

MATH 2270-2, SPRING 2006, PRACTICE EXAM 1

0. Go over all HW problems with solution key on the website. And go over the True/False questions on each chapter to prepare for T/F questions.

1. Determine whether the following transformations are linear or not. (Justify your answer by proving completely or giving a counterexample.)

(1) $T : \mathbb{R}^3 \rightarrow \mathbb{R}$, defined by $T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = x_1 + x_2 - x_3$.

(2) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by $T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 1 \\ x_2 - 2 \end{bmatrix}$.

2. Let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = 2x_2 - 4x_3 \right\}$. Show that this is a subspace of \mathbb{R}^3 and find a matrix A such that $V = \mathbf{im}(A)$.

3. Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent vectors in \mathbb{R}^n . Prove that $\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \vec{v}_3 + \vec{v}_1$ are linearly independent.

4. Let $A = \begin{bmatrix} 3 & 0 & 1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}$.

(1) Compute AB and BA .

(2) Find a basis of $\mathbf{ker}(A)$ and a basis of $\mathbf{im}(A)$.

5. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation such that

$$T \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad T \left(\begin{bmatrix} 3 \\ -4 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right)$ be a basis of \mathbb{R}^2 .

(1) Explain why \mathcal{B} forms a basis of \mathbb{R}^2 .

(2) Find the \mathcal{B} -matrix of T .

(3) Find $T \left(\begin{bmatrix} -1 \\ -2 \end{bmatrix} \right)$.

6. Determine whether the following statement is true or false and justify your answer by proving or giving a counterexample.

If a square $n \times n$ matrix A satisfies $A^2 = A$, either $A = I_n$ or $A = O$.

7. Consider a 6×10 matrix A such that $\dim(\mathbf{ker}(A)) = 3$. What is $\dim(\mathbf{im}(A)) = \mathbf{rank}(A)$?