $R^p\Gamma_{K,T}(V)$, $p\in\mathbb{Z}_+$. To do this, we have to calculate the relative Lie algebra cohomology modules from 1.6. The calculation is based on the following observations. First, the matrix coefficient map $V\longrightarrow R(K,V)$ defines a linear map γ of $R(K,V)$ into itself, given by

$$\gamma(F)(k) = c(F(k))(k) = \pi(k)F(k),\quad k\in K,$$

for $F\in R(K,V)$. This map is clearly an isomorphism of linear spaces and its inverse is given by

$$\delta(F)(k) = \pi(k^{-1})F(k),\quad k\in K,$$

for $F \in R(K, V)$. We can define the following actions on $R(K, V)$: the representation $1 \otimes \pi$ of $\mathfrak{g}$ which is the tensor product of the trivial representation on $R(K)$ and the natural representation on V, the representation τ of K which is the tensor product of the right regular representation of K on $R(K)$ with the natural representation on V, and the representation μ of K which is the tensor product of the left regular representation of K on $R(K)$ with the trivial representation on V.

By a direct calculation we check that the following result holds.

1.8. Lemma. *For any V in $\mathcal{M}(\mathfrak{g}, K)$, the linear map $\gamma : R(K, V) \longrightarrow R(K, V)$ is a linear space automorphism. Also,*

(i) *γ intertwines the representation $1 \otimes \pi$ of $\mathfrak{g}$ with ν;*
(ii) *γ intertwines the representation τ of K with ρ;*
(iii) *γ intertwines the representation μ of K with λ.*

Therefore, to calculate $H^p(\mathfrak{k}, T; R(K, V))$, we can assume that the actions of $\mathfrak{k}$ and T are given by μ. In this case, we have

$$R^p\Gamma_{K,T}(V) = H^p(\mathfrak{k}, T; R(K, V)) = H^p(\mathfrak{k}, T; R(K)) \otimes V.$$

Here the relative Lie algebra cohomology of $R(K)$ is calculated with respect to the left regular action. The action of $\mathfrak{g}$ on the last module is given as the tensor product of the trivial action on the first factor and the natural action on V, while the action of K is given as the tensor product of the action induced by the right regular representation on $R(K)$ with the natural action on V.

Let $K^\dagger$ be the subgroup of K generated by the identity component K_0 of K and T.

Assume first that K is reductive. Then the left regular representation on $R(K)$ is a direct sum of irreducible finite-dimensional representations of $K^\dagger$. It is well known that, for any nontrivial finite-dimensional irreducible representation F of $K^\dagger$, the relative Lie algebra cohomology modules $H^p(\mathfrak{k}, T; F)$ are zero for any $p \in \mathbb{Z}_+$. Let $\mathrm{Ind}_{K^\dagger}^K(1)$ be the space of functions on K that are constant on right $K^\dagger$-cosets. It follows that the inclusion of $\mathrm{Ind}_{K^\dagger}^K(1)$ into $R(K)$ induces isomorphisms

$$H^p(\mathfrak{k}, T; \mathrm{Ind}_{K^\dagger}^K(1)) = H^p(\mathfrak{k}, T; R(K)), \quad p \in \mathbb{Z}_+.$$

Therefore, in this case we have

$$R^p\Gamma_{K,T}(V) = H^p(\mathfrak{k}, T; \mathrm{Ind}_{K^\dagger}^K(1)) \otimes V = H^p(\mathfrak{k}, T; \mathbb{C}) \otimes \mathrm{Ind}_{K^\dagger}^K(1) \otimes V, \quad p \in \mathbb{Z}_+,$$

and the action of K is the tensor product of the trivial action on the first factor, the right regular action on the second factor and the natural action on the third factor. The action of $\mathfrak{g}$ is given by the tensor product of trivial actions on the first two factors with the natural action on the third factor.

Now we drop the assumption that K is reductive. Denote by U the unipotent radical in K. Since T is reductive, $T \cap U = \{1\}$. Moreover, there exists a Levi factor L of K, such that $T \subset L$. Denote by $L^\dagger$ the subgroup of L generated by the identity component L_0 of L and T. By 1.5, we have the spectral sequence

$$R^p\Gamma_{K,L}(R^q\Gamma_{L,T}(V)) \Rightarrow R^{p+q}\Gamma_{K,T}(V).$$

Since L is reductive, by the above discussion we have

$$R^q\Gamma_{L,T}(V) = H^q(\mathfrak{l}, T; \mathbb{C}) \otimes \operatorname{Ind}_{L^\dagger}^{L}(1) \otimes V, \quad q \in \mathbb{Z}_+.$$

Since the restriction to L induces an isomorphism of $\operatorname{Ind}_{K^\dagger}^{K}(1)$ with $\operatorname{Ind}_{L^\dagger}^{L}(1)$, we also have

$$R^q\Gamma_{L,T}(V) = H^q(\mathfrak{l}, T; \mathbb{C}) \otimes \operatorname{Ind}_{K^\dagger}^{K}(1) \otimes V, \quad q \in \mathbb{Z}_+.$$

Hence, it remains to study $R^p\Gamma_{K,L}(V)$, $p \in \mathbb{Z}_+$, for V in $\mathcal{M}(\mathfrak{g}, K)$. Let $\mathfrak{u}$ and $\mathfrak{l}$ be the Lie algebras of U and L respectively. Let $N^\cdot(\mathfrak{u})$ be the standard complex of $\mathfrak{u}$. Then $N^p(\mathfrak{u}) = \mathcal{U}(\mathfrak{u}) \otimes_{\mathbb{C}} \bigwedge^{-p} \mathfrak{u}$, $p \in \mathbb{Z}$, are algebraic K-modules for the tensor products of the adjoint actions on $\mathcal{U}(\mathfrak{u})$ and $\bigwedge^\cdot \mathfrak{u}$. Therefore, if we write $N^p(\mathfrak{u})$ as $\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{l})} \bigwedge^{-p} \mathfrak{u}$, $p \in \mathbb{Z}$, we see that they are algebraic L-modules and also $\mathcal{U}(\mathfrak{k})$-modules for the left multiplication in the first factor. In this way we see that

$$\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{l})} \textstyle\bigwedge^{-\cdot} \mathfrak{u}$$

is a projective resolution of $\mathbb{C}$ in $\mathcal{M}(\mathfrak{k}, L)$. It is usually called the relative standard complex for the pair $(\mathfrak{k}, \mathfrak{l})$. It follows that for any object W in $\mathcal{M}(\mathfrak{k}, L)$ we have

$$\begin{aligned} H^p(\mathfrak{k}, L; W) &= \operatorname{Ext}^p_{(\mathfrak{k},L)}(\mathbb{C}, W) = H^p(\operatorname{Hom}_{(\mathfrak{k},L)}(\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{l})} \textstyle\bigwedge^\cdot \mathfrak{u}, W)) \\ &= H^p(\operatorname{Hom}_L(\textstyle\bigwedge^\cdot \mathfrak{u}, W)) = H^p(\operatorname{Hom}_{\mathbb{C}}(\textstyle\bigwedge^\cdot \mathfrak{u}, W)^L) \end{aligned}$$

for $p \in \mathbb{Z}_+$. Since L is reductive, we have

$$H^p(\mathfrak{k}, L; W) = H^p(\mathfrak{u}, W)^L, \quad p \in \mathbb{Z}_+.$$

In particular, we have

$$H^p(\mathfrak{k}, L; R(K)) = H^p(\mathfrak{u}, R(K))^L, \quad p \in \mathbb{Z}_+.$$

The quotient map $K \longrightarrow K/U$ induces a natural inclusion of $R(K/U)$ into $R(K)$ as algebraic K-modules for the left regular action.

1.9. Lemma. *We have*

$$H^p(\mathfrak{u}, R(K)) = \begin{cases} R(K/U) & \text{if } p = 0; \\ 0 & \text{if } p > 0. \end{cases}$$

Proof. We prove a slightly more general statement which allows induction. Let N be a normal unipotent subgroup of K and $\mathfrak{n}$ the Lie algebra of N. We claim that

$$H^p(\mathfrak{n}, R(K)) = \begin{cases} R(K/N) \text{ if } p = 0; \\ 0 \text{ if } p > 0. \end{cases}$$

Since $H^0(\mathfrak{n}, R(K))$ consists of functions constant on N-cosets, the statement for $p = 0$ is evident.

Hence, we just have to establish the vanishing for $p > 0$. The map $(u, l) \longmapsto u \cdot l$ from $U \times L$ into K is an isomorphism of varieties. Moreover, if we assume that N acts by left multiplication on the first factor in $U \times L$ and by left multiplication on K, it is an N-equivariant isomorphism. Therefore, the corresponding algebraic representations of N on $R(K)$ and $R(U \times L) = R(U) \otimes R(L)$ are isomorphic. This implies that

$$H^p(\mathfrak{n}, R(K)) = H^p(\mathfrak{n}, R(U) \otimes R(L)) = H^p(\mathfrak{n}, R(U)) \otimes R(L), \quad p \in \mathbb{Z}_+,$$

as linear spaces. Assume first that N is abelian. Since U is an affine space, $R(U)$ is a polynomial algebra. Moreover, $H^p(\mathfrak{n}, R(U))$, $p \in \mathbb{Z}_+$, is just the cohomology of the Koszul complex with coefficients in this algebra. By the polynomial version of the Poincaré lemma we see that $H^p(\mathfrak{n}, R(U)) = 0$ for $p > 0$.

Now we proceed by induction on $\dim N$. If $\dim N > 0$, the commutator subgroup $N' = (N, N)$ is a unipotent group and $\dim N' < \dim N$. Moreover, N' is a normal subgroup of K. Therefore, its Lie algebra $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$ is an ideal in $\mathfrak{k}$. By the Hochschild-Serre spectral sequence of Lie algebra cohomology we have

$$H^p(\mathfrak{n}/\mathfrak{n}', H^q(\mathfrak{n}', R(K))) \Rightarrow H^{p+q}(\mathfrak{n}, R(K)).$$

Also, by the induction assumption this spectral sequence collapses, i.e., we have

$$H^p(\mathfrak{n}/\mathfrak{n}', R(K/N')) = H^p(\mathfrak{n}, R(K))$$

for $p \in \mathbb{Z}_+$. Finally, by the first part of the proof, the left side is 0 if $p > 0$ since N/N' is abelian. □

As an immediate consequence, we have

$$H^p(\mathfrak{k}, L; R(K)) = H^p(\mathfrak{u}, R(K))^L = \begin{cases} R(K/U)^L = R(L)^L = \mathbb{C} \text{ for } p = 0; \\ 0 \text{ for } p > 0. \end{cases}$$

It follows that for V in $\mathcal{M}(\mathfrak{g}, K)$, we have

$$\Gamma_{K,L}(V) = V$$

and

$$R^p\Gamma_{K,L}(V) = 0 \text{ for } p > 0.$$

Therefore, the spectral sequence we considered earlier collapses, and we immediately get the following consequence.

1.10. Proposition. *Let T be a reductive subgroup of K and L a Levi factor of K containing T. Let V be a module in $\mathcal{M}(\mathfrak{g}, K)$. Then we have*

$$R^p\Gamma_{K,T}(V) = H^p(\mathfrak{l}, T; \mathbb{C}) \otimes \mathrm{Ind}_{K^\dagger}^{K}(1) \otimes V, \quad p \in \mathbb{Z}_+.$$

Now we prove a result about derived categories of Harish-Chandra modules which reduces the case of general pairs $(\mathfrak{g}, K)$ to the study of categories with reductive group K.

Assume that K is arbitrary. Let U be the unipotent radical and L a Levi factor of K. The category $\mathcal{M}(\mathfrak{g}, K)$ is a subcategory of $\mathcal{M}(\mathfrak{g}, L)$. Moreover, since U is connected, any $(\mathfrak{g}, L)$-morphism between two objects in $\mathcal{M}(\mathfrak{g}, K)$ is automatically a $(\mathfrak{g}, K)$-morphism. Therefore, $\mathcal{M}(\mathfrak{g}, K)$ is a full subcategory of $\mathcal{M}(\mathfrak{g}, L)$. Moreover, $(\mathfrak{g}, L)$-subobjects and $(\mathfrak{g}, L)$-quotients of any object in $\mathcal{M}(\mathfrak{g}, K)$ are objects in $\mathcal{M}(\mathfrak{g}, K)$.

