The dipole distribution is frequently represented as the derivative of the delta distribution $\Delta_y = -\delta'_y$ since

$$\phi'(0) = \lim_{\epsilon \to 0} \frac{\phi(\epsilon) - \phi(-\epsilon)}{2\epsilon} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} (\langle \delta_\epsilon, \phi \rangle - \langle \delta_{-\epsilon}, \phi \rangle)$$

but since δ_y is not a function, it is certainly not differentiable in the usual sense. We define the derivative of δ_y momentarily.

One often sees δ_y written as $\delta(x-y)$ and also $\int_{-\infty}^{\infty} \delta(x-y)\phi(x)dx = \phi(y)$. It should always be kept in mind that this is simply a notational convenience and is not meant to represent an actual integral or that $\delta(x-y)$ is an actual function. The notation $\delta(x-y)$ is a **symbolic function** for the delta distribution δ_y.

The correct way to view δ_y is as an operator on the set of test functions. One should never refer to pointwise values of δ_y since it is not a function, but an operator on functions. The operation $\langle \delta_y, \phi \rangle = \phi(y)$ makes perfectly good sense and we have violated no rules of integration or function theory to make this definition.

The fact that some operators can be viewed as being generated by functions through normal integration should not confuse the issue. δ_y is not such an operator. Another operator that is operator valued but not pointwise valued is the operator (not a linear functional) $L = d/dx$. We know that d/dx cannot be evaluated at the point $x = 3$ for example, but d/dx can be evaluated pointwise only after it has first acted on a differentiable function $u(x)$. Thus, $du/dx = u'(x)$ can be evaluated at $x = 3$ only after the operand $u(x)$ is known. Similarly, $\langle \delta_y, \phi \rangle$ can be evaluated only after $\phi(x)$ is known.

Although distributions are not always representable as integrals, their properties are nonetheless always defined to be consistent with the corresponding property of inner products. The following are some properties of distributions that result from this association.

1. If t is a distribution and $f \in C^\infty$ then ft is a distribution whose action is defined by $\langle ft, \phi \rangle = \langle t, f\phi \rangle$. For example, if f is continuous $f(x)\delta_0 = f(0)\delta_0$. If f is continuously differentiable at 0, $f\delta'_0 = -f'(0)\delta_0 + f(0)\delta'_0$. This follows since

$$\langle f\delta'_0, \phi \rangle = \langle \delta'_0, f\phi \rangle = - \left.(f\phi)'\right|_{x=0} = -f'(0)\phi(0) - f(0)\phi'(0).$$

2. Two distributions t_1 and t_2 are said to be equal on the interval $a < x < b$ if for all test functions ϕ with support in $[a, b]$, $\langle t_1, \phi \rangle = \langle t_2, \phi \rangle$. Therefore it is often said (and this is unfortunately misleading) that $\delta(x) = 0$ for $x \neq 0$.

3. The usual rules of integration are always assumed to hold. For example, by change of scale $t(\alpha x)$ we mean

$$\langle t(\alpha x), \phi \rangle = \frac{1}{|\alpha|} \left\langle t, \phi\left(\frac{x}{\alpha}\right) \right\rangle$$

and the shift of axes $t(x-y)$ is taken to mean

$$\langle t(x-y), \phi\rangle = \langle t, \phi(x+y)\rangle.$$

even though pointwise values of t may not have meaning. It follows for example that $\delta(x-y) = \delta_y$ and $\delta(ax) = \delta(x)/|a| = \frac{1}{|a|}\delta_0$.

4. The derivative t' of a distribution t is defined by $\langle t', \phi\rangle = -\langle t, \phi'\rangle$ for all test functions $\phi \in D$. This definition is natural since, for differentiable functions

$$\langle f', \phi\rangle = \int_{-\infty}^{\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx = -\langle f, \phi'\rangle.$$

Since $\phi(x)$ has compact support, the integration by parts has no boundary contributions at $x = \pm\infty$.

If t is a distribution, then t' is also a distribution. If $\{\phi_n\}$ is a zero sequence in D, then $\{\phi_n'\}$ is also a zero sequence, so that

$$\langle t', \phi_n\rangle = -\langle t, \phi_n'\rangle \to 0 \text{as } n \to \infty.$$

It follows that for any distribution t, the nth distributional derivative $t^{(n)}$ exists and its action is

$$\langle t^{(n)}, \phi\rangle = (-1)^n \langle t, \phi^{(n)}\rangle.$$

Thus, any L^2 function has distributional derivatives of all orders, provided the function is identified with its corresponding distribution.

Examples:

1. The Heaviside distribution (4.4) has derivative

$$\langle H', \phi\rangle = -\int_0^{\infty} \phi'(x)dx = \phi(0)$$

since ϕ has compact support, so that $H' = \delta_0$.

2. The derivative of the δ_0 distribution is

$$\langle \delta_0', \phi\rangle = -\langle \delta_0, \phi'\rangle = -\phi'(0) = \langle \Delta_0, \phi\rangle,$$

which is the dipole distribution at 0.

3. Suppose $f(x)$ is continuously differentiable except at the discrete points $x_1, x_2, \ldots, x_n$ at which $f(x)$ has jump discontinuities Δf_1, Δf_2, $\ldots, \Delta f_n$, respectively, where $\Delta f_k = f(x_k^+) - f(x_k^-)$. Its distribution has derivative given by

$$\begin{aligned}
\langle f', \phi\rangle = & -\langle f, \phi'\rangle = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx \\
= & -\int_{-\infty}^{x_1} f(x)\phi'(x)dx - \int_{x_1}^{x_2} f(x)\phi'(x)dx \\
& \ldots - \int_{x_n}^{\infty} f(x)\phi'(x)dx \\
= & \int_{-\infty}^{\infty} \frac{df}{dx}\phi(x)dx + \sum_{k=1}^{n} \Delta f_k \phi(x_k),
\end{aligned}$$