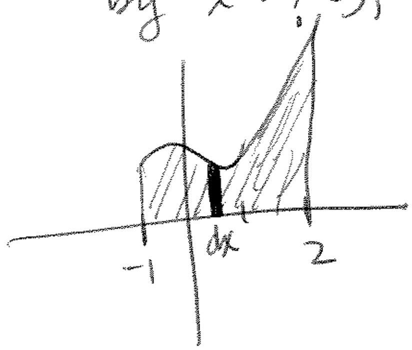
5.1 Area of a Plane Region

$$A = \left\{ \begin{array}{l} \text{area under a curve } f(x) \\ \text{from } x=a \text{ to } x=b \end{array} \right\} = \int_a^b f(x)\,dx$$

Ex 1 Find area of region under $f(x) = x^3 - x + 2$ (bounded below by x-axis), on $[-1, 2]$.



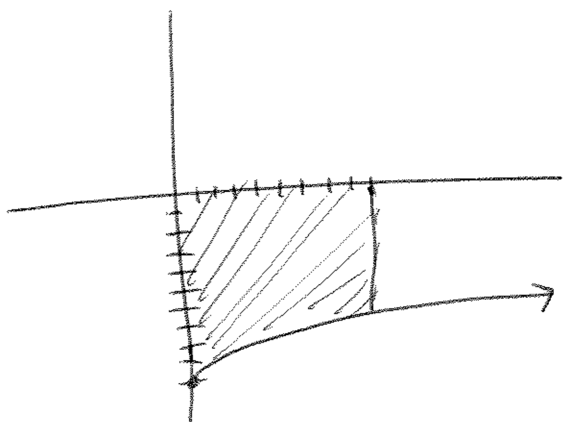
$$A = \int_{-1}^{2} x^3 - x + 2 \quad dx$$

Process

1. Sketch graph.
2. Slice into thin pieces + label. (decide dx or dy)
3. Decide on integration bounds.
4. Take integral of function.

5.1 (continued)

Ex 2 Find area between $y = \sqrt{x} - 10$ & $y = 0$, between $x = 0$ & $x = 9$.



5.1 (continued)

Ex 3 Find area between $y=x^2-9$ + $y=(2x-1)(x+3)$.

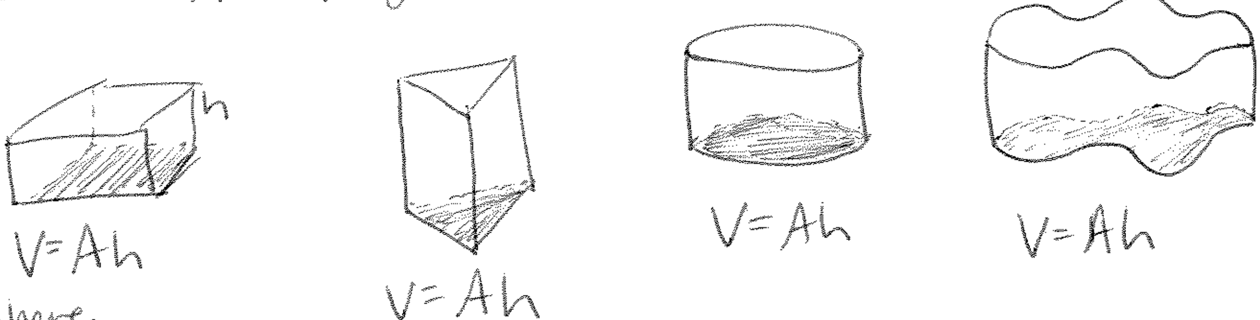
5.1 (continued)

Ex 4 Find the area of the region bounded by $x = y^2 - 2y$ and $x - y - 4 = 0$.

5.2 Volumes of Solids (Slabs, Disks, Washers)

Definite integral = ∞ sum of thin slices of something where the slice has a thickness of dx (or dy)

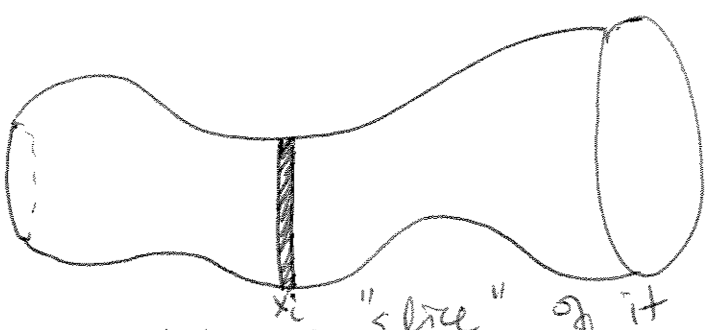
Volume of right prisms/cylinders $\Rightarrow$



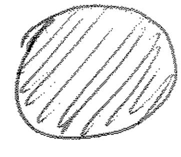
where
A = area of base

$V_{penny} = A_{penny\,face}\,\Delta x$ where Δx = thickness of penny

To find volume of a stack of n pennies,
$$V = \sum_{i=1}^{n} A\,\Delta x$$
(looks like a Riemann sum ☺)



Imagine we have a 3d solid like this + we want to find the volume.