1.11. Lemma. *The category $\mathcal{M}(\mathfrak{g}, K)$ is a thick subcategory of $\mathcal{M}(\mathfrak{g}, L)$.*

Proof. It remains to show that the subcategory $\mathcal{M}(\mathfrak{g}, K)$ of $\mathcal{M}(\mathfrak{g}, L)$ is closed under extensions. Consider a short exact sequence

$$0 \to V \to V' \to V'' \to 0$$

in $\mathcal{M}(\mathfrak{g}, L)$. Then the adjointness morphism of the zero-th Zuckerman functor of a module into the module implies the commutativity of the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_{K,L}(V) & \longrightarrow & \Gamma_{K,L}(V') & \longrightarrow & \Gamma_{K,L}(V'') & \longrightarrow & R^1\Gamma_{K,L}(V) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & V' & \longrightarrow & V'' & \longrightarrow & 0
\end{array} .$$

Assume that V and V'' are objects in $\mathcal{M}(\mathfrak{g}, K)$. Then, by 1.10, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & \Gamma_{K,L}(V') & \longrightarrow & V'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & V & \longrightarrow & V' & \longrightarrow & V'' & \longrightarrow & 0
\end{array} ,$$

and the middle vertical arrow is also an isomorphism, i.e., $V' \cong \Gamma_{K,L}(V')$. □

Let $D^*(\mathcal{M}(\mathfrak{g}, K))$, where $*$ is either b, $+$, $-$ or nothing, be the derived category of $\mathcal{M}(\mathfrak{g}, K)$ consisting of bounded, bounded from below, bounded from above or arbitrary complexes, respectively.

We can consider the full subcategory $D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathcal{M}(\mathfrak{g}, L))$ of all complexes in $D^*(\mathcal{M}(\mathfrak{g}, L))$ with cohomology in $\mathcal{M}(\mathfrak{g}, K)$. By a standard argument using 1.11 and the long exact sequence of cohomology modules attached to a distinguished triangle, we can conclude that for any distinguished triangle in $D^*(\mathcal{M}(\mathfrak{g}, L))$, if two vertices are in $D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathcal{M}(\mathfrak{g}, L))$, the third one is there too, i.e., $D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathcal{M}(\mathfrak{g}, L))$ is a triangulated subcategory of $D^*(\mathcal{M}(\mathfrak{g}, L))$.

In addition, we have the natural forgetful functor For from $D^*(\mathcal{M}(\mathfrak{g}, K))$ to $D^*(\mathcal{M}(\mathfrak{g}, L))$ and its image is inside $D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathcal{M}(\mathfrak{g}, L))$. Denote by α the induced functor from $D^*(\mathcal{M}(\mathfrak{g}, K))$ into $D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathcal{M}(\mathfrak{g}, L))$.

1.12. Theorem. *The functor* $\alpha : D^*(\mathcal{M}(\mathfrak{g}, K)) \to D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathcal{M}(\mathfrak{g}, L))$ *is an equivalence of categories.*

Proof. Clearly, it is sufficient to prove this statement for the derived categories of unbounded complexes. Since the functor $R\Gamma_{K,L}$ has finite right cohomological dimension by 1.6, the adjointness of the forgetful functor For : $\mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}, L)$ and $\Gamma_{K,L} : \mathcal{M}(\mathfrak{g}, L) \to \mathcal{M}(\mathfrak{g}, K)$ implies that $R\Gamma_{K,L}$ is also the right adjoint of the forgetful functor from $D(\mathcal{M}(\mathfrak{g}, K)) \to D(\mathcal{M}(\mathfrak{g}, L))$. Since, by 1.10, the objects in $\mathcal{M}(\mathfrak{g}, K)$ are $\Gamma_{K,L}$-acyclic, we have

$$R\Gamma_{K,L}(\mathrm{For}(V^\cdot)) = \Gamma_{K,L}(\mathrm{For}(V^\cdot)) = V^\cdot$$

for any complex $V^\cdot$ in $D(\mathcal{M}(\mathfrak{g}, K))$. In addition, we have the adjointness morphism $\mathrm{For}(R\Gamma_{K,L}(U^\cdot)) \to U^\cdot$ for any $U^\cdot$ in $D(\mathcal{M}(\mathfrak{g}, L))$. If $U^\cdot$ is a complex in $D_{\mathcal{M}(\mathfrak{g},K)}(\mathcal{M}(\mathfrak{g}, L))$, its cohomology modules are $\Gamma_{K,L}$-acyclic by 1.10. Therefore, by a standard argument, $H^p(R\Gamma_{K,L}(U^\cdot)) = \Gamma_{K,L}(H^p(U^\cdot)) = H^p(U^\cdot)$, for $p \in \mathbb{Z}$, and the adjointness morphism is a quasiisomorphism. Hence, α is an equivalence of categories. □

It follows that we can view $D^*(\mathcal{M}(\mathfrak{g}, K))$ as a triangulated subcategory in $D^*(\mathcal{M}(\mathfrak{g}, L))$.

Now we can discuss the consequences of 1.12 with respect to Zuckerman functors. Let H be a subgroup of K and T its Levi factor. Then we have the following commutative diagram

$$\begin{array}{ccc} D^+(\mathcal{M}(\mathfrak{g}, T)) & \xrightarrow{R\Gamma_{H,T}} & D^+(\mathcal{M}(\mathfrak{g}, H)) \\ {\scriptstyle R\Gamma_{K,T}}\downarrow & & \downarrow{\scriptstyle R\Gamma_{K,H}} \\ D^+(\mathcal{M}(\mathfrak{g}, K)) & = & D^+(\mathcal{M}(\mathfrak{g}, K)) \end{array} .$$

Finally, by replacing the top left corner with $D^+_{\mathcal{M}(\mathfrak{g},H)}(\mathcal{M}(\mathfrak{g}, T))$ and inverting the top horizontal arrow, we get the commutative diagram

$$\begin{array}{ccc} D^+_{\mathcal{M}(\mathfrak{g},H)}(\mathcal{M}(\mathfrak{g}, T)) & \xleftarrow{\alpha} & D^+(\mathcal{M}(\mathfrak{g}, H)) \\ {\scriptstyle R\Gamma_{K,T}}\downarrow & & \downarrow{\scriptstyle R\Gamma_{K,H}} \\ D^+(\mathcal{M}(\mathfrak{g}, K)) & = & D^+(\mathcal{M}(\mathfrak{g}, K)) \end{array} ,$$

i.e., $R\Gamma_{K,H}$ is the restriction of $R\Gamma_{K,T}$ to $D^+(\mathcal{M}(\mathfrak{g}, H))$. Since the right cohomological dimension of $R\Gamma_{K,T}$ is $\le \dim(K/T)$ by 1.6, the right cohomological dimension of $R\Gamma_{K,H}$ is also $\le \dim(K/T)$. Therefore, both functors extend to the categories of unbounded complexes, and we have the following result.

1.13. Theorem. *The Zuckerman functor $R\Gamma_{K,H}$ is the restriction of $R\Gamma_{K,T}$ to the subcategory $D(\mathcal{M}(\mathfrak{g}, H))$ of $D(\mathcal{M}(\mathfrak{g}, T))$.*

2. Equivariant derived categories

As we already remarked in the introduction, in certain instances the construction of Zuckerman functors is not sufficiently flexible for applications. The problem lies in the construction of the derived category $D(\mathcal{M}(\mathfrak{g}, K))$. In this section we discuss a more appropriate construction due to Beilinson and Ginzburg [3].

The first, and critical, step is a "two-step" definition of Harish-Chandra modules.

A triple (V, π, ν) is called a *weak Harish-Chandra module* for the pair $(\mathfrak{g}, K)$ if:

(W1) V is a $\mathcal{U}(\mathfrak{g})$-module with an action π;
(W2) V is an algebraic K-module with an action ν;
(W3) for any $\xi \in \mathfrak{g}$ and $k \in K$ we have

$$\pi(\phi(k)\xi) = \nu(k)\pi(\xi)\nu(k)^{-1};$$

i.e., the $\mathfrak{g}$-action map $\mathfrak{g} \otimes V \longrightarrow V$ is K-equivariant.

The action ν of K differentiates to an action of $\mathfrak{k}$ which we denote also by ν. We put $\omega(\xi) = \nu(\xi) - \pi(\xi)$ for $\xi \in \mathfrak{k}$. The following simple observation is critical.

2.1. Lemma. *Let V be a weak Harish-Chandra module. Then*

(i) *ω is a representation of $\mathfrak{k}$ on V;*
(ii) *ω is K-equivariant, i.e.,*

$$\omega(\mathrm{Ad}(k)\xi) = \nu(k)\omega(\xi)\nu(k)^{-1}$$

for $\xi \in \mathfrak{k}$ and $k \in K$;
(iii)

$$[\omega(\xi), \pi(\eta)] = 0,$$

for $\eta \in \mathfrak{g}$ and $\xi \in \mathfrak{k}$.

Proof. By (W3), the representation π of $\mathfrak{k}$ is K-equivariant. Since the representation ν of $\mathfrak{k}$ is obviously K-equivariant, (ii) follows immediately. By differentiating (W3) we also get

$$[\pi(\xi), \pi(\eta)] = \pi([\xi, \eta]) = [\nu(\xi), \pi(\eta)]$$

for $\xi \in \mathfrak{k}$ and $\eta \in \mathfrak{g}$. This implies that

$$[\omega(\xi), \pi(\eta)] = 0$$

for $\xi \in \mathfrak{k}$ and $\eta \in \mathfrak{g}$; i.e., (iii) holds.

Hence, we have

$$[\omega(\xi),\omega(\eta)] = [\omega(\xi),\nu(\eta)] - [\omega(\xi),\pi(\eta)] = [\omega(\xi),\nu(\eta)] =$$
$$[\nu(\xi),\nu(\eta)] - [\pi(\xi),\nu(\eta)] = \nu([\xi,\eta]) - [\pi(\xi),\pi(\eta)] = \nu([\xi,\eta]) - \pi([\xi,\eta]) = \omega([\xi,\eta]),$$

for any $\xi, \eta \in \mathfrak{k}$. □

We see that a weak Harish-Chandra module V is a Harish-Chandra module if and only if $\omega = 0$. A morphism $\alpha : V \longrightarrow W$ of two weak Harish-Chandra modules is a linear map that is a morphism for both $\mathcal{U}(\mathfrak{g})$- and K-module structures. We denote by $\mathcal{M}(\mathfrak{g}, K)_w$ the category of all weak Harish-Chandra modules for the pair $(\mathfrak{g}, K)$. Clearly, the category $\mathcal{M}(\mathfrak{g}, K)$ of Harish-Chandra modules is a full subcategory of $\mathcal{M}(\mathfrak{g}, K)_w$. Also, $\mathcal{M}(\mathfrak{g}, K)_w$ is an abelian category.

Now we define a functor from $\mathcal{M}(\mathfrak{g})$ into $\mathcal{M}(\mathfrak{g}, K)_w$. Let V be a $\mathfrak{g}$-module. We consider the linear space $R(K, V)$ with the following actions of $\mathfrak{g}$ and K,

(i)
$$(\pi(\xi)F)(k) = \pi_V(\phi(k)\xi)F(k), \qquad k \in K,$$
for $\xi \in \mathfrak{g}$ and $F \in R(K, V)$;

(ii)
$$(\nu(k)F)(h) = F(hk), \quad h \in K,$$
for $k \in K$ and $F \in R(K, V)$.

As remarked in §1, the action π is K-equivariant. Hence, $\mathrm{Ind}_w(V) = R(K, V)$ is a weak Harish-Chandra module. If for a $\mathfrak{g}$-morphism $\alpha : V \longrightarrow W$ we define $\mathrm{Ind}_w(\alpha) = 1 \otimes \alpha$, Ind_w becomes an exact functor from $\mathcal{M}(\mathfrak{g})$ into $\mathcal{M}(\mathfrak{g}, K)_w$.

