MATH 2270-2, SPRING 2006, PRACTICE EXAM 1

0. Go over all HW problems with solution key on the website. And go over the True/False questions on each chapter to prepare for T/F questions.

1. Determine whether the following transformations are linear or not. (Justify your answer by proving completely or giving a counterexample.)

   (1) $T : \mathbb{R}^3 \rightarrow \mathbb{R}$, defined by $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 + x_2 - x_3$.

   (2) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 1 \\ x_2 - 2 \end{pmatrix}$.

2. Let $V = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 | x_1 = 2x_2 - 4x_3 \}$. Show that this is a subspace of $\mathbb{R}^3$ and find a matrix $A$ such that $V = \text{im}(A)$.

3. Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent vectors in $\mathbb{R}^n$. Prove that $\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \vec{v}_3 + \vec{v}_1$ are linearly independent.

4. Let $A = \begin{bmatrix} 3 & 0 & 1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}$.

   (1) Compute $AB$ and $BA$.

   (2) Find a basis of $\ker(A)$ and a basis of $\text{im}(A)$.

5. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation such that

   $T \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $T \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

   Let $B = \left( \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right)$ be a basis of $\mathbb{R}^2$.

   (1) Explain why $B$ forms a basis of $\mathbb{R}^2$.

   (2) Find the $B$-matrix of $T$.

   (3) Find $T \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

6. Determine whether the following statement is true or false and justify your answer by proving or giving a counterexample.

   If a square $n \times n$ matrix $A$ satisfies $A^2 = A$, either $A = I_n$ or $A = O$.

7. Consider a $6 \times 10$ matrix $A$ such that $\text{dim(}\ker(A)) = 3$. What is $\text{dim(}\text{im}(A)) = \text{rank}(A)$?

Date: February 4, 2006.