We can take a "slice" of it with a thickness Δx.
The very thin slice will look like  .

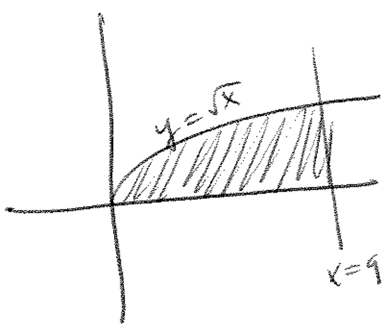
$$\Rightarrow V \approx \sum_{i=1}^{n} A(x_i)\,\Delta x \quad + \quad V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta x$$

$$\Rightarrow \boxed{V = \int_a^b A(x)\,dx}$$
where $A(x)$ is the area of the circular slice

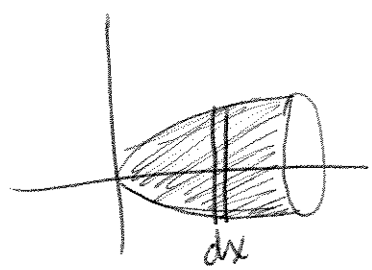
5.2 (continued)

We will now find the volume of a <u>solid of revolution</u>; i.e. a 3d solid generated by revolving a 2d curve about an axis in a 2d plane.

<u>Ex 1</u> Find the volume of the solid of revolution obtained by revolving the region bounded by $y=\sqrt{x}$, the x-axis + the line $x=9$ about the x-axis.



revolved about x-axis



Each slice is  with $A=\pi r^2$.

$$\Rightarrow V=\int_0^9 \pi r^2 \, dx$$

but $r=$ vertical dist. from x-axis to $y=\sqrt{x}$

$$=\pi\int_0^9 (\sqrt{x})^2\,dx$$

$$=\pi\int_0^9 x\,dx = \frac{\pi}{2}x^2\Big|_0^9 = \frac{\pi}{2}(81-0)$$

$$=\boxed{\frac{81\pi}{2}}$$

Disk method process

1. Sketch the graph.
2. Decide on a dy or dx thickness for each slice.
3. Find limits of integration.
4. Determine area function for each slice.
5. Integrate! :)

Hint #4: Area $=\pi r^2$

5.2 (continued)

Ex 2 Find the volume of the solid generated by revolving the region enclosed by $x = \frac{2}{y}$, $y = 2$, $y = 6$, $x = 0$ about the y-axis.

5.2 (continued)

Ex 3 Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 6x$ + $y = 6x^2$.

Washer method

Each slice here is a washer rather than a circle.

So, Area $= \pi\left[r_{outer}^2 - r_{inner}^2\right]$

Otherwise, this is same as Disk method process.

Ex 4 Find the volume of the solid generated by revolving about the line $y=2$ the region in the 1st quadrant bounded by the parabolas $3x^2-16y+48=0$ + $x^2-16y+80=0$ + the y-axis. (Hint: always measure radius from axis of revolution!)

5.3 Volumes of Solids (Shells)

There are 2 methods used to find volume of a solid of revolution

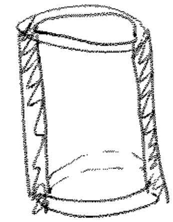
1. Disk/Washer method
2. Shell method

[For most problems, the washer method will be easier. But, the shell method is needed for other cases where the washer method is too ugly.]