Let V be a weak Harish-Chandra module. As in §1, we see that the matrix coefficient map c_V from V into $\mathrm{Ind}_w(V)$, defined by $c_V(v)(k) = \nu_V(k)v$ for $v \in V$ and $k \in K$, is a $(\mathfrak{g}, K)$-morphism. Furthermore, the maps c_V define a natural transformation of the identity functor on $\mathcal{M}(\mathfrak{g}, K)_w$ into the composition of Ind_w and the forgetful functor from $\mathcal{M}(\mathfrak{g}, K)_w$ into $\mathcal{M}(\mathfrak{g})$.

On the other hand, for any W in $\mathcal{M}(\mathfrak{g})$, we define a linear map $e_W : \mathrm{Ind}_w(W) \longrightarrow W$ by $e_W(F) = F(1)$ for $F \in \mathrm{Ind}_w(W)$. As in §1, we see that e_W is a morphism of $\mathfrak{g}$-modules, and that the maps e_W define a natural transformation of the composition of the forgetful functor from $\mathcal{M}(\mathfrak{g}, K)_w$ into $\mathcal{M}(\mathfrak{g})$ with Ind_w into the identity functor on $\mathcal{M}(\mathfrak{g})$.

Proceeding as in the proof of 1.2, we get

2.2. Lemma. *The functor* $\mathrm{Ind}_w : \mathcal{M}(\mathfrak{g}) \longrightarrow \mathcal{M}(\mathfrak{g}, K)_w$ *is right adjoint to the forgetful functor from* $\mathcal{M}(\mathfrak{g}, K)_w$ *into* $\mathcal{M}(\mathfrak{g})$.

This immediately leads to the following result analogous to 1.3.

2.3. Lemma. *The category* $\mathcal{M}(\mathfrak{g}, K)_w$ *has enough injectives.*

Let U be in $\mathcal{M}(\mathfrak{g}, K)_w$. Denote by

$$U^{\mathfrak{k}} = \{u \in U \mid \omega(\xi)u = 0,\ \xi \in \mathfrak{k}\}.$$

Then, by 2.1, $U^{\mathfrak{k}}$ is the largest Harish-Chandra submodule of U. Clearly, for any Harish-Chandra module V, we have

$$\operatorname{Hom}_{(\mathfrak{g},K)}(V,U) = \operatorname{Hom}_{(\mathfrak{g},K)}(V,U^{\mathfrak{k}}).$$

Therefore, $U \longmapsto U^{\mathfrak{k}}$ is the right adjoint of the forgetful functor $\mathcal{M}(\mathfrak{g},K) \longrightarrow \mathcal{M}(\mathfrak{g},K)_w$.

Therefore, the composition $V \longmapsto \operatorname{Ind}_w(V)^{\mathfrak{k}}$ is the right adjoint of the forgetful functor from the category $\mathcal{M}(\mathfrak{g},K)$ into $\mathcal{M}(\mathfrak{g})$. This is the Zuckerman functor Γ_K.

2.4. Lemma. *For any V in $\mathcal{M}(\mathfrak{g})$, we have*

$$\Gamma_K(V) = \operatorname{Ind}_w(V)^{\mathfrak{k}}.$$

To make this more explicit we calculate the ω-action on $\operatorname{Ind}_w(V)$. We have

$$\begin{aligned}(\lambda(\operatorname{Ad}(k)\xi)F)(k) &= (\gamma(\operatorname{Ad}(k)\xi)F)(k) + \pi_V(\operatorname{Ad}(k)\xi)F(k) \\ &= (\pi(\xi)F)(k) - (\nu(\xi)F)(k) = -(\omega(\xi)F)(k), \quad k \in K,\end{aligned}$$

for $\xi \in \mathfrak{k}$, where we denoted by γ the left regular representation of $\mathfrak{k}$ on $R(K)$ tensored by the trivial representation on V. Hence, we established the following formula:

$$(\omega(\xi)F)(k) = -(\lambda(\operatorname{Ad}(k)\xi)F)(k), \qquad k \in K.$$

This implies that the largest Harish-Chandra submodule $\Gamma_K(V) = \operatorname{Ind}_w(V)^{\mathfrak{k}}$ of $\operatorname{Ind}_w(V)$ can be characterized as the submodule of all λ-invariants in $\operatorname{Ind}_w(V)$.

This agrees with the construction in §1.

Denote by $D(\mathfrak{g},K)_w = D(\mathcal{M}(\mathfrak{g},K)_w)$ the derived category of the category $\mathcal{M}(\mathfrak{g},K)_w$ of weak Harish-Chandra modules. Then we can consider the forgetful functors $D(\mathcal{M}(\mathfrak{g},K)) \longrightarrow D(\mathfrak{g},K)_w$ and $D(\mathfrak{g},K)_w \longrightarrow D(\mathcal{M}(\mathfrak{g}))$. Their composition is the forgetful functor from $D(\mathcal{M}(\mathfrak{g},K))$ into $D(\mathcal{M}(\mathfrak{g}))$. Therefore, the right adjoint functor $R\Gamma_K : D(\mathcal{M}(\mathfrak{g})) \longrightarrow D(\mathcal{M}(\mathfrak{g},K))$ is the composition of the right adjoint functor $V^{\cdot} \longmapsto \operatorname{Ind}_w(V^{\cdot})$ from $D(\mathcal{M}(\mathfrak{g}))$ into $D(\mathfrak{g},K)_w$ with the right derived functor of the functor $U \longmapsto U^{\mathfrak{k}}$. This leads us back to the setup of §1.

Instead of proceeding like in the last step, Beilinson and Ginzburg interpret the condition $\omega = 0$, which makes a weak Harish-Chandra module an ordinary Harish-Chandra module, as a homotopic condition.

An *equivariant Harish-Chandra complex* $V^{\cdot}$ is a complex of weak Harish-Chandra modules equipped with a linear map i from $\mathfrak{k}$ into graded linear maps from $V^{\cdot}$ to $V^{\cdot}$ of degree -1. This map satisfies the following conditions:

(E1) i_ξ, $\xi \in \mathfrak{k}$, are $\mathfrak{g}$-morphisms, i.e.,

$$\pi(\eta) i_\xi = i_\xi \pi(\eta), \text{ for } \eta \in \mathfrak{g};$$

(E2) i_ξ, $\xi \in \mathfrak{k}$, are K-equivariant, i.e.

$$i_{\mathrm{Ad}(k)\xi} = \nu(k) i_\xi \nu(k^{-1}) \text{ for } k \in K;$$

(E3)

$$i_\xi i_\eta + i_\eta i_\xi = 0$$

for $\xi, \eta \in \mathfrak{k}$;

(E4)

$$d i_\xi + i_\xi d = \omega(\xi)$$

for $\xi \in \mathfrak{k}$.

Clearly, (E4) implies that cohomology modules of equivariant Harish-Chandra complexes are Harish-Chandra modules.

A morphism ϕ of equivariant Harish-Chandra complexes is a morphism of complexes of weak Harish-Chandra modules that also satisfies

$$\phi \circ i_\xi = i_\xi \circ \phi$$

for all $\xi \in \mathfrak{k}$. Let $C^*(\mathfrak{g}, K)$ be the abelian category of equivariant Harish-Chandra complexes with the appropriate boundedness condition. Two morphisms $\phi, \psi : V^\cdot \to W^\cdot$ in this category are *homotopic* if there exists a homotopy Σ of the corresponding complexes of weak Harish-Chandra modules that in addition satisfies

$$\Sigma \circ i_\xi = -i_\xi \circ \Sigma$$

for any $\xi \in \mathfrak{k}$. We denote by $K^*(\mathfrak{g}, K)$ the corresponding homotopic category of equivariant complexes. This category has a natural structure of a triangulated category [18]. Quasiisomorphisms form a localizing class of morphisms in $K^*(\mathfrak{g}, K)$. The localization of $K^*(\mathfrak{g}, K)$ with respect to quasiisomorphisms is the *equivariant derived category* $D^*(\mathfrak{g}, K)$ of Harish-Chandra modules. Clearly, $D^*(\mathfrak{g}, K)$ inherits the structure of a triangulated category from $K^*(\mathfrak{g}, K)$, but a priori $D^*(\mathfrak{g}, K)$ doesn't have to be a derived category of an abelian category (still, in this particular case, the reader should consult 2.14 at this point).

We have a natural functor $\iota : D^*(\mathcal{M}(\mathfrak{g}, K)) \to D^*(\mathfrak{g}, K)$ that maps a complex of Harish-Chandra modules $V^\cdot$ into the equivariant Harish-Chandra complex $V^\cdot$ with $i_\xi = 0$ for all $\xi \in \mathfrak{k}$.

In particular, for a Harish-Chandra module V we denote by $D(V)$ the complex

$$\ldots \to 0 \to V \to 0 \to \ldots$$

where V is in degree zero, and the corresponding equivariant Harish-Chandra complex. A straightforward modification of the standard argument proves that $D : \mathcal{M}(\mathfrak{g}, K) \to D(\mathfrak{g}, K)$ is fully faithful, i.e,

$$\mathrm{Hom}_{(\mathfrak{g},K)}(V, W) = \mathrm{Hom}_{D(\mathfrak{g},K)}(D(V), D(W))$$

for any two Harish-Chandra modules V and W. Hence, if we equip $D^*(\mathfrak{g}, K)$ with the standard truncation functors, its core is isomorphic to $\mathcal{M}(\mathfrak{g}, K)$.

Therefore, we have a natural sequence of functors

$$D^*(\mathcal{M}(\mathfrak{g}, K)) \xrightarrow{\iota} D^*(\mathfrak{g}, K) \to D^*(\mathfrak{g}, K)_w \to D^*(\mathcal{M}(\mathfrak{g}))$$

where the last two are just the corresponding forgetful functors.

In various applications, like the ones we discuss later in the paper, it is necessary to consider simple variants of the above construction. Let $\mathcal{Z}(\mathfrak{g})$ be the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. Let $\mathfrak{h}$ be an (abstract) Cartan algebra of $\mathfrak{g}$ ([14], §2). Denote by W the Weyl group of the root system Σ of $\mathfrak{g}$ in $\mathfrak{h}^*$. By a classical result of Harish-Chandra, the space of maximal ideals $\mathrm{Max}(\mathcal{Z}(\mathfrak{g}))$ is isomorphic to the space $\mathfrak{h}^*/W$ of W-orbits in $\mathfrak{h}^*$. Let θ be a W-orbit in $\mathfrak{h}^*$, and denote by J_θ the corresponding maximal ideal of $\mathcal{Z}(\mathfrak{g})$. Let $\lambda \in \theta$. We denote by χ_λ the unique homomorphism of $\mathcal{Z}(\mathfrak{g})$ into $\mathbb{C}$ with its kernel equal to J_θ. We denote by $\mathcal{U}_\theta$ the quotient of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ by the ideal generated by J_θ. Then we can view the category $\mathcal{M}(\mathcal{U}_\theta)$ of $\mathcal{U}_\theta$-modules as a full subcategory of the category $\mathcal{M}(\mathfrak{g})$. Following the classical terminology, the objects of $\mathcal{M}(\mathcal{U}_\theta)$ are just $\mathcal{U}(\mathfrak{g})$-modules with *infinitesimal character* χ_λ.

Since the image of ϕ is in $\mathrm{Int}(\mathfrak{g})$, the group K acts trivially on $\mathcal{Z}(\mathfrak{g})$, hence also on $\mathrm{Max}(\mathcal{Z}(\mathfrak{g}))$. Therefore, we can define the category $\mathcal{M}(\mathcal{U}_\theta, K)$ as the full subcategory of $\mathcal{M}(\mathfrak{g}, K)$ of Harish-Chandra modules with infinitesimal character χ_λ. Clearly, for any Harish-Chandra module V in $\mathcal{M}(\mathfrak{g}, K)$, the module

$$\mathcal{U}_\theta \otimes_{\mathcal{U}(\mathfrak{g})} V = V/J_\theta V$$

is in $\mathcal{M}(\mathcal{U}_\theta, K)$. Therefore, we have the right exact functor $P_\theta : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathcal{U}_\theta, K)$. It is straightforward to check the following result.

