Instructions: You may treat this take–home final just like a homework assignment. You *may* seek outside help from *any* source including me, the Internet, other students, other teachers, etc.. Just like homework, if you do not have time to typeset your answers via LaTeX, then submit your answers on paper, but you will suffer a 10% penalty. Please submit your answers as a PDF attachment via email, or printed on paper to my mailbox in JWB 228 by 5:00 pm, Friday 5/2.

1. A field is a set which allows you to perform arithmetic with its elements. More precisely, a field is a set, F, that satisfies three criteria:

   1. F is an abelian group under addition $(+)$.
   2. $F - \{0\}$ is an abelian group under mulitiplication $(\times)$.
   3. Addition and multiplication are related through the distributive law:

   $$a(b+c) = ab + ac.$$

   Some examples of fields are $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$. We can enlarge a field by *adjoining* (explained in the next problem) a new number that is not already in the set. Adjoining is *not* the same as unioning the field and the new number.

   For example, $\mathbb{Q} \cup \{\sqrt{2}\}$ is not a field because this new set does not contain the additive inverse of this new number, $-\sqrt{2}$. But neither is the set $\mathbb{Q} \cup \{\sqrt{2}, -\sqrt{2}\}$ a field. This set does not contain the multiplicative inverses of the two new elements. OK, well perhaps if we compute the multiplicative inverses and then union them to $\mathbb{Q}$ we will get a new field!

   $$(\sqrt{2})^{-1} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

   Similarly, we see that $(-\sqrt{2})^{-1} = -\sqrt{2}/2$. Maybe if we union all four of these new numbers to $\mathbb{Q}$, then we will get a field.

   $$S = \mathbb{Q} \cup \left\{ \sqrt{2}, -\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\}.$$

   (a) Explain why the process described in the preceding paragraph is doomed to fail. That is, explain why the above process is unusable in practice for extending a field to a larger field.

   Solution: Your answer here...

   (b) Give a specific example of an expression obtained by combining two elements from S (defined above) via one of the four operations of arithmetic which is *not* in the set S. That is show that S is not closed under one of its four arithmetic operations.

   Hint: An extremely simple expression will suffice. The simpler the better for my sanity's sake.

   Solution: Your answer here...

2. Adjoining a new number to a field is how we create larger fields. For example, we can adjoin $\sqrt{2}$ to $\mathbb{Q}$, and we write $\mathbb{Q}(\sqrt{2})$ which is pronounced "Q adjoin root 2", and defined by

$$\mathbb{Q}(\sqrt{2}) = \left\{ a + b\sqrt{2} \,\middle|\, a, b \in \mathbb{Q} \right\}.$$

Now suppose we wish to extend the field $\mathbb{Q}(\sqrt{2})$ by adjoining $\sqrt{3}$. Before we do this we should ask "Is $\sqrt{3}$ already in $\mathbb{Q}(\sqrt{2})$?" If it is already a member of that set, then adjoining it is superfluous.

Show that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$.

Hint 1: Use proof by contradiction, that is assume the opposite: $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$, and see if this leads to a contradiction. Start by assuming,

$$\sqrt{3} = a + b\sqrt{2} \quad \text{for some } a, b \in \mathbb{Q},$$

then square both sides, to get:

$$\begin{aligned} 3 &= (a + b\sqrt{2})(a + b\sqrt{2}), \\ 3 &= (a^2 + 2b^2) + (2ab)\sqrt{2}, \\ 3 + 0\sqrt{2} &= (a^2 + 2b^2) + (2ab)\sqrt{2}. \end{aligned}$$

That last equation decomposes into two equations. Show that these two equations lead to a contradiction.

Hint 2: What do you know must be true if the product of two rational numbers is 0?

Solution: Your answer here...