We know $V = Ah$ (A = area of base)

For a shell, then

$$\Rightarrow V = (\pi r_o^2 - \pi r_i^2)h$$



$$= \pi(r_o^2 - r_i^2)h$$

r_o = outer radius

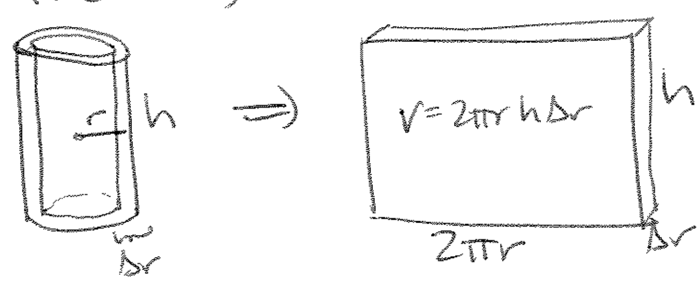
r_i = inner radius

$$= \pi(r_o - r_i)(r_o + r_i)h$$

$$= 2\pi\underbrace{\left(\frac{r_o + r_i}{2}\right)}_{\text{avg radius}}\underbrace{(r_o - r_i)}_{\Delta r}h$$

$$= 2\pi r \Delta r h$$

If we cut our shell down the side, we get:

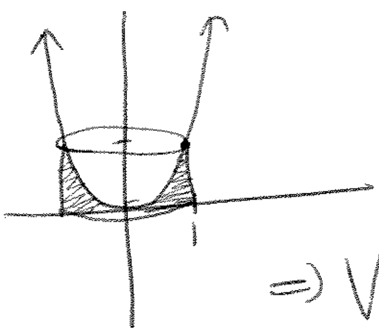


Now, we can think of adding up a bunch of "small thickness" shells to get the volume of a solid cylinder. $\boxed{V = 2\pi \int_a^b x f(x)\,dx}$

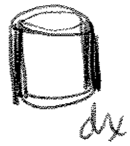
(*See nice picture on pg 288 of book)

5.3 (continued)

Ex 1 Find the volume of the solid generated when the region bounded by $y=x^2$, $x=1$, $y=0$ is revolved about the y-axis. (Use the shell method.)



Our shell will be like



i.e. the shell thickness is dx.

$$\Rightarrow V = 2\pi \int_0^1 \text{radius} \cdot \text{ht} \quad dx$$

radius of shell $= x$ (measured from y-axis)

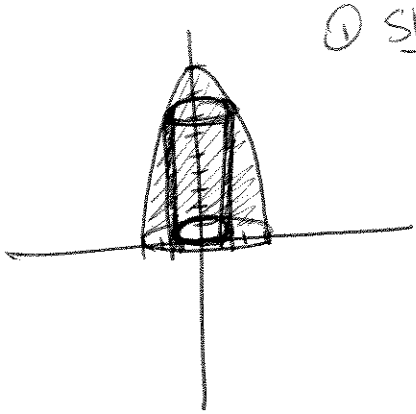
ht of shell $= y$ (measured from x-axis)

$$\Rightarrow \quad V = 2\pi \int_0^1 x\,(x^2)\,dx$$

5.3 (continued)

Ex 2 Find the volume of the solid generated when the region bounded by $y = 9 - x^2$ $(x \geq 0)$, $x = 0$, $y = 0$ is revolved about the y-axis.

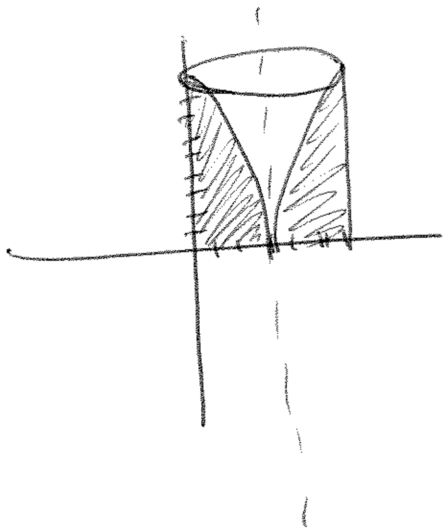
① Shell method:



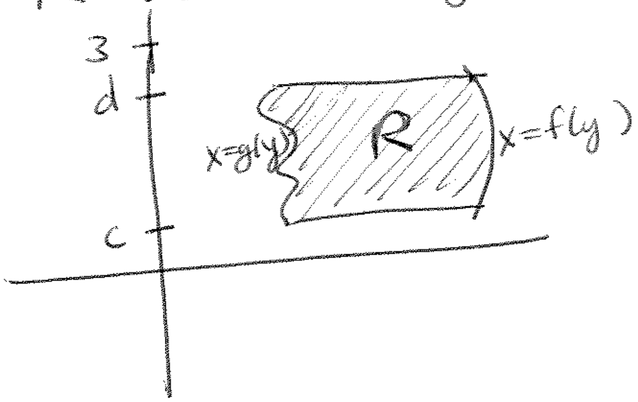
② Disk method:

5.3 (continued)

Ex 3 Find the volume of the solid generated when the region bounded by $y = 9 - x^2$ $(x \geq 0)$, $x = 0$, $y = 0$ is revolved about the line $x = 3$.