2.5. Lemma. *The functor $P_\theta : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathcal{U}_\theta, K)$ is the left adjoint of the forgetful functor from $\mathcal{M}(\mathcal{U}_\theta, K)$ into $\mathcal{M}(\mathfrak{g}, K)$.*

Analogously, we can define the category $\mathcal{M}(\mathcal{U}_\theta, K)_w$ of weak Harish-Chandra modules with infinitesimal character χ_λ, and the corresponding derived categories $D^*(\mathcal{U}_\theta, K)_w$ and $D^*(\mathcal{U}_\theta, K)$. In addition, the definition of P_θ obviously extends to the corresponding categories of weak Harish-Chandra modules and we have an obvious analogue of 2.5. Let U be an algebraic K-module. Then $\mathcal{U}(\mathfrak{g}) \otimes U$, equipped with the $\mathcal{U}(\mathfrak{g})$-action by left multiplication on the first factor and the tensor product of natural algebraic actions of K, is a weak Harish-Chandra module. Since it is a flat $\mathcal{U}(\mathfrak{g})$-module, it is also acyclic for P_θ. Since any weak Harish-Chandra module V is a quotient of $\mathcal{U}(\mathfrak{g}) \otimes V$, P_θ has its left derived functor $LP_\theta : D^-(\mathfrak{g}, K)_w \to D^-(\mathcal{U}_\theta, K)_w$. Moreover, the homological dimension of $\mathcal{U}(\mathfrak{g})$ is finite, so LP_θ extends to the derived categories of unbounded complexes. Also, this functor is the left adjoint of the forgetful functor from $D(\mathcal{U}_\theta, K)_w$ into $D(\mathfrak{g}, K)_w$. Finally, it also induces the left derived functor[1]

$$LP_\theta : D(\mathfrak{g}, K) \to D(\mathcal{U}_\theta, K).$$

This is again the left adjoint of the forgetful functor:

2.6. Proposition. *The functor $LP_\theta : D(\mathfrak{g}, K) \to D(\mathcal{U}_\theta, K)$ is the left adjoint of the forgetful functor from $D(\mathcal{U}_\theta, K)$ into $D(\mathfrak{g}, K)$.*

Clearly, if H is a closed subgroup of K, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(\mathfrak{g}, K) & \xrightarrow{P_\theta} & \mathcal{M}(\mathcal{U}_\theta, K) \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathfrak{g}, H) & \xrightarrow{P_\theta} & \mathcal{M}(\mathcal{U}_\theta, H) \end{array} .$$

This leads to the commutative diagram

$$\begin{array}{ccc} D(\mathfrak{g}, K) & \xrightarrow{LP_\theta} & D(\mathcal{U}_\theta, K) \\ \downarrow & & \downarrow \\ D(\mathfrak{g}, H) & \xrightarrow[LP_\theta]{} & D(\mathcal{U}_\theta, H) \end{array}$$

where the vertical arrows represent forgetful functors.

The following result is proved in [17].

2.7. Theorem. *The forgetful functor from $D^+(\mathfrak{g}, K)$ into $D^+(\mathfrak{g}, H)$ (resp. from $D^+(\mathcal{U}_\theta, K)$ into $D^+(\mathcal{U}_\theta, H)$) has the right adjoint $R\Gamma^{equi}_{K,H} : D^+(\mathfrak{g}, H) \to D^+(\mathfrak{g}, K)$ (resp. $R\Gamma^{equi}_{K,H} : D^+(\mathcal{U}_\theta, H) \to D^+(\mathcal{U}_\theta, K)$).*

If H is reductive, the amplitude of $R\Gamma^{equi}_{K,H}$ is finite. In this situation, the above claims hold also for unbounded equivariant derived categories.

We are going to discuss the construction of this *equivariant Zuckerman functor* $R\Gamma^{equi}_{K,H}$ in the next section.

As in the case of ordinary Zuckerman functors, the following result holds.

2.8. Proposition. *Let $T \subset H$ be algebraic subgroups of K. Then we have the isomorphism of functors*

$$R\Gamma^{equi}_{K,T} = R\Gamma^{equi}_{K,H} \circ R\Gamma^{equi}_{H,T}.$$

If H is reductive, the functor $R\Gamma^{equi}_{K,H}$ extends to the equivariant derived categories of unbounded complexes, and preserves its adjointness property. Therefore, by taking the adjoints of the above diagram, we get the following result (which also follows from the explicit formula for $R\Gamma^{equi}_{K,H}$ we are going to discuss in the next section).

[1]Since equivariant derived categories are not derived categories in the standard sense, this requires some additional care (see [8], [17]).

2.9. Lemma. *If H is a reductive subgroup of K, the following diagram commutes*

$$\begin{array}{ccc} D(\mathcal{U}_\theta, H) & \longrightarrow & D(\mathfrak{g}, H) \\ {\scriptstyle R\Gamma^{equi}_{K,H}}\big\downarrow & & \big\downarrow{\scriptstyle R\Gamma^{equi}_{K,H}} \\ D(\mathcal{U}_\theta, K) & \longrightarrow & D(\mathfrak{g}, K) \end{array} .$$

This explains the ambiguity in our notation.

Another simple consequence of the construction of the equivariant Zuckerman functor and 1.6 is the following result ([17], 6.2.7).

2.10. Lemma. *Assume that H is a reductive subgroup of K. Let V be a Harish-Chandra module in $\mathcal{M}(\mathfrak{g}, H)$. Then*

$$H^p(R\Gamma^{equi}_{K,H}(D(V))) = R^p\Gamma_{K,H}(V)$$

for any $p \in \mathbb{Z}_+$.

Let K be an arbitrary algebraic group and L a Levi subgroup of K. As we remarked in 1.11, $\mathcal{M}(\mathfrak{g}, K)$ is a thick subcategory of $\mathcal{M}(\mathfrak{g}, L)$. Clearly, the same applies to the subcategory $\mathcal{M}(\mathcal{U}_\theta, K)$ of $\mathcal{M}(\mathcal{U}_\theta, L)$. Therefore, we can define full triangulated subcategories $D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathfrak{g}, L)$ and $D^*_{\mathcal{M}(\mathcal{U}_\theta,K)}(\mathcal{U}_\theta, L)$ of $D^*(\mathfrak{g}, L)$, resp. $D^*(\mathcal{U}_\theta, L)$, consisting of equivariant Harish-Chandra complexes with cohomology in $\mathcal{M}(\mathfrak{g}, K)$, resp. $\mathcal{M}(\mathcal{U}_\theta, K)$.

Clearly, we have natural forgetful functors $D^*(\mathfrak{g}, K) \to D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathfrak{g}, L)$ and $D^*(\mathcal{U}_\theta, K) \to D^*_{\mathcal{M}(\mathcal{U}_\theta,K)}(\mathcal{U}_\theta, L)$. The following result is an equivariant analogue of 1.12.

2.11. Theorem. *The natural forgetful functors*

$$D^*(\mathfrak{g}, K) \to D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathfrak{g}, L) \text{ and } D^*(\mathcal{U}_\theta, K) \to D^*_{\mathcal{M}(\mathcal{U}_\theta,K)}(\mathcal{U}_\theta, L)$$

are equivalences of categories.

Proof. The proofs of these equivalences are identical. Therefore, we discuss the case of modules over the enveloping algebra.

Also, it is sufficient to prove this statement for the derived categories of unbounded complexes. The functor $R\Gamma^{equi}_{K,L}$ is the right adjoint of the forgetful functor For : $D(\mathfrak{g}, K) \to D(\mathfrak{g}, L)$. Therefore, we have the natural transformation of the identity functor on $D(\mathfrak{g}, K)$ into $R\Gamma^{equi}_{K,L} \circ$ For. Assume that $V^\cdot$ is an equivariant complex in $D(\mathfrak{g}, K)$. Then we have a natural morphism $V^\cdot \to R\Gamma^{equi}_{K,L}(V^\cdot)$. Since $H^p(V^\cdot)$, $p \in \mathbb{Z}$, are Harish-Chandra modules in $\mathcal{M}(\mathfrak{g}, K)$, they are $\Gamma_{K,L}$-acyclic by 1.10. By a standard argument using 2.10, this implies that

$$H^p(R\Gamma^{equi}_{K,L}(V^\cdot)) = \Gamma_{K,L}(H^p(V^\cdot)) = H^p(V^\cdot)$$

and the natural morphism $V^\cdot \to R\Gamma^{equi}_{K,L}(V^\cdot)$ is a quasiisomorphism.

On the other hand, we have the adjointness morphism $\mathrm{For}(R\Gamma^{equi}_{K,L}(U^\cdot)) \to U^\cdot$ for any $U^\cdot$ in $D(\mathfrak{g},L)$. If $U^\cdot$ is a complex in $D_{\mathcal{M}(\mathfrak{g},K)}(\mathfrak{g},L)$, its cohomology modules are $\Gamma_{K,L}$-acyclic by 1.10. Therefore, as before, $H^p(R\Gamma^{equi}_{K,L}(U^\cdot)) = H^p(U^\cdot)$, for $p \in \mathbb{Z}$, and the adjointness morphism is a quasiisomorphism.

Hence, $D^*(\mathfrak{g},K) \to D^*_{\mathcal{M}(\mathfrak{g},K)}(\mathfrak{g},L)$ is an equivalence of categories. □

It follows that we can view $D^*(\mathfrak{g},K)$ and $D^*(\mathcal{U}_\theta,K)$ as triangulated subcategories in $D^*(\mathfrak{g},L)$ and $D^*(\mathcal{U}_\theta,L)$ respectively.

Now we can discuss the consequences of 2.11 with respect to equivariant Zuckerman functors. Let H be a subgroup of K and T a Levi factor of H. Then we have the following commutative diagram

$$\begin{array}{ccc} D^+(\mathfrak{g},T) & \xrightarrow{R\Gamma^{equi}_{H,T}} & D^+(\mathfrak{g},H) \\ {\scriptstyle R\Gamma^{equi}_{K,T}}\downarrow & & \downarrow{\scriptstyle R\Gamma^{equi}_{K,H}} \\ D^+(\mathfrak{g},K) & = & D^+(\mathfrak{g},K) \end{array} .$$

Finally, by replacing the top left corner with $D^+_{\mathcal{M}(\mathfrak{g},H)}(\mathfrak{g},T)$ and inverting the top horizontal arrow, we get the commutative diagram:

$$\begin{array}{ccc} D^+_{\mathcal{M}(\mathfrak{g},H)}(\mathfrak{g},T) & \longleftarrow & D^+(\mathfrak{g},H) \\ {\scriptstyle R\Gamma^{equi}_{K,T}}\downarrow & & \downarrow{\scriptstyle R\Gamma^{equi}_{K,H}} \\ D^+(\mathfrak{g},K) & = & D^+(\mathfrak{g},K) \end{array} ,$$

i.e., $R\Gamma^{equi}_{K,H}$ is the restriction of $R\Gamma^{equi}_{K,T}$ to $D^+(\mathfrak{g},H)$. Since the amplitude of $R\Gamma^{equi}_{K,T}$ is finite by 2.7, the amplitude of $R\Gamma^{equi}_{K,H}$ is also finite. Both functors extend to the categories of unbounded complexes, and we have the following result.

2.12. Theorem. *The equivariant Zuckerman functor $R\Gamma^{equi}_{K,H}$ is the restriction of $R\Gamma^{equi}_{K,T}$ to the subcategory $D(\mathfrak{g},H)$ (resp. $D(\mathcal{U}_\theta,H)$) of $D(\mathfrak{g},T)$ (resp. $D(\mathcal{U}_\theta,T)$).*

2.13. Remark. We can now eliminate the assumption of H being reductive from 2.7, 2.9 and 2.10. We already noted that above for 2.7. For 2.9 it follows from 2.12, and for 2.10 from 2.12 and 1.13.

Finally, we quote a result of Bernstein and Lunts [5] which completely explains 2.10.[2]

[2] Bernstein and Lunts treat the case of derived categories bounded from below. The general case follows from [17].

