8.1 Indeterminate Forms of Type 0/0

Remember limits:

e.g. $\lim_{x \to 2} \frac{x^2-2x}{x^2+3x-10}$ we can't just plug in $x=2$ because it gives $\frac{0}{0}$, so we have to simplify more

$$= \lim_{x \to 2} \frac{x(x-2)}{(x-2)(x+5)} = \lim_{x \to 2} \frac{x}{x+5} = \frac{2}{7}$$

Also, remember that for $\lim_{x \to 0} \frac{\sin x}{x}$, we had to argue geometrically, because we couldn't manipulate this function algebraically.

L'Hopital's Rule

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists (either finite or $\pm\infty$), then

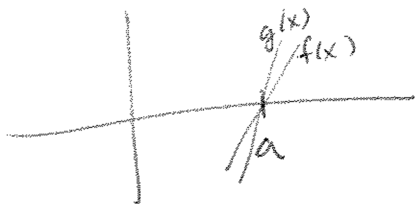
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Ex 1 $\lim_{x \to 0} \frac{\sin x}{x}$

8.1 (cont)

Geometric interpretation of L'Hopital's Rule $\Rightarrow$

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then both graphs either go through the pt $(a,0)$ or have a hole there (that's approached).



For all x super close to a, we can approximate $f(x)$ + $g(x)$ as lines there.

So, as $x\to a$, $f(x) \simeq m(x-a)$ and $g(x) \simeq n(x-a)$ where m = slope of $f(x)$ as $x\to a$ + n = slope of $g(x)$ as $x \to a$.

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{m(x-a)}{n(x-a)} = \lim_{x\to a} \frac{m}{n} = \frac{m}{n} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

That is, the limit of the quotient = ratio of slopes as $x \to a$.

Ex 2 $\displaystyle \lim_{x\to 2} \frac{x^2-2x}{x^2+3x-10} \overset{L}{=} \lim_{x\to 2} \frac{2x-2}{2x+3} = \frac{2}{7}$ ✓

Ex 3 $\displaystyle \lim_{x\to 0} \frac{e^x-e^{-x}}{2\sin x}$

8.1 (cont)

Ex 4 $\displaystyle\lim_{x \to 0^+} \frac{7^{\sqrt{x}} - 1}{2^{\sqrt{x}} - 1}$

Ex 5 $\displaystyle\lim_{x \to 0} \frac{\sin x - \tan x}{x^2 \sin x}$

8.1 (cont)

Ex 6 $\lim_{x \to 0^-} \dfrac{\sin x + \tan x}{e^x + e^{-x} - 2}$

Ex 7 $\lim_{x \to 0^+} \dfrac{\int_0^x \sqrt{t}\cos t \, dt}{x^2}$

8.2 Other Indeterminate Forms

Try $\lim_{x\to\infty} \frac{x}{e^x} \to \frac{\infty}{\infty}$. We have seen rational functions before that originally went to $\frac{\infty}{\infty}$ & we argued that the limit depended on which one went to ∞ "fastest!" So, we were ultimately talking about a rate.

For example, $\lim_{x\to\infty} \frac{x^2+4x-5}{3x^2-2x} \sim \frac{x^2}{3x^2} = \frac{1}{3}$.

Can we use same basic argument for $\lim_{x\to\infty} \frac{x}{e^x}$? Since $y=e^x$ goes to ∞ a lot faster than $y=x$, then $\lim_{x\to\infty} \frac{x}{e^x} = 0$, right?

Can we leverage L'Hopital's Rule? Yes!

L'Hopital's Rule for $\frac{\infty}{\infty}$

If $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (finite or $\pm\infty$).

$$\Rightarrow \lim_{x\to\infty} \frac{x}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

8.2 (cont)

Ex 1 $\lim_{x\to\infty} \frac{x^9}{e^x}$

Ex 2 $\lim_{x\to\infty} \frac{(\ln x)^2}{2^x}$

8.2 (cont)

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

Ex 3 $\lim_{x \to 0} 3x^2 \csc^2 x$

Ex 4 $\lim_{x \to \pi/2} (\tan x - \sec x)$

8.2 (cont)

Indeterminate Forms: 0^0, ∞^0, 1^∞

Ex 5 $\lim_{x \to 0} (\cos x)^{1/x^2}$

Let $y = (\cos x)^{1/x^2} \Rightarrow \ln y = \frac{1}{x^2} \ln(\cos x)$

$\lim_{x \to 0} (\ln y) = \lim_{x \to 0} \frac{\ln(\cos x)}{x^2}$

8.2 (cont)