Ex 4 A region R is shown below. Set up an integral for the volume obtained by revolving R about the given line.



(a) The y-axis.

(b) The x-axis.

(c) The line $y = 3$.

5.4 Length of a Plane Curve

Plane curve ⇒ a curve that lies in a 2d plane.
we can define a plane curve using
parametric equations; ie. by defining y + x both
as functions of a parameter.

For example, we know from trig that
$y = \sin\theta$ + $x = \cos\theta$ on unit circle.
and $\sin^2\theta + \cos^2\theta = 1$
$\Rightarrow y^2 + x^2 = 1$
+ we know $x^2+y^2=1$ is usual eqn for unit circle in a Cartesian coordinate system.

So, we can define a unit circle as
① $x^2+y^2=1$
or ② $x = \cos\theta$ where θ is our "parameter"
$y = \sin\theta$, $\theta \in [0, 2\pi)$

Ex 1 Sketch the graph of the curve given by
$x = 3t^2+2$ + $y = 2t^2 - 1$ $1 \le t \le 4$

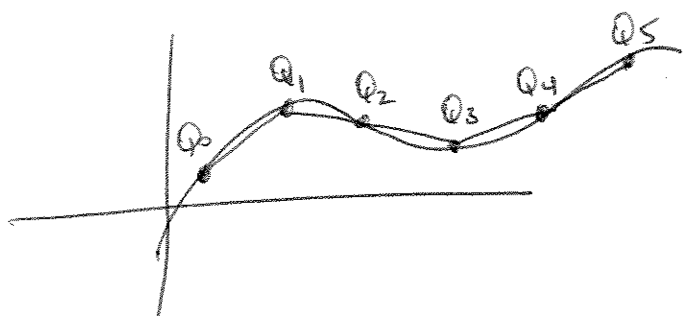
5.4 (continued)

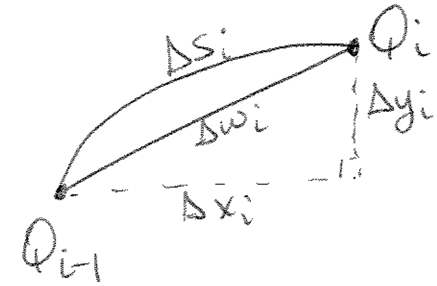
When we trace a plane curve given parametrically, we use arrowheads (visually) to indicate <u>orientation</u> of curve, i.e. how it travels as our parameter increases.

<u>Defn</u> A plane curve is <u>smooth</u> if it is given by a pair of parametric eqns $x=f(t)$, $y=g(t)$, $t\in[a,b]$, where f' & g' exist + are continuous on $[a,b]$, and $f'(t)$ + $g'(t)$ are not simultaneously zero on (a,b).

<u>Arc length</u> (typically s = arc length)

We can approximate length of a plane curve by adding up lengths of linear segments, between Q_i (pts on curve)





linear distance from Q_{i-1} to $Q_i \Rightarrow$

$$\Delta w_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

but $\Delta x_i = f(t_i) - f(t_{i-1})$
+ $\Delta y_i = g(t_i) - g(t_{i-1})$

$$\Rightarrow \Delta w_i = \sqrt{[f(t_i)-f(t_{i-1})]^2 + [g(t_i)-g(t_{i-1})]^2}$$

From MVT for Derivatives, we know $\bar{t}_i$ + $\hat{t}_i$ exist

$\ni \quad f(t_i) - f(t_{i-1}) = f'(\bar{t}_i)\Delta t_i \quad$ w/ $\Delta t_i = t_i - t_{i-1}$

$+ \quad g(t_i) - g(t_{i-1}) = g'(\hat{t}_i)\Delta t_i$

5.4 (continued)

$$\Rightarrow \Delta w_i = \sqrt{[f'(\bar{t}_i)\Delta t_i]^2 + [g'(\hat{t}_i)\Delta t_i]^2}$$

$$= \sqrt{([f'(\bar{t}_i)]^2 + [g'(\hat{t}_i)]^2)\Delta t_i^2}$$

$$= \sqrt{[f'(\bar{t}_i)]^2 + [g'(\hat{t}_i)]^2}\ \Delta t_i$$

Approximate length of curve $= \sum_{i=1}^{n} \Delta w_i$

$$= \sum_{i=1}^{n} \sqrt{[f'(\bar{t}_i)]^2 + [g'(\hat{t}_i)]^2}\ \Delta t_i$$

Arc length

$$\Rightarrow \boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt}$$

where L = arclength of plane curve given by $x = f(t)$ & $y = g(t)$, $a \le t \le b$

i.e. $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt = \boxed{\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}$

If $y = f(x)$ (so no parametric eqns), then

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.}$$

Likewise, if $x = g(y)$, then

$$\boxed{L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy}$$

5.4 (continued)

Ex 1 (A classic ☺) Find the circumference of the circle $x^2+y^2=r^2$.