2.14. Theorem. *The natural functor $\iota : D(\mathcal{M}(\mathfrak{g}, K)) \longrightarrow D(\mathfrak{g}, K)$ is an equivalence of categories.*

Proof. For reductive K, the claim is proved in [17], Section 5.8. For arbitrary K, let L be a Levi factor of K. The equivalence $\iota : D(\mathcal{M}(\mathfrak{g}, L)) \to D(\mathfrak{g}, L)$ induces an equivalence of subcategories $D_{\mathcal{M}(\mathfrak{g},K)}(\mathcal{M}(\mathfrak{g}, L))$ and $D_{\mathcal{M}(\mathfrak{g},K)}(\mathfrak{g}, L)$. The first of these is equivalent to $D(\mathcal{M}(\mathfrak{g}, K))$ by 1.12, and the second is equivalent to $D(\mathfrak{g}, K)$ by 2.11. The induced functor is clearly $\iota : D(\mathcal{M}(\mathfrak{g}, K)) \longrightarrow D(\mathfrak{g}, K)$, and it is an equivalence. □

3. Equivariant Zuckerman functors

In this section we sketch the construction of equivariant Zuckerman functors $R\Gamma^{equi}_{K,H}$. The details can be found in [17]. For simplicity, we will describe the definitions and the arguments for $\mathcal{U}(\mathfrak{g})$-modules. However, analogous statements hold also for the variant with $\mathcal{U}_\theta$-modules, with identical proofs.

Let $N^{\cdot}(\mathfrak{k}) = \mathcal{U}(\mathfrak{k}) \otimes \bigwedge^{-\cdot} \mathfrak{k}$ be the standard complex of $\mathfrak{k}$. It can be viewed as an equivariant $(\mathfrak{k}, H)$-complex in the following way: $\mathfrak{k}$ acts by left multiplication on the first factor, H acts by the tensor product of the adjoint actions on both factors, and the map i is given by

$$i_\xi(u \otimes \lambda) = -u \otimes \lambda \wedge \xi,$$

for $\xi \in \mathfrak{h}$ and $u \otimes \lambda \in N^{\cdot}(\mathfrak{k})$.

The standard complex has a natural structure of an algebra. It is generated by the subalgebra $\mathcal{U}(\mathfrak{k}) \otimes 1$ isomorphic to the enveloping algebra $\mathcal{U}(\mathfrak{k})$ and the subalgebra $1 \otimes \bigwedge \mathfrak{k}$ isomorphic to the exterior algebra $\bigwedge \mathfrak{k}$. The multiplication is defined by the relations

$$(u \otimes 1) \cdot (1 \otimes \lambda) = u \otimes \lambda,$$

for $u \in \mathcal{U}(\mathfrak{k})$ and $\lambda \in \bigwedge \mathfrak{k}$, and

$$(1 \otimes \xi) \cdot (\eta \otimes 1) - (\eta \otimes 1) \cdot (1 \otimes \xi) = 1 \otimes [\xi, \eta]$$

for $\xi, \eta \in \mathfrak{k}$. With this multiplication and with its natural grading and differential, $N^{\cdot}(\mathfrak{k})$ becomes a *differential graded algebra*, namely a graded algebra which is also a complex of vector spaces, such that for any two homogeneous elements x and y,

$$d(x \cdot y) = dx \cdot y + (-1)^{\deg x} x \cdot dy.$$

Furthermore, the principal antiautomorphism of $\mathcal{U}(\mathfrak{k})$ and the linear isomorphism of $\mathfrak{k} \subset \bigwedge \mathfrak{k}$ defined by $\xi \longmapsto -\xi$ extend to a *principal antiautomorphism* ι of $N^{\cdot}(\mathfrak{k})$; it satisfies

$${}^{\iota}(x \cdot y) = (-1)^{\deg x \deg y}\ {}^{\iota}y \cdot {}^{\iota}x$$

for homogeneous $x, y \in N^{\cdot}(\mathfrak{k})$.

Any equivariant $(\mathfrak{g}, K)$-complex can be viewed as a graded module over $N^{\cdot}(\mathfrak{k})$, if we let $\mathfrak{k} \subset \mathcal{U}(\mathfrak{k})$ act via the ω-action, and $\mathfrak{k} \subset \bigwedge \mathfrak{k}$ via the map i. We will denote this action of $N^{\cdot}(\mathfrak{k})$ by ω again.

Let $V^{\cdot}$ be an equivariant $(\mathfrak{g}, H)$-complex. If we forget the H-action and the equivariant structure, $R(K, V^{\cdot}) = R(K) \otimes V^{\cdot} = \mathrm{Ind}_w(V^{\cdot})$ is a complex of weak Harish-Chandra modules for $(\mathfrak{g}, K)$. On the other hand, we can consider $R(K, V^{\cdot})$ as an equivariant $(\mathfrak{k}, H)$-complex with respect to the λ-action of §1. That is, $\mathfrak{k}$ and H act by the left regular representation tensored by the natural action on V, and the map i on $R(K, V^{\cdot})$ is given by

$$i_{R(K,V^{\cdot}),\xi}(f \otimes v) = f \otimes i_{V^{\cdot},\xi}(v)$$

for $\xi \in \mathfrak{h}$, $f \in R(K)$ and $v \in V^{\cdot}$ (here $i_{V^{\cdot}}$ is the i-map of $V^{\cdot}$ as an equivariant $(\mathfrak{g}, H)$-complex).

We can now consider the complex of vector spaces

$$\Gamma^{equi}_{K,H}(V^{\cdot}) = \mathrm{Hom}^{\cdot}_{(\mathfrak{k},H)}(N^{\cdot}(\mathfrak{k}), R(K, V^{\cdot})).$$

Here $\mathrm{Hom}^{\cdot}_{(\mathfrak{k},H)}$ consists of graded linear maps of equivariant $(\mathfrak{k}, H)$-complexes $N^{\cdot}(\mathfrak{k})$ and $R(K, V^{\cdot})$ described above. These linear maps f intertwine the actions of $\mathfrak{k}$ and H, and,

$$f \circ i_{N^{\cdot}(\mathfrak{k}),\xi} = (-1)^{\deg f} i_{R(K,V^{\cdot}),\xi} \circ f$$

for $\xi \in \mathfrak{h}$. The differential of this complex is given by

$$df = d_{R(K,V^{\cdot})} \circ f - (-1)^{\deg f} f \circ d_{N^{\cdot}(\mathfrak{k})},$$

for a homogeneous $f \in \Gamma^{equi}_{K,H}(V^{\cdot})$.

The $(\mathfrak{g}, K)$-action on $R(K, V^{\cdot})$ defines, by composition, a $(\mathfrak{g}, K)$-action on $\Gamma^{equi}_{K,H}(V^{\cdot})$. In this way, $\Gamma^{equi}_{K,H}(V^{\cdot})$ becomes a complex of weak $(\mathfrak{g}, K)$-modules. Moreover, if we define the map i by

$$(i_\xi f)(u \otimes \lambda)(k) = -(-1)^{\deg f} f((1 \otimes \mathrm{Ad}(k)\xi) \cdot (u \otimes \lambda))(k),$$

for $\xi \in \mathfrak{k}$, a homogeneous $f \in \Gamma^{equi}_{K,H}(V^{\cdot})$, $u \otimes \lambda \in N^{\cdot}(\mathfrak{k})$ and $k \in K$, the complex $\Gamma^{equi}_{K,H}(V^{\cdot})$ becomes an equivariant $(\mathfrak{g}, K)$-complex. One can check that $\Gamma^{equi}_{K,H}$ is a functor from the category of equivariant $(\mathfrak{g}, H)$-complexes $C(\mathfrak{g}, H)$ into $C(\mathfrak{g}, K)$. It also induces a functor between the corresponding homotopic categories $K(\mathfrak{g}, H)$ and $K(\mathfrak{g}, K)$.

Moreover, we have

3.1. Theorem. *The functor* $\Gamma^{equi}_{K,H}$ *is right adjoint to the forgetful functor from* $C(\mathfrak{g},K)$ *into* $C(\mathfrak{g},H)$ *(resp.* $K(\mathfrak{g},K)$ *into* $K(\mathfrak{g},H)$*).*

Proof. We just define the adjointness morphisms, and leave tedious checking to the reader. For an equivariant $(\mathfrak{g},K)$-complex $V^{\cdot}$, we define $\Phi_{V^{\cdot}} : V^{\cdot} \longrightarrow \Gamma^{equi}_{K,H}(V^{\cdot})$ by

$$\Phi_{V^{\cdot}}(v)(u\otimes\lambda)(k) = (-1)^{\deg v \deg \lambda}\omega_{V^{\cdot}}({}^{\iota}(u\otimes\lambda))\nu_{V^{\cdot}}(k)v,$$

for homogeneous $v \in V^{\cdot}$, $\lambda \in \bigwedge \mathfrak{k}$, $u \in \mathcal{U}(\mathfrak{k})$ and $k \in K$. The other adjointness morphism is much simpler: for an equivariant $(\mathfrak{g},H)$-complex $W^{\cdot}$, we define $\Psi_{W^{\cdot}} : \Gamma^{equi}_{K,H}(W^{\cdot}) \longrightarrow W^{\cdot}$ by

$$\Psi_{W^{\cdot}}(f) = f(1\otimes 1)(1)$$

for $f \in \Gamma^{equi}_{K,H}(W^{\cdot})$. $\square$

By results of Bernstein and Lunts, $\Gamma^{equi}_{K,H}$ always has a right derived functor

$$R\Gamma^{equi}_{K,H} : D^+(\mathfrak{g},H) \longrightarrow D^+(\mathfrak{g},K).$$

The proof of this uses the existence of *K-injective* resolutions of equivariant complexes. Their existence is established in [5] (see [17] for a more detailed account). However, these resolutions are very complicated and it is unclear when they are bounded above. The following theorem gives an explicit formula for $R\Gamma^{equi}_{K,H}$ in case H is reductive.

3.2. Lemma. *Assume that H is reductive. Then for any acyclic equivariant* $(\mathfrak{g},H)$*-complex* $V^{\cdot}$*, the complex* $\Gamma^{equi}_{K,H}(V^{\cdot})$ *is acyclic.*

Proof. It is obvious that $R(K,V^{\cdot})$ is an acyclic equivariant $(\mathfrak{k},H)$-complex. Therefore, it is enough to prove that the functor

$$\operatorname{Hom}^{\cdot}_{(\mathfrak{k},H)}(N^{\cdot}(\mathfrak{k}),-),$$

from equivariant $(\mathfrak{k},H)$-complexes to complexes of vector spaces, preserves acyclicity.

This follows from the fact that the $(\mathfrak{k},H)$-complex $N^{\cdot}(\mathfrak{k})$, is *K-projective* (see [4], [18][3] or [17]). It is proved by induction, using the *Hochschild-Serre filtration* $\mathrm{F}_{\cdot} N^{\cdot}(\mathfrak{k})$, associated to the subalgebra $\mathfrak{h}$, of the standard complex $N^{\cdot}(\mathfrak{k})$ ([17], 6.1). Namely, one can see that the graded pieces corresponding to this filtration are K-projective. Then one shows that the short exact sequences

$$0 \longrightarrow \mathrm{F}_{p-1} N^{\cdot}(\mathfrak{k}) \longrightarrow \mathrm{F}_{p} N^{\cdot}(\mathfrak{k}) \longrightarrow \mathrm{Gr}_{p} N^{\cdot}(\mathfrak{k}) \longrightarrow 0$$

define distinguished triangles in the homotopic category of equivariant $(\mathfrak{k},H)$-complexes. If two vertices of a distinguished triangle are K-projective, then so is the third vertex. Since the filtration is finite, the theorem follows. $\square$

Hence, the functor $\Gamma^{equi}_{K,H}$ preserves acyclic complexes. It follows that it also preserves quasiisomorphisms. Therefore, it is well defined on morphisms in the derived category. Hence, $\Gamma^{equi}_{K,H}$ defines a functor on the level of derived categories which is equal to $\Gamma^{equi}_{K,H}$ on objects, i.e., we have the following result.

[3] Verdier calls such objects "free on the left."