Ex 6 $\lim_{x \to 0^+} x^x$

Indeterminate Forms

$\left.\begin{array}{l} \frac{0}{0} \\ \frac{\pm\infty}{\pm\infty} \\ 0\cdot\infty \\ \infty-\infty \\ 0^0 \\ \infty^0 \\ 1^\infty \end{array}\right\}$ use L'Hopital's Rule

NOT Indeterminate

$1^0 \to 1$

$0^\infty \to 0$

$\infty^\infty \to \infty$

$\infty\cdot\infty \to \infty$

$\infty+\infty \to \infty$

$\frac{0}{\infty} \to 0$

$\frac{\infty}{0} \to \infty$

8.3 Improper Integrals

Improper Integrals ⇒ like a definite integral except one or both of the limits of integration are $\pm\infty$.

Defn $\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)dx$

$\int_{a}^{\infty} f(x)dx = \lim_{b \to \infty} \int_{a}^{b} f(x)dx$

converge if limit exists and is finite

diverge if limit dne (or goes to $\pm\infty$)

Ex 1 $\int_{-\infty}^{2} e^{x}\,dx$

8.3 (cont)

Ex 2 $\displaystyle\int_1^\infty \frac{dx}{\sqrt{\pi x}}$

Ex 3 $\displaystyle\int_1^\infty \frac{x}{(1+x^2)^2}\,dx$

<u>Defn</u> If $\int_{-\infty}^{0} f(x)\,dx$ and $\int_{0}^{\infty} f(x)\,dx$ converge, then $\int_{-\infty}^{\infty} f(x)\,dx$ converges and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

Otherwise, $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

<u>Ex 4</u> $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+16)^2}$

8.3 (cont)

Ex 5 $\int_1^\infty \frac{1}{x^p}\,dx$

if $p=1$:

if $p\neq 1$:

8.3 (cont)

Probability Density Function (PDF) (needed for continuous random variables)

PDF is defined on $(-\infty, \infty) \ni f(x) \geq 0 \ \forall x$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

probability that random variable x is between a and b is $\int_a^b f(x)\,dx$

mean $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$ [This is weighted average like we saw in center of mass from Calc I!]

variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$ (measures how spread out distribution is)

Ex 6 PDF $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a<x<b \\ 0 & \text{if } x \leq a \text{ or } x \geq b \end{cases}$

(a) show $\int_{-\infty}^{\infty} f(x)\,dx = 1$

8.3 (cont)

Ex6 (cont)

(b) Find μ + σ^2.

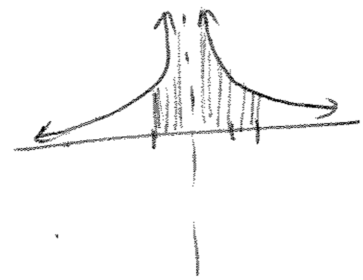
(c) If $a=0$ + $b=10$, find probability that X is less than 2.

8.4 Improper Integrals ; Infinite Integrands

Look at $\int_{-1}^{2} \frac{1}{x^4}\,dx$. Can we calculate it as

$$\frac{x^{-3}}{3}\Big|_{1}^{2} = \frac{2^{-3}}{3} - \frac{(-1)^{-3}}{3} = \frac{1}{24} + \frac{1}{3} = \frac{9}{24} = \frac{3}{8} \ ?$$

Wrong Look at the graph of $\frac{1}{x^4} = f(x)$



It's unbounded at $x=0$!

We have conveniently ignored problems like this so far.

Defn Let $f(x)$ be continuous on $[a,b]$ + $\lim\limits_{x\to b^-} |f(x)| = \infty \Rightarrow \int_a^b f(x)\,dx = \lim\limits_{t\to b^-} \int_a^t f(x)\,dx$

if the limit exists and is finite.

Otherwise, it diverges.

Ex 1 $\int_1^3 \frac{dx}{(x-1)^{4/3}}$

8.4 (cont)

Ex 2 $\displaystyle\int_0^9 \frac{dx}{\sqrt{9-x}}$

Ex 3 $\displaystyle\int_0^1 x^{-p}\,dx \qquad p \geq 1$

if $p=1$:

if $p \neq 1$:

8.4 (cont)

Defn If f is continuous on $[a,b]$ except at $x=c$ where $a<c<b$ & $\lim_{x\to c} |f(x)| = \infty$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ if both integrals converge. Otherwise it diverges.

Ex 4 $\int_{-5}^{0} \frac{1}{(x+3)^2}\,dx$

8.4 (cont)

Ex 5 $\displaystyle\int_{-3}^{1} \frac{5}{(x+2)^{3/5}}\,dx$