We can represent this w/ parametric eqns

$x=r\cos\theta \quad y=r\sin\theta \quad \theta\in[0,2\pi]$

$\frac{dx}{d\theta}=-r\sin\theta$

$\frac{dy}{d\theta}=r\cos\theta$

$$\Rightarrow L=\int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{2\pi}\sqrt{r^2\sin^2\theta+r^2\cos^2\theta}\,d\theta$$

$$=\int_0^{2\pi}\sqrt{r^2(\sin^2\theta+\cos^2\theta)}\,d\theta$$

$$=\int_0^{2\pi}\sqrt{r^2}\,d\theta$$

$$=\int_0^{2\pi}r\,d\theta$$

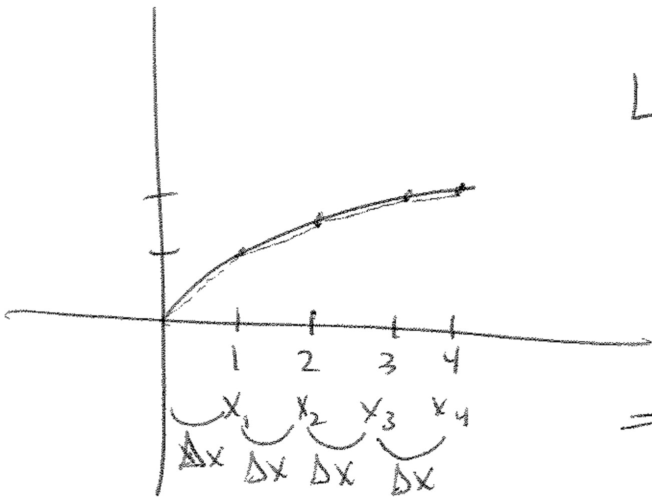
$$=r\int_0^{2\pi}d\theta$$

$$=r(\theta)\Big|_0^{2\pi}=r(2\pi-0)=2\pi r \;//$$

Ex 2 Find length of line segment on $2y-2x+3=0$ between $y=1$ + $y=3$. (Check using distance formula.)

5.4 (continued)

Ex 3 (a) Estimate the arc length of curve $f(x)=\sqrt{x}$ from $x=0$ to $x=4$ by 4 line segments.



$$L \simeq \sum_{i=1}^{4} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

but for this problem

$$\Delta x_i = 1 \quad + \quad \Delta y_i = f(x_i) - f(x_{i-1})$$

$$\Rightarrow L \simeq \sum_{i=1}^{4} \sqrt{1 + [f(x_i) - f(x_{i-1})]^2}$$

$$\Rightarrow L \simeq \sqrt{1+[\sqrt{1}-\sqrt{0}]^2} + \sqrt{1+[\sqrt{2}-\sqrt{1}]^2} + \sqrt{1+[\sqrt{3}-\sqrt{2}]^2} + \sqrt{1+[\sqrt{4}-\sqrt{3}]^2}$$

$$= \sqrt{1+1} + \sqrt{1+2-2\sqrt{2}+1} + \sqrt{1+3-2\sqrt{6}+2} + \sqrt{1+4-4\sqrt{3}+3}$$

$$= \sqrt{2} + \sqrt{4-2\sqrt{2}} + \sqrt{6-2\sqrt{6}} + \sqrt{8-4\sqrt{3}}$$

$$\simeq$$

(b) Find arc length now.

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_0^4 \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx$$

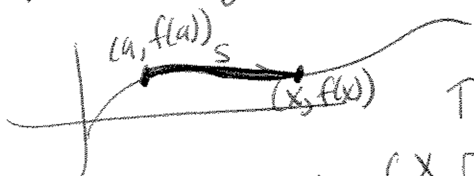
$$= \int_0^4 \sqrt{1+\frac{1}{4x}}\,dx = \int_0^4 \sqrt{\frac{4x+1}{4x}}\,dx$$

Now what?