3.3. Theorem. *Assume that H is reductive. Then for any equivariant $(\mathfrak{g}, H)$-complex $V^\cdot$, we have*

$$R\Gamma^{equi}_{K,H}(V^\cdot) = \Gamma^{equi}_{K,H}(V^\cdot) = \operatorname{Hom}^\cdot_{(\mathfrak{k},H)}(N^\cdot(\mathfrak{k}), R(K, V^\cdot)).$$

In particular, $R\Gamma^{equi}_{K,H}$ has finite amplitude.

4. Localization of Zuckerman functors

In this section we sketch the localization of equivariant Zuckerman functors. The details will appear in [15].

First we recall the basic constructions and results of the localization theory of Beilinson and Bernstein (cf. [1], [14]). Let X be the flag variety of $\mathfrak{g}$. For any λ in the dual $\mathfrak{h}^*$ of the abstract Cartan algebra $\mathfrak{h}$, Beilinson and Bernstein construct a *twisted sheaf of differential operators* $\mathcal{D}_\lambda$ on X and an algebra homomorphism $\mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(X, \mathcal{D}_\lambda)$. They prove that

$$\Gamma(X, \mathcal{D}_\lambda) = \mathcal{U}_\theta \quad \text{and} \quad H^p(X, \mathcal{D}_\lambda) = 0 \text{ for } p > 0,$$

where $\theta = W \cdot \lambda$. Let $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ be the category of quasicoherent $\mathcal{D}_\lambda$-modules on X. Then, for any object $\mathcal{V}$ in $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ its global sections are an object in $\mathcal{M}(\mathcal{U}_\theta)$, i.e., the functor of global sections Γ is a left exact functor from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}(\mathcal{U}_\theta)$. It has a left adjoint $\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ defined by

$$\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$$

for any $\mathcal{U}_\theta$-module V. This functor is the *localization functor.*

The functor Γ has finite right cohomological dimension. Therefore, it defines a functor $R\Gamma : D(\mathcal{D}_\lambda) \longrightarrow D(\mathcal{U}_\theta)$ from the derived category of $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into the derived category of $\mathcal{M}(\mathcal{U}_\theta)$. On the other hand, the functor Δ_λ has finite left cohomological dimension if and only if the orbit θ is regular [12]. Therefore, in general, we have the left derived functor $L\Delta_\lambda : D^-(\mathcal{U}_\theta) \longrightarrow D^-(\mathcal{D}_\lambda)$ which is the left adjoint of $R\Gamma : D^-(\mathcal{D}_\lambda) \longrightarrow D^-(\mathcal{U}_\theta)$. If θ is regular, $L\Delta_\lambda$ extends to $D(\mathcal{U}_\theta)$ and is the left adjoint of $R\Gamma : D(\mathcal{D}_\lambda) \longrightarrow D(\mathcal{U}_\theta)$. Moreover, for regular θ, we have the following result [2].

4.1. Theorem. *If $\theta \in \mathfrak{h}^*$ is regular, the functor $R\Gamma : D(\mathcal{D}_\lambda) \longrightarrow D(\mathcal{U}_\theta)$ is an equivalence of categories. Its quasi-inverse is $L\Delta_\lambda : D(\mathcal{U}_\theta) \longrightarrow D(\mathcal{D}_\lambda)$.*

Clearly, the group $\operatorname{Int}(\mathfrak{g})$ acts algebraically on the sheaf of algebras $\mathcal{D}_\lambda$. Let K be an algebraic group satisfying the conditions from §1. Then K acts algebraically on X and $\mathcal{D}_\lambda$. $\mathcal{V}$ is a *weak Harish-Chandra sheaf* on X if

- (i) $\mathcal{V}$ is a quasicoherent $\mathcal{D}_\lambda$-module;
- (ii) $\mathcal{V}$ is a K-equivariant $\mathcal{O}_X$-module (cf. [16]);
- (iii) the action morphism $\mathcal{D}_\lambda \otimes_{\mathcal{O}_X} \mathcal{V} \longrightarrow \mathcal{V}$ is a morphism of K-equivariant $\mathcal{O}_X$-modules.

A weak Harish-Chandra sheaf $\mathcal{V}$ is a *Harish-Chandra sheaf*[4] if the differential of the K-action on $\mathcal{V}$ agrees with the action of $\mathfrak{k}$ given by the map $\mathfrak{k} \longrightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{D}_\lambda$. A morphism of weak Harish-Chandra sheaves is a $\mathcal{D}_\lambda$-module morphism which is also a morphism of K-equivariant $\mathcal{O}_X$-modules. We denote by $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$ the abelian category of weak Harish-Chandra sheaves, and by $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)$ its full subcategory of Harish-Chandra sheaves.

For any $\mathcal{V}$ in $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$, the module $\Gamma(X, \mathcal{V})$ of global sections of $\mathcal{V}$ is in $\mathcal{M}(\mathcal{U}_\theta, K)_w$. Conversely, the localization $\Delta_\lambda(V)$ of a weak Harish-Chandra module from $\mathcal{M}(\mathcal{U}_\theta, K)_w$ is in $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$. Also,

$$\mathrm{Hom}_{(\mathcal{D}_\lambda, K)}(\Delta_\lambda(U), \mathcal{V}) = \mathrm{Hom}_{(\mathcal{U}_\theta, K)}(U, \Gamma(X, \mathcal{V})),$$

for any U in $\mathcal{M}(\mathcal{U}_\theta, K)_w$ and $\mathcal{V}$ in $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$, i.e., the functors Δ_λ and Γ are again an adjoint pair. Moreover, if $\mathcal{V}$ is a Harish-Chandra sheaf, $\Gamma(X, \mathcal{V})$ is a Harish-Chandra module. Also, if V is a Harish-Chandra module, $\Delta_\lambda(V)$ is a Harish-Chandra sheaf.

Let Σ be the root system in $\mathfrak{h}^*$ attached to $\mathfrak{g}$, and let $Q(\Sigma)$ be the corresponding root lattice. For any $\nu \in Q(\Sigma)$, let $\mathcal{O}(\nu)$ be the corresponding $\mathrm{Int}(\mathfrak{g})$-homogeneous invertible $\mathcal{O}_X$-module on X. Then we have the natural *twist* functor $\mathcal{V} \longmapsto \mathcal{V}(\nu) = \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{O}(\nu)$ from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}(\mathcal{D}_{\lambda+\nu})$ ([14], §4). This functor is clearly an equivalence of categories and its quasi-inverse is $\mathcal{W} \longmapsto \mathcal{W}(-\nu)$. The twist functor preserves (weak) Harish-Chandra sheaves, hence it induces equivalences of $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$ with $\mathcal{M}_{qc}(\mathcal{D}_{\lambda+\nu}, K)_w$ (resp. $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)$ with $\mathcal{M}_{qc}(\mathcal{D}_{\lambda+\nu}, K)$).

For a root $\alpha \in \Sigma$, let $\alpha\check{}$ be its dual root. By the Borel-Weil theorem, there exists a unique set of positive roots Σ^+ in Σ such that $H^p(X, \mathcal{O}(\nu))$ vanish for $p > 0$ for all $\nu \in Q(\Sigma)$ satisfying $\alpha\check{}(\nu) \leq 0$ for $\alpha \in \Sigma^+$. We say that $\lambda \in \mathfrak{h}^*$ is *antidominant* if $\alpha\check{}(\lambda)$ is not a positive integer for $\alpha \in \Sigma^+$. If λ is antidominant and regular, the functor $\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$ is an equivalence of categories (cf. [14], 3.7). Hence, it induces an equivalence of $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$ with $\mathcal{M}(\mathcal{U}_\theta, K)_w$. By the analogue of 2.2, we know that the forgetful functor $\mathcal{M}(\mathcal{U}_\theta, K)_w \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$ has a right adjoint Ind_w and the adjointness morphism $V \longrightarrow \mathrm{Ind}_w(V)$ is a monomorphism. Therefore, for any regular antidominant λ, the forgetful functor $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ has a right adjoint $Ind_w : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$ and the adjointness morphism $\mathcal{V} \longrightarrow Ind_w(\mathcal{V})$ is a monomorphism. By applying the twist functor we deduce that this statement holds for arbitrary λ in $\mathfrak{h}^*$.

The functor Ind_w can be described in geometric terms. Let $p : K \times X \longrightarrow X$ be the projection to the second variable. Let $\mu : K \times X \longrightarrow X$ be the action morphism, i.e., $\mu(k, x) = k \cdot x$ for $k \in K$ and $x \in X$. For a morphism f of algebraic varieties, we denote by f^* and f_* the inverse image and the direct image functors between the corresponding categories of $\mathcal{O}$-modules.

[4] In [14] we assumed that a Harish-Chandra sheaf is a coherent $\mathcal{D}_\lambda$-module. No such restriction is convenient in our present setting.

4.2. Lemma. *For any $\mathcal{V}$ in $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$, we have $Ind_w(\mathcal{V}) = \mu_*(p^*(\mathcal{V}))$ as $\mathcal{O}_X$-modules.*

Since μ is an affine morphism,

$$H^p(X, Ind_w(\mathcal{V})) = H^p(X, \mu_*(p^*(\mathcal{V}))) = H^p(K \times X, p^*(\mathcal{V})) \\ = R(K) \otimes H^p(X, \mathcal{V}) = \mathrm{Ind}_w(H^p(X, \mathcal{V})),$$

for $\mathcal{V}$ in $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ and $p \in \mathbb{Z}_+$. In particular, if $\mathcal{I}$ is an injective quasicoherent $\mathcal{D}_\lambda$-module, $\mathcal{I}$ is Γ-acyclic and $H^p(X, Ind_w(\mathcal{I})) = 0$ for $p > 0$, i.e., $Ind_w(\mathcal{I})$ is also Γ-acyclic.

4.3. Lemma.

(i) *The category $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$ has enough injectives.*
(ii) *Injective weak Harish-Chandra sheaves are acyclic for the functor of global sections $\Gamma(X, -)$.*

Proof. (i) Let $\mathcal{V}$ be a weak Harish-Chandra sheaf. Since the category $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ has enough injectives, there exist an injective quasicoherent $\mathcal{D}_\lambda$-module $\mathcal{I}$ and a monomorphism $\mathcal{V} \longrightarrow \mathcal{I}$. Since Ind_w is exact, $Ind_w(\mathcal{V}) \longrightarrow Ind_w(\mathcal{I})$ is a monomorphism of weak Harish-Chandra sheaves. Therefore, $\mathcal{V} \longrightarrow Ind_w(\mathcal{I})$ is a monomorphism of weak Harish-Chandra sheaves. On the other hand, since Ind_w is the right adjoint of an exact functor, it preserves injectives. This implies that $Ind_w(\mathcal{I})$ is an injective weak Harish-Chandra sheaf.

(ii) Let $\mathcal{J}$ be an injective weak Harish-Chandra sheaf. Then, by the above argument, $\mathcal{J}$ is a submodule of $Ind_w(\mathcal{I})$ for some injective quasicoherent $\mathcal{D}_\lambda$-module $\mathcal{I}$. Therefore, $Ind_w(\mathcal{I}) = \mathcal{J} \oplus \mathcal{W}$ for some weak Harish-Chandra sheaf $\mathcal{W}$. But this implies that $H^p(X, \mathcal{J})$ is a direct summand of $H^p(X, Ind_w(\mathcal{I}))$ for $p \in \mathbb{Z}_+$. Hence, $H^p(X, \mathcal{J}) = 0$ for $p > 0$. □

Let $D^*(\mathcal{D}_\lambda, K)_w = D^*(\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w)$ be the derived category of the abelian category of weak Harish-Chandra sheaves. Since $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)_w$ has enough injectives, the right derived functor $R\Gamma : D^+(\mathcal{D}_\lambda, K)_w \longrightarrow D^+(\mathcal{U}_\theta, K)_w$ of Γ exists. Moreover, by 4.3(ii), the following diagram is commutative:

$$\begin{array}{ccc} D^+(\mathcal{D}_\lambda, K)_w & \xrightarrow{R\Gamma} & D^+(\mathcal{U}_\theta, K)_w \\ \downarrow & & \downarrow \\ D^+(\mathcal{D}_\lambda) & \xrightarrow[R\Gamma]{} & D^+(\mathcal{U}_\theta) \end{array} ,$$

where the vertical arrows are the forgetful functors, and the lower horizontal arrow is the standard cohomology functor. This explains the ambiguity in our notation. Moreover, since the right cohomological dimension is finite, the standard truncation argument extends this statement to derived categories of unbounded complexes.