5.4 (continued) (Surface Area)

Differential of Arc length

Let $f(x)$ be continuously differentiable on $[a,b]$. Start measuring arc length from $(a, f(a))$, up to $(x, f(x))$, where $a \in \mathbb{R}$.



Then, our arc length is a function of x.

i.e. $$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$\Rightarrow s'(x) = \frac{d}{dx} \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

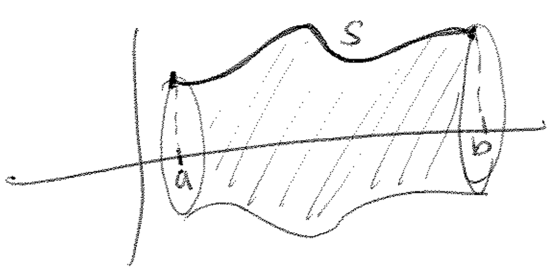
$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \boxed{ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$ arc length differential

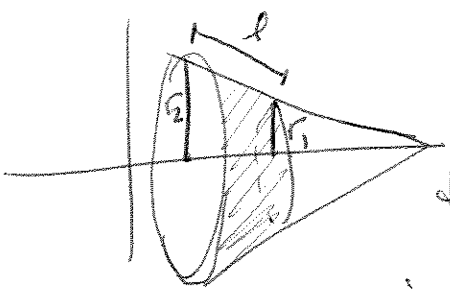
$$\boxed{\text{or } \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Surface Area (of Surface of Revolution)

Now, we'll take a plane curve + rotate it about an axis to create a hollow 3d solid. We're interested in its surface area.



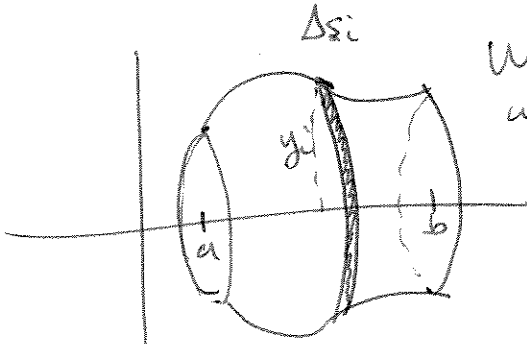
5.4 (continued)



<u>frustum</u> of a cone is a small piece of the cone.

& we know $A = 2\pi \left(\frac{r_1 + r_2}{2} \right) l$

i.e. $A = 2\pi$ (avg radius of frustum) $\cdot$ (slant ht)



We can find surface area by adding up a bunch of little frustum areas!

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi y_i \Delta s_i = \int_a^b 2\pi y \, ds$$

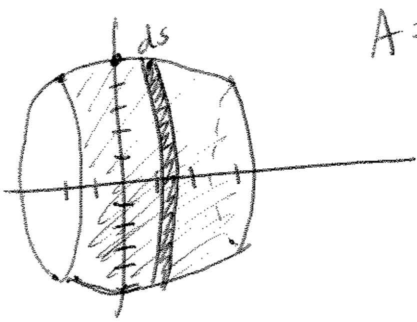
$$= \int_a^b 2\pi f(x) \, ds$$

$$\boxed{A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx}$$

$$\boxed{\text{OR} \quad A = 2\pi \int_a^b g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt \quad \text{if parametric eqns}}$$

<u>Ex 1</u> Find the area of the surface generated by revolving $y = \sqrt{25 - x^2}$ $\forall x \in [-2, 3]$ about the x-axis.

$$A = 2\pi \int_a^b y \sqrt{1 + (y')^2} \, dx$$



5.4 (continued)

Ex 2 Find area of the surface generated by revolving $x = 1 - t^2$, $y = 2t$, $t \in [0,1]$ about the x-axis.

5.5 Work

Work = Force · Distance (work done by a force)

$W = FD$

(If force measured in newtons, distance in meters, then work units are joules. If force in pds + distance in ft, then work is ft-pds.)

Force is sometimes variable, in which case we need to approximate the work done in little chunks + then add up all the "chunks" of work. Aha, another case for a definite integral. :)

$$W = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \underbrace{F(x_i)}_{\text{Force at } x_i} \underbrace{\Delta x}_{\text{little bit of distance}} \Longrightarrow \boxed{\int_a^b F(x)\,dx = W}$$

W = work
F(x) = force functn

Springs

Hooke's law says $F(x) = kx$ where k = spring constant, F(x) = force necessary to keep a spring stretched (or compressed) x units beyond (or short of) its natural length.

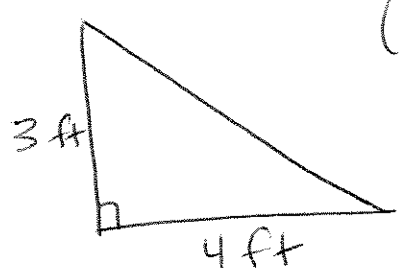
Ex 1 (#1) A force of 6 pds is required to keep a spring stretched ½ ft beyond its normal length. Find the spring constant. And find the work done in stretching the spring ½ ft beyond its natural length.

5.5 (continued)

Ex 2 A force of 1.8 newtons is required to keep a spring of natural length of 0.5 meter compressed to a length of 0.3 m. Find the work done in compressing the spring from its natural length to a length of 0.2 m.

5.5 (continued)

Ex 3 (#10) A tank w/ the triangular cross section (as shown) has a length of 10 ft & is full of water. The water is to be pumped to a height of 5 feet above the top of the tank. Find the work done in emptying the tank.



(Hint: Work is still $W = Fd$ and $F =$ weight and $\delta = 62.4\ \text{lbs}/\text{ft}^3$ is density of water.)

5.6 Moments, Center of Mass

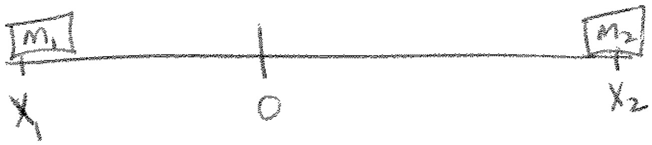
called lever arm

teeter-totter

m_1, m_2 masses (wts)
d_1, d_2 distance from fulcrum

This stays balanced only if $m_1 d_1 = m_2 d_2$.

If we put seesaw on x-axis w/ fulcrum at origin, then to stay balanced we need to satisfy

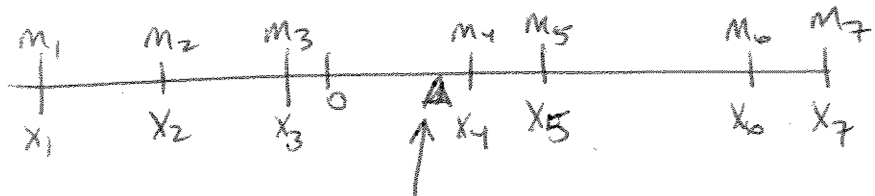


$$x_1 m_1 + x_2 m_2 = 0$$

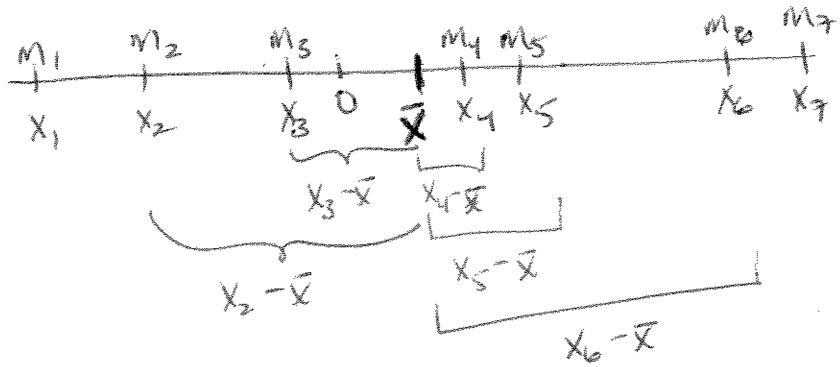
(since $x_1 = -d_1$)

<u>moment of a particle wrt a pt</u> $\Rightarrow$ product of mass m of the particle with its <u>directed</u> distance from a pt. (This measures tendency to produce a rotatn about that pt.)

total moment M for a bunch of masses $= \sum_{i=1}^{n} x_i m_i$



Where does fulcrum need to be placed to balance? Let's call it $\bar{x}$.



5.6 (continued)

Then, for balance at $\bar{x}$, we need

$$(x_1-\bar{x})m_1 + (x_2-\bar{x})m_2 + \dots + (x_n-\bar{x})m_n = 0$$

$$\iff x_1m_1 + x_2m_2 + \dots + x_nm_n = \bar{x}m_1 + \bar{x}m_2 + \dots + \bar{x}m_n$$

$$\iff x_1m_1 + x_2m_2 + \dots + x_nm_n = \bar{x}(m_1 + m_2 + \dots + m_n)$$

$$\bar{x} = \frac{x_1m_1 + x_2m_2 + \dots + x_nm_n}{m_1+m_2+\dots+m_n} = \boxed{\frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \bar{x}}$$

balance pt, a.k.a. center of mass, is just M (total moment w.r.t origin) divided by m (total mass)

Center of mass

i.e. balance pt.