Now, let V be a weak Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)_w$. Let $P_w(V) = \mathcal{U}_\theta \otimes_{\mathbb{C}} V$ be the module on which K acts by the tensor product of the action ϕ on $\mathcal{U}_\theta$ with the natural action on V, and $\mathcal{U}_\theta$ by the multiplication in the first factor. Then $P_w(V)$ is a weak Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)_w$ since

$$k\cdot(ST\otimes v) = \phi(k)(ST)\otimes\nu(k)v = \phi(k)(S)\phi(k)(T)\otimes\nu(k)v = \phi(k)(S)(k\cdot(T\otimes v))$$

for $S, T \in \mathcal{U}_\theta$, $k \in K$ and $v \in V$. Moreover, the natural map $p : P_w(V) \longrightarrow V$ given by $p(T \otimes v) = \pi(T)v$, $T \in \mathcal{U}_\theta$, $v \in V$, satisfies

$$p(ST \otimes v) = \pi(ST)v = \pi(S)\pi(T)v = \pi(S)p(T \otimes v)$$

and

$$p(k\cdot(T\otimes v)) = p(\phi(k)T\otimes\nu(k)v) = \pi(\phi(k)T)\nu(k)v = \nu(k)\pi(T)v = \nu(k)p(T\otimes v)$$

for all $S, T \in \mathcal{U}_\theta$, $k \in K$ and $v \in V$; i.e., p is an epimorphism of weak Harish-Chandra modules. Clearly, $P_w(V)$ is a free $\mathcal{U}_\theta$-module and therefore Δ_λ-acyclic. This implies that $\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta, K)_w \longrightarrow \mathcal{M}(\mathcal{D}_\lambda, K)_w$ has a left derived functor $L\Delta_\lambda : D^-(\mathcal{U}_\theta, K)_w \longrightarrow D^-(\mathcal{D}_\lambda, K)_w$ and in addition the following diagram commutes:

$$\begin{array}{ccc} D^-(\mathcal{U}_\theta, K)_w & \xrightarrow{L\Delta_\lambda} & D^-(\mathcal{D}_\lambda, K)_w \\ \downarrow & & \downarrow \\ D^-(\mathcal{U}_\theta) & \xrightarrow[L\Delta_\lambda]{} & D^-(\mathcal{D}_\lambda) \end{array},$$

where the vertical arrows are forgetful functors and the lower horizontal arrow is the usual localization functor. In addition, we have

$$\operatorname{Hom}_{D^-(\mathcal{D}_\lambda,K)_w}(L\Delta_\lambda(U^\cdot), \mathcal{V}^\cdot) = \operatorname{Hom}_{D^-(\mathcal{U}_\theta,K)_w}(U^\cdot, R\Gamma(X, \mathcal{V}^\cdot)).$$

From the above discussion, it also follows that $R\Gamma \circ L\Delta_\lambda \cong 1$ on $D^-(\mathcal{U}_\theta, K)_w$, since this is obviously true on modules $P_w(V)$.

In addition, if θ is regular, the left cohomological dimension of Δ_λ is finite and $L\Delta_\lambda$ extends to $D(\mathcal{U}_\theta, K)_w$. Moreover, we have the following result, which is a variant of 4.1.

4.4. Theorem. *Let θ be regular. Then $R\Gamma : D(\mathcal{D}_\lambda, K)_w \longrightarrow D(\mathcal{U}_\theta, K)_w$ is an equivalence of categories. Its quasiinverse is $L\Delta_\lambda : D(\mathcal{U}_\theta, K)_w \longrightarrow D(\mathcal{D}_\lambda, K)_w$.*

Remark. The preceding results show that weak Harish-Chandra modules behave nicely with respect to the cohomology and localization functors, in sharp contrast to the case of Harish-Chandra modules. To see this, the reader should consider the case of $\mathcal{M}_{qc}(\mathcal{D}_\lambda, K)$, with $K = \operatorname{Int}(\mathfrak{g})$, which is clearly a semisimple abelian category.

In complete analogy with the constructions in §2, we can define equivariant complexes of Harish-Chandra sheaves, and corresponding categories $C^*(\mathcal{D}_\lambda, K)$ and $K^*(\mathcal{D}_\lambda, K)$. By localizing $K^*(\mathcal{D}_\lambda, K)$ with respect to quasi-isomorphisms we get the equivariant derived category $D^*(\mathcal{D}_\lambda, K)$ of Harish-Chandra sheaves.

The natural functors Γ from $K(\mathcal{D}_\lambda, K)$ into $K(\mathcal{U}_\theta, K)$ and Δ_λ from $K^-(\mathcal{U}_\theta, K)$ into $K^-(\mathcal{D}_\lambda, K)$ have right, resp. left, derived functors. More precisely, we have the following result.

4.5. Theorem. *The functors Γ and Δ_λ define the corresponding derived functors*

$$R\Gamma : D(\mathcal{D}_\lambda, K) \longrightarrow D(\mathcal{U}_\theta, K)$$

and

$$L\Delta_\lambda : D^-(\mathcal{U}_\theta, K) \longrightarrow D^-(\mathcal{D}_\lambda, K)$$

such that the following diagrams commute

$$\begin{array}{ccc} D(\mathcal{D}_\lambda, K) & \xrightarrow{R\Gamma} & D(\mathcal{U}_\theta, K) \\ \downarrow & & \downarrow \\ D(\mathcal{D}_\lambda, K)_w & \xrightarrow[R\Gamma]{} & D(\mathcal{U}_\theta, K)_w \end{array} ;$$

$$\begin{array}{ccc} D^-(\mathcal{U}_\theta, K) & \xrightarrow{L\Delta_\lambda} & D^-(\mathcal{D}_\lambda, K) \\ \downarrow & & \downarrow \\ D^-(\mathcal{U}_\theta, K)_w & \xrightarrow[L\Delta_\lambda]{} & D^-(\mathcal{D}_\lambda, K)_w \end{array} ;$$

where the vertical arrows represent forgetful functors.

Moreover, $L\Delta_\lambda$ is the left adjoint of $R\Gamma$, i.e.,

$$\operatorname{Hom}_{D^-(\mathcal{D}_\lambda,K)}(L\Delta_\lambda(U^\cdot), \mathcal{V}^\cdot) = \operatorname{Hom}_{D^-(\mathcal{U}_\theta,K)}(U^\cdot, R\Gamma(X, \mathcal{V}^\cdot))$$

for $U^\cdot$ in $D^-(\mathcal{U}_\theta, K)$ and $\mathcal{V}^\cdot$ in $D^-(\mathcal{D}_\lambda, K)$.

If θ is regular, Δ_λ has finite left cohomological dimension and $L\Delta_\lambda$ extends to $D(\mathcal{U}_\theta, K)$. This leads to the following equivariant version of 4.1.

4.6. Theorem. *Let θ be regular. Then $R\Gamma : D(\mathcal{D}_\lambda, K) \to D(\mathcal{U}_\theta, K)$ is an equivalence of categories. Its quasi-inverse is $L\Delta_\lambda : D(\mathcal{U}_\theta, K) \to D(\mathcal{D}_\lambda, K)$.*

For any $\nu \in Q(\Sigma)$, the twist functor $\mathcal{V} \longmapsto \mathcal{V}(\nu)$ induces equivalences of the corresponding derived categories, i.e., the equivalences $D(\mathcal{D}_\lambda, K)_w \to D(\mathcal{D}_{\lambda+\nu}, K)_w$ and $D(\mathcal{D}_\lambda, K) \to D(\mathcal{D}_{\lambda+\nu}, K)$.

Let H be a closed subgroup of K. Then we have the "forgetful" functor For : $D(\mathcal{D}_\lambda, K) \to D(\mathcal{D}_\lambda, H)$. Clearly, it commutes with twists, i.e., the following diagram is commutative:

$$\begin{array}{ccc} D(\mathcal{D}_\lambda, K) & \xrightarrow{\text{For}} & D(\mathcal{D}_\lambda, H) \\ {\scriptstyle -(\nu)}\downarrow & & \downarrow{\scriptstyle -(\nu)} \\ D(\mathcal{D}_{\lambda+\nu}, K) & \xrightarrow[\text{For}]{} & D(\mathcal{D}_{\lambda+\nu}, H) \end{array}$$

for any $\nu \in Q(\Sigma)$. Moreover, by 4.5 and 4.6, we have a commutative diagram

$$\begin{array}{ccc} D(\mathcal{D}_\lambda, K) & \xrightarrow{\text{For}} & D(\mathcal{D}_\lambda, H) \\ {\scriptstyle R\Gamma}\downarrow & & \downarrow{\scriptstyle R\Gamma} \\ D(\mathcal{U}_\theta, K) & \xrightarrow[\text{For}]{} & D(\mathcal{U}_\theta, H) \end{array},$$

where the vertical arrows are equivalences for λ regular. This, combined with 2.7 and 2.13, implies the following result.

4.7. Theorem.

(i) *The forgetful functor* For : $D(\mathcal{D}_\lambda, K) \to D(\mathcal{D}_\lambda, H)$ *has a right adjoint* $\Gamma^{geo}_{K,H} : D(\mathcal{D}_\lambda, H) \to D(\mathcal{D}_\lambda, K)$ *of finite amplitude.*

(ii) *The functor* $\Gamma^{geo}_{K,H}$ *commutes with twists, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} D(\mathcal{D}_\lambda, H) & \xrightarrow{\Gamma^{geo}_{K,H}} & D(\mathcal{D}_\lambda, K) \\ {\scriptstyle -(\nu)}\downarrow & & \downarrow{\scriptstyle -(\nu)} \\ D(\mathcal{D}_{\lambda+\nu}, H) & \xrightarrow[\Gamma^{geo}_{K,H}]{} & D(\mathcal{D}_{\lambda+\nu}, K) \end{array}$$

for any $\nu \in Q(\Sigma)$.

(iii) *The following diagram is commutative:*

$$\begin{array}{ccc} D(\mathcal{D}_\lambda, H) & \xrightarrow{\Gamma^{geo}_{K,H}} & D(\mathcal{D}_\lambda, K) \\ {\scriptstyle R\Gamma}\downarrow & & \downarrow{\scriptstyle R\Gamma} \\ D(\mathcal{U}_\theta, H) & \xrightarrow[R\Gamma^{equi}_{K,H}]{} & D(\mathcal{U}_\theta, K) \end{array}.$$

We call the functor $\Gamma^{geo}_{K,H}$ the *geometric Zuckerman functor*. By 4.7 (iii), we can view it as the localization of the equivariant Zuckerman functor $R\Gamma^{equi}_{K,H}$. This functor can be described in $\mathcal{D}$-module theoretic terms using techniques analogous to [4] (the details will appear in [15]). It can be viewed as a generalization of Bernstein's functor of "integration along K-orbits".

5. Cohomology of standard Harish-Chandra sheaves

First we recall the construction of the standard Harish-Chandra sheaves (cf. [14], §6). Let $\lambda \in \mathfrak{h}^*$. Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$.[5] Then the direct image of τ with respect to the inclusion $Q \to X$ is the standard Harish-Chandra sheaf $\mathcal{I}(Q, \tau)$. Since τ is holonomic, $\mathcal{I}(Q, \tau)$ is also a holonomic $\mathcal{D}_\lambda$-module and therefore of finite length. This implies that its cohomologies $H^p(X, \mathcal{I}(Q, \tau))$, $p \in \mathbb{Z}_+$, are Harish-Chandra modules of finite length [13]. In this section, we calculate these cohomology modules in terms of "classical" Zuckerman functors.

Fix $x \in Q$. Denote by $\mathfrak{b}_x$ the Borel subalgebra of $\mathfrak{g}$ corresponding to x, and by S_x the stabilizer of x in K. Then the geometric fiber $T_x(\tau)$ of τ at x is an irreducible finite-dimensional representation ω of S_x. We can view it as an S_x-equivariant connection over the S_x-orbit $\{x\}$. Therefore, we can consider the standard Harish-Chandra sheaf $\mathcal{I}(\omega) = \mathcal{I}(\{x\}, \omega)$. It is an S_x-equivariant $\mathcal{D}_\lambda$-module. The following lemma is critical.