For a continuous mass distribution along a line (like in a wire) $\Rightarrow$ $\boxed{\bar{x} = \frac{M}{m} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx}}$ (where $\delta(x)$ = density function)

Ex 1 John + Mary, weighing 180 lbs + 110 lbs respectively, sit at opposite ends of a 12-ft teeter totter w/ the fulcrum in the middle. Where should their 80-lb son sit in order for the board to balance?

5.6 (continued)

Ex 2 A straight wire 7 units long has density $\delta(x) = 1 + x^3$ at a pt x units from one end. Find the distance from this end to the center of mass.

5.6 (continued)

Now, consider a discrete set on 2d masses.

Then, to find the center of mass (i.e. the geometric center) $(\bar{x}, \bar{y})$, we'll have

$$\bar{x} = \frac{M_y}{m} \quad \& \quad \bar{y} = \frac{M_x}{m}$$

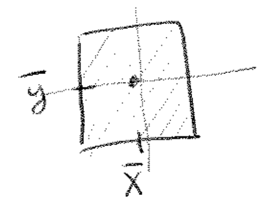
where $M_y = \sum_{i=1}^{n} x_i m_i \quad M_x = \sum_{i=1}^{n} y_i m_i$

and $m = \sum_{i=1}^{n} m_i$

Ex 3 The masses and coordinates of a system of particles are given by the following: 5, (−3, 2); 6, (−2, −2); 2, (3, 5); 7, (4, 3); 1, (7, −1). Find the moments of this system wrt the coord. axes & find center of mass.

5.6 (continued)

Now, consider a continuous 2d region (we'll call it a lamina) that has constant (homogeneous) density everywhere. Then, to find the center of mass $(\bar{x}, \bar{y})$, we'll have (still)



$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

but $M_y = \delta \int_a^b x[f(x)-g(x)]\,dx$

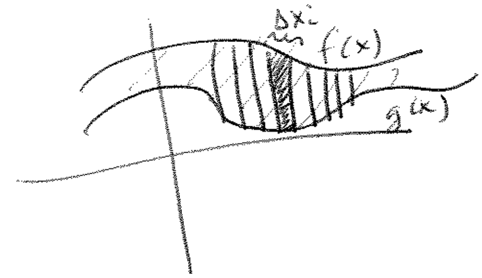
$M_x = \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)]\,dx$

and $m = \delta \int_a^b [f(x) - g(x)]\,dx$

where δ = density of lamina

because m used to be $m = \sum_{i=1}^n m_i$ which becomes

$m = \sum_{i=1}^n \delta \Delta x_i \Delta y_i = \delta \sum_{i=1}^n \Delta x_i (f(x_i) - g(x_i))$



$\Rightarrow M_y = \sum_{i=1}^n x_i m_i = \sum_{i=1}^n x_i (f(x_i) - g(x_i)) \delta \Delta x_i$

$= \delta \int_a^b x (f(x) - g(x))\,dx$

and $\Rightarrow M_x = \sum_{i=1}^n y_i m_i = \sum_{i=1}^n \overbrace{\left(\frac{f(x_i)+g(x_i)}{2}\right)}^{\text{avg } y\text{-value}} \left(\delta (f(x_i) - g(x_i)) \Delta x_i\right)$

$= \frac{\delta}{2} \sum_{i=1}^n [f^2(x_i) - g^2(x_i)] \Delta x_i$

$= \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)]\,dx$

5.6 (continued)

$$\Rightarrow \bar{x} = \frac{M_y}{m} = \frac{\delta \int_a^b x\,(f(x)-g(x))\,dx}{\delta \int_a^b [f(x)-g(x)]\,dx} = \frac{\int_a^b x\,[f(x)-g(x)]\,dx}{\int_a^b [f(x)-g(x)]\,dx}$$

and $$\bar{y} = \frac{M_x}{m} = \frac{1}{2}\left(\frac{\int_a^b [f^2(x)-g^2(x)]\,dx}{\int_a^b [f(x)-g(x)]\,dx} \right)$$

★ Note: It doesn't depend at all on density! Only depends on shape $\Rightarrow$ geometric problem

center of mass = centroid

$(\bar{x}, \bar{y})$

Ex 4 Find the centroid of the region bounded by $y = x^2$ and $y = x+2$