5.1. Lemma.

$$\Gamma^{geo}_{K,S_x}(D(\mathcal{I}(\omega))) = D(\mathcal{I}(Q, \tau))[-\dim Q].$$

The proof of this lemma follows from the geometric description of the functor Γ^{geo}_{K,S_x} which we mentioned at the end of §4. This construction makes sense on any smooth algebraic variety with a K-action. By specialization, the linear form $\lambda+\rho$ determines a linear form on $\mathfrak{b}_x$. By restriction, it determines a linear form μ on the Lie algebra $\mathfrak{s}_x$ of S_x. In turn, μ determines a homogeneous twisted sheaf of differential operators $\mathcal{D}_{Q,\mu}$ on Q. Therefore, we can consider the equivariant derived categories $D^b(\mathcal{D}_{Q,\mu}, S_x)$ and $D^b(\mathcal{D}_{Q,\mu}, K)$ and the functor $\Gamma^{geo}_{K,S_x} : D^b(\mathcal{D}_{Q,\mu}, S_x) \to D^b(\mathcal{D}_{Q,\mu}, K)$. Also, the following diagram commutes

$$\begin{array}{ccc} D^b(\mathcal{D}_{Q,\mu}, S_x) & \xrightarrow{i_{Q,+}} & D^b(\mathcal{D}_\lambda, S_x) \\ \Gamma^{geo}_{K,S_x}\Big\downarrow & & \Big\downarrow\Gamma^{geo}_{K,S_x} \\ D^b(\mathcal{D}_{Q,\mu}, K) & \xrightarrow{i_{Q,+}} & D^b(\mathcal{D}_\lambda, K) \end{array}.$$

Let $j_x : \{x\} \to Q$ be the natural immersion. Then we have the $\mathcal{D}$-module direct image module $\mathcal{J}(\omega) = j_{x,+}(\omega)$. To establish 5.1, by the above diagram, it is enough to show that

$$\Gamma^{geo}_{K,S_x}(D(\mathcal{J}(\omega))) = D(\tau)[-\dim Q].$$

Clearly, $\mathcal{D}_{Q,0}$ is the sheaf of differential operators $\mathcal{D}_Q$ on Q. If ω is trivial, $\tau = \mathcal{O}_Q$, and we put $\mathcal{J} = \mathcal{J}(\omega) = j_{x,+}(\mathbb{C})$. Moreover, the general formula follows by tensoring with τ, from the special case

$$\Gamma^{geo}_{K,S_x}(D(\mathcal{J})) = D(\mathcal{O}_Q)[-\dim Q].$$

[5] Here ρ is the half-sum of positive roots in Σ^+.

We define an action of $K \times K$ on K by

$$(k,l) \cdot h = khl^{-1}, \text{ for } k,h,l \in K.$$

Consider the orbit map $o_x : K \longrightarrow Q$ given by $o_x(k) = k \cdot x$ for $k \in K$. Let $\mathcal{D}_K$ be the sheaf of differential operators on K. Then the inverse image o_x^* is a functor from $D^b(\mathcal{D}_Q, K)$ into $D^b(\mathcal{D}_K, K \times S_x)$. It is an equivalence of categories (compare [4]). Also, o_x^* induces an equivalence of $D^b(\mathcal{D}_Q, S_x)$ with $D^b(\mathcal{D}_K, S_x \times S_x)$. Therefore we have the following commutative diagram:

$$\begin{array}{ccc} D^b(\mathcal{D}_Q, S_x) & \xrightarrow{o_x^*} & D^b(\mathcal{D}_K, S_x \times S_x) \\ {\scriptstyle \Gamma^{geo}_{K,S_x}} \downarrow & & \downarrow {\scriptstyle \Gamma^{geo}_{K \times S_x, S_x \times S_x}} \\ D^b(\mathcal{D}_Q, K) & \xrightarrow{o_x^*} & D^b(\mathcal{D}_K, K \times S_x) \end{array} .$$

The map $k \mapsto k^{-1}$ of K induces the equivalences $D^b(\mathcal{D}_K, S_x \times S_x) \longrightarrow D^b(\mathcal{D}_K, S_x \times S_x)$ and $D^b(\mathcal{D}_K, K \times S_x) \longrightarrow D^b(\mathcal{D}_K, S_x \times K)$. Let $\pi : K \longrightarrow pt$ be the projection of K onto a point pt. Then, as before, the inverse image $\pi^* : D^b(\mathcal{D}_{pt}, S_x) \longrightarrow D^b(\mathcal{D}_K, S_x \times K)$ is an equivalence of categories. Let $q : Q \longrightarrow pt$. This leads to the following commutative diagram:

$$\begin{array}{ccc} D^b(\mathcal{D}_Q, S_x) & \xrightarrow{o_x^*} & D^b(\mathcal{D}_K, S_x \times S_x) \\ {\scriptstyle q^*} \uparrow & & \uparrow {\scriptstyle \mathrm{For}} \\ D^b(\mathcal{D}_{pt}, S_x) & \xrightarrow{\pi^*} & D^b(\mathcal{D}_K, S_x \times K) \end{array}$$

where the horizontal arrows are equivalences of categories. This diagram implies that q^* has a right adjoint Φ and that the following diagram commutes:

$$\begin{array}{ccc} D^b(\mathcal{D}_Q, S_x) & \xrightarrow{o_x^*} & D^b(\mathcal{D}_K, S_x \times S_x) \\ {\scriptstyle \Phi} \downarrow & & \downarrow {\scriptstyle \Gamma^{geo}_{S_x \times K, S_x \times S_x}} \\ D^b(\mathcal{D}_{pt}, S_x) & \xrightarrow{\pi^*} & D^b(\mathcal{D}_K, S_x \times K) \end{array} .$$

Since q is a smooth morphism, the shifted $\mathcal{D}$-module direct image functor $q_+[-\dim Q]$ is the right adjoint of the inverse image q^*. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} D^b(\mathcal{D}_Q, S_x) & \xrightarrow{\mathrm{For}} & D^b(\mathcal{D}_Q) \\ {\scriptstyle \Phi} \downarrow & & \downarrow {\scriptstyle q_+[-\dim Q]} \\ D^b(\mathcal{D}_{pt}, S_x) & \xrightarrow{\mathrm{For}} & D^b(\mathcal{D}_{pt}) \end{array} .$$

This allows us to calculate $\Gamma^{geo}_{K,S_x}(D(\mathcal{J}))$. Following the above equivalences and forgetting the equivariant structure, we see that it corresponds to

$$q_+(D(\mathcal{J}))[-\dim Q] = D((q_+ \circ j_{x,+})(\mathbb{C}))[-\dim Q] = D(\mathbb{C})[-\dim Q]$$

in $D^b(\mathcal{D}_{pt})$. By following the equivalences in the reverse order, we see that this object corresponds to $D(\mathcal{O}_Q)[-\dim Q]$. This completes the sketch of the proof of 5.1.[6]

The formula from 5.1 immediately leads to the following result:

$$R\Gamma(D(\mathcal{I}(Q,\tau))) = R\Gamma(\Gamma^{geo}_{K,S_x}(D(\mathcal{I}(\omega))))[\dim Q] = \Gamma^{equi}_{K,S_x}(R\Gamma(D(\mathcal{I}(\omega))))[\dim Q].$$

On the other hand, if δ is a finite-dimensional algebraic representation of S_x compatible with $\lambda + \rho$, we can view it as a $(\mathfrak{b}_x, S_x)$-module where $\mathfrak{b}_x$ acts by $\lambda + \rho$. Let Ω_X be the invertible $\mathcal{O}_X$-module of top degree differential forms on X. Then its geometric fiber $T_x(\Omega_X)$ is one-dimensional and the Borel subgroup B_x of $\operatorname{Int}(\mathfrak{g})$ acts on it by a character. The differential of this action is equal to the specialization of 2ρ. Therefore, $\delta \otimes T_x(\Omega_X^{-1})$ can be viewed as a $(\mathfrak{b}_x, S_x)$-module where $\mathfrak{b}_x$ acts by the specialization of $\lambda - \rho$. Let

$$M(\delta) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_x)} (\delta \otimes T_x(\Omega_X^{-1})),$$

where $\mathfrak{g}$ acts by left multiplication on the first factor and S_x by the tensor product of the action ϕ on $\mathcal{U}(\mathfrak{g})$ with the natural representation on $\delta \otimes T_x(\Omega_X^{-1})$. Then $M(\delta)$ is a Harish-Chandra module for the pair $(\mathfrak{g}, S_x)$. As a $\mathcal{U}(\mathfrak{g})$-module, $M(\delta)$ is a direct sum of $\dim \delta$ copies of the Verma module $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_x)} \mathbb{C}_{\lambda-\rho}$.

The following result is well known (see, for example, [13]).

5.2. Lemma. *We have*

$$H^p(X, \mathcal{I}(\omega)) = \begin{cases} M(\omega) \text{ for } p = 0; \\ 0 \text{ for } p > 0. \end{cases}$$

Therefore, the above relation implies that

$$R\Gamma(D(\mathcal{I}(Q,\tau))) = \Gamma^{equi}_{K,S_x}(D(\Gamma(\mathcal{I}(\omega))))[\dim Q] = \Gamma^{equi}_{K,S_x}(D(M(\omega)))[\dim Q].$$

Taking the cohomology of this complex and using the generalization of 2.10 from 2.13, we get

$$H^p(X, \mathcal{I}(Q,\tau)) = H^{p+\dim Q}(\Gamma^{equi}_{K,S_x}(D(M(\omega))) = R^{p+\dim Q}\Gamma_{K,S_x}(M(\omega))).$$

This proves the following result which computes the cohomology of standard Harish-Chandra sheaves.

[6]The starting point of our investigation was Bernstein's argument to prove a special case of the duality theorem of [11]. Bernstein explained that argument in a seminar at the Institute for Advanced Study in the fall of 1985. It can be used to prove 5.1. If λ is antidominant and regular, the functor Γ and the localization functor Δ_λ are exact and the formula in 5.1 follows from the results explained in ([6], II.4). The general case of 5.1 follows immediately, since Γ^{geo}_{K,S_x} commutes with the twists by 4.7.

5.3. Theorem. *Let $\lambda \in \mathfrak{h}^*$, Q a K-orbit in X and τ an irreducible K-homogeneous connection compatible with $\lambda + \rho$. Let $x \in Q$ and let S_x be the stabilizer of x in K. Let ω be the representation of S_x in the geometric fiber $T_x(\tau)$. Then we have*

$$H^p(X, \mathcal{I}(Q,\tau)) = R^{p+\dim Q}\Gamma_{K,S_x}(M(\omega))$$

for any $p \in \mathbb{Z}$.

Assume now that, in addition, K is reductive. Denote by T a Levi factor of S_x and by U_x the unipotent radical of S_x. Then we have

$$\dim(K/T) - \dim Q = \dim K - \dim T - (\dim K - \dim S_x) = \dim S_x - \dim T = \dim U_x.$$

By 1.13, we have

$$R^p\Gamma_{K,S_x}(M(\omega)) = R^p\Gamma_{K,T}(M(\omega)), \quad p \in \mathbb{Z}.$$

Denote by $V \longmapsto V\tilde{}$ the contragredient functor on the categories $\mathcal{M}(\mathfrak{g}, T)$ and $\mathcal{M}(\mathfrak{g}, K)$. Then, by the duality theorem for derived Zuckerman modules (see, for example, [9], [17]), we have

$$\begin{aligned} H^p(X, \mathcal{I}(Q,\tau))\tilde{} &= R^{p+\dim Q}\Gamma_{K,T}(M(\omega))\tilde{} \\ &= R^{\dim(K/T)-\dim Q-p}\Gamma_{K,T}(M(\omega)\tilde{}) = R^{\dim U_x - p}\Gamma_{K,T}(M(\omega)\tilde{}), \quad p \in \mathbb{Z}. \end{aligned}$$

This is exactly the statement of the duality theorem of Hecht, Miličić, Schmid and Wolf [